524.

ON THE DEFICIENCY OF CERTAIN SURFACES.

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IF a given point or curve is to be an ordinary or singular point or curve on a surface of the order n, this imposes on the surface a certain number of conditions, which number may be termed the "Postulation"; thus "Postulation of a given curve *quà* i-tuple curve on a surface n" will denote the number of conditions to be satisfied by the surface in order that the given curve may be an i-tuple curve on the surface.

The "deficiency" (Flächengeschlecht) of a given surface of the order n is

$=\frac{1}{6}(n-1)(n-2)(n-3)$ *less* deficiency-value of the several singularities; viz. as shown by Dr Noether, if the surface has a given i-tuple curve, the deficiency-value hereof is

= Postulation of the curve *quà* $(i-1)$ tuple curve on a surface $n-4$;

and if the surface has an i-conical point, the deficiency-value hereof is

= Postulation of the point *quà* $(i-2)$ conical point on a surface $n-4$; viz. this is $=\frac{1}{6}i(i-1)(i-2)$, and is thus independent of the order of the surface.

I remark that if the tangent-cone at the i-conical point has δ double lines and κ cuspidal lines, then the deficiency-value is

$$=\tfrac{1}{6}i(i-1)(i-2)+(i-2)(n-i-1)(\delta+\kappa).$$

In the case of a double or cuspidal curve i is $=2$, and the deficiency-value is

= Postulation of given curve *quà* simple curve on a surface $n-4$;

and so for an ordinary conical point i is $=2$, and the deficiency-value is $=0$: results which were first obtained by Dr Clebsch.

I found in this manner the expression for the deficiency of a surface n having a double and cuspidal curve and the other singularities considered in my "Memoir on the Theory of Reciprocal Surfaces," *Phil. Trans.* vol. CLIX. (1869), [411, *Coll. Math. Papers*, vol. VI. p. 356]; viz. this was

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}y + i - \tfrac{1}{8}\theta,$$

where we have

b, order of double curve,

q, class of Do.,

c, order of cuspidal curve,

r, class of Do.,

β, number of intersections of the two curves, stationary points on b,

y, number of intersections, stationary points on c,

i, number of intersections, not stationary points on either curve,

θ, number of certain singular points on c, the nature of which I do not completely understand; it is here taken to be $=0$.

Before going further I remark that

Postulation of right line *quà* i-tuple on surface n

$$\begin{aligned} &= \tfrac{1}{2}i(i+1)n - \tfrac{1}{6}i(i+1)(2i-5), \\ &= \tfrac{1}{6}i(i+1)(3n-2i+5). \end{aligned}$$

Whence if a surface n has an i-tuple right line, the deficiency-value hereof is

$$= \tfrac{1}{6}i(i-1)(3n-2i-5),$$

or we have

$$\begin{aligned} D &= \tfrac{1}{6}(n-1)(n-2)(n-3) - \tfrac{1}{6}i(i-1)(3n-2i-5) \\ &= \tfrac{1}{6}(i-n+1)(i-n+2)(2i+n-3); \end{aligned}$$

so that $D=0$ if either $i=n-1$ or $i=n-2$; the former case is that of a scroll (skew surface) with a $(n-1)$ tuple right line, the latter that of a surface with a $(n-2)$ tuple line: whence (as shown by Dr Noether) such surface is rationally transformable into a plane.

For a surface of the order n with an i-conical point where the tangent cone has δ double lines and κ cuspidal lines, we have

$$\begin{aligned} D &= \tfrac{1}{6}(n-1)(n-2)(n-3) - \{\tfrac{1}{6}i(i-1)(i-2) + (i-2)(n-i-1)(\delta+\kappa)\} \\ &= \tfrac{1}{6}(n-i-1)\{n^2 + n(i-5) + i^2 - 4i + 6 - 6(i-2)(\delta+\kappa)\}; \end{aligned}$$

viz. for $i=n-1$ this is $D=0$ (in fact, a surface n with a $(n-1)$ conical point is at once seen to be rationally transformable into a plane): and for $i=n$, that is, for a cone of the order n, we have

$$D = -\tfrac{1}{2}(n-1)(n-2) + (n-2)(\delta+\kappa) - (n-3)(\delta+\kappa),$$

where the last term $-(n-3)(\delta+\kappa)$ is added because in the present case the surface has the δ double lines and the κ cuspidal lines.

The formula therefore gives

$$D = -\tfrac{1}{2}(n-1)(n-2) + \delta + \kappa,$$

viz. this is equal to the deficiency of the plane sections *taken negatively.*

I find that the same property exists *first* in the case of a scroll (skew surface) having only a double curve; and *secondly* in the case of a torse (developable surface) having a cuspidal curve with the ordinary singularities; and this being so there can I think be no doubt but that it is true for any scroll or torse whatever—viz. that for any ruled surface whatever the deficiency is equal to that of the plane section taken negatively.

First, for the scroll, we have

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)b + \tfrac{1}{2}q + 2t,$$

which should be

$$= -\tfrac{1}{2}(n-1)(n-2) + b.$$

Salmon's equations give in the case of a scroll

$$3t = (n-4)\{3b - n(n-2)\},$$

$$q = n(n-2)(n-5) - 2(n-6)b,$$

and with these values the relation is at once verified.

Secondly, for the torse; changing the notation into that used for the singularities of the curve and torse, we have

$$D = \tfrac{1}{6}(r-1)(r-2)(r-3) - (r-3)(x+m) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + \alpha,$$

which should be

$$= -\tfrac{1}{2}(m-1)(m-2) + h + \beta.$$

We have $q = r(n-3) - 3\alpha$, and substituting this value and expressing everything in terms of r, m, n by means of the formulæ

$$x = \tfrac{1}{2}(r^2 - r - n - 3m),$$

$$\alpha = m - 3r + 3n,$$

$$\beta = n - 3r + 3m,$$

$$t = \tfrac{1}{6}\{r^3 - 3r^2 - 58r - 3r(n+3m) + 42n + 78m\},$$

$$\gamma = rm + 12r - 14m - 6n,$$

$$h = \tfrac{1}{2}(m^2 - 10m - 3n + 8r),$$

we have after all reductions

$$D = -\tfrac{1}{2}(m+n) + r - 1 = -\tfrac{1}{2}(m-1)(m-2) + h + \beta.$$

We have thus a class of surfaces of *negative deficiency*; viz. any rational transformation of a cone for which the plane section has a given (positive) deficiency produces such a surface: and I think it may be assumed conversely that a surface of negative deficiency is always the rational transformation of a cone for which the deficiency is equal to that of the surface taken with the reverse sign. As an instance, take a quintic surface having a nodal conic and two 3-conical (cubiconical) points (this of course implies that the line joining the two cubiconical points is a line on the surface); the formula for the deficiency is ($n=5,\ b=2,\ q=2,\ r=0,\ t=0$)

$$D=\tfrac{1}{6}(n-1)(n-2)(n-3)-(n-3)b+\tfrac{1}{2}q+2t-2,$$

(viz. a term -1 for each of the cubiconical points)

$$=4-4+1\ \ -2,\ \ =-1.$$

Such a surface can be obtained as the quadric inverse of a cubic cone; viz. taking for the vertex the point $x:y:z:w=\alpha:\beta:\gamma:\delta$ and the cone to pass through the point $x=0,\ y=0,\ z=0$, the equation of the cone is

$$(\delta x-\alpha w,\ \ \delta y-\beta w,\ \ \delta z-\gamma w)^3=0,$$

where $(\alpha,\ \beta,\ \gamma)^3=0$.

Taking Q a quadric function $(x,\ y,\ z)^2$, the transformation in question consists in the change of $x,\ y,\ z,\ w$ into $xw,\ yw,\ zw,\ Q$; viz. the new equation, rejecting the factor w which divides out, is

$$\frac{1}{w}(\delta xw-\alpha Q,\ \ \delta yw-\beta Q,\ \ \delta zw-\gamma Q)^3=0,$$

which is a quintic surface, having the two cubiconical points $x=0,\ y=0,\ z=0$ and $x:y:z:w=\alpha:\beta:\gamma:\frac{1}{\delta}Q_0$ (where Q_0 is the value of Q on writing therein α,β,γ in place of $x,\ y,\ z$): and having the nodal conic $w=0,\ Q=0$.

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