523.

ON THE TRANSFORMATION OF UNICURSAL SURFACES.

[From the *Mathematische Annalen*, vol. III. (1871), pp. 469—474.]

I CONSIDER the question of the transformation (Abbildung auf einer Ebene) of unicursal surfaces. Taking (x, y, z, w) for the coordinates of a point on the surface, (x', y', z') for those of the corresponding point on the plane; then if X', Y', Z', W' denote each of them a function $(x', y', z')^{n'}$, the equations of transformation are

$$x : y : z : w = X' : Y' : Z' : W':$$

and assuming that each of the curves

$$X' = 0, \quad Y' = 0, \quad Z' = 0, \quad W' = 0$$

(or, what is the same thing, the general curve

$$aX' + bY' + cZ' + dW' = 0)$$

passes once through each of α_1 points, twice through each of α_2 points, ..., r times through each of α_r points (for convenience I write α_r instead of α_r'); and writing also

$$n = n'^2 - \Sigma r^2 \alpha_r,$$

$$0 = \tfrac{1}{2}n'(n'+3) - 3 - \Theta - \Sigma \tfrac{1}{2} r(r+1)\alpha_r,$$

(where Θ is $= 0$ or positive except in the case of special relations between the positions of the fixed points $\alpha_1, \alpha_2, \ldots, \alpha_r$), which equations give

$$-n = 3n' - 6 - 2\Theta - \Sigma r\alpha_r;$$

then the order of the surface is $= n$, and the order of the nodal curve is $b = \tfrac{1}{2}(n-2)(n-3) + \Theta$. I assume that the nodal curve has h apparent double points and t actual triple points, but no stationary points, so that q being the class, we have $q = b^2 - b - 2h - 6t$; and I endeavour to find these numbers q, t, h.

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For this purpose, imagine through the nodal curve a surface of the order k, which therefore meets the surface besides in a curve of the order $nk-2b$; this curve I call the k-thic residue of the nodal curve, or simply the "residue." The projection (Abbildung) of the complete intersection kn is a curve of the order kn' passing kr times through each of the points α_r: this is made up of the projection of the nodal curve *once*, and of the projection of the residue. But as shown by Dr Clebsch the projection of the nodal curve is of the order $(n-4)n'+3$, and it passes $(n-4)r+1$ times through each of the points α_r; hence the projection of the residue is of the order $(k-n+4)n'-3$, and it passes $(k-n+4)r-1$ times through each of the points α_r. I assume that the projection of the residue is the *general* curve which satisfies the foregoing conditions, viz. that the residue, and its projection as defined by the foregoing conditions, *depend each of them on the same number of constants.* The necessity for this is I confess by no means obvious: but take as an illustration Steiner's quartic surface as transformed by the equations $x : y : z : w = x'^2 : y'^2 : z'^2 : (x'+y'+z')^2$: the nodal curve consists of three lines meeting in a point, the quadric residue is the remaining intersection of the surface by a quadric cone passing through the three lines; and the projection thereof is a line; the quadric cone, and therefore the conic, each depend upon 2 constants; and the line which is the projection of the conic depends upon the same number (2) of constants: at all events I make the assumption provisionally.

Now in the projection of the residue, we have twice the number of constants

$$=[(k-n+4)n'-3](k-n+4)n' - \Sigma\,[(k-n+4)r-1](k-n+4)r\alpha_r,$$

viz. this is

$$=(k-n+4)^2(n'^2-\Sigma r^2\alpha_r)+(k-n+4)(-3n'+\Sigma r\alpha_r),$$

or, what is the same thing, it is

$$=(k-n+4)^2 n + (k-n+4)(n-6-2\Theta),$$

viz. reducing, and replacing Θ by its value $=-\tfrac{1}{2}(n-2)(n-3)+b$, the number in question is

$$=k^2n+k(-n^2+4n-2b)+2(n-4)b.$$

Now k being $=$ or $>n-3$, the curve of intersection of a given surface n by a surface k depends on

$$\tfrac{1}{6}(k+1)(k+2)(k+3)-\tfrac{1}{6}(k-n+1)(k-n+2)(k-n+3)-1$$

constants; and making the surface k to pass through the curve b we have to subtract herefrom $(k+1)b-\tfrac{1}{2}g-2t$; that is, for the residue, twice the number of constants is

$$=\tfrac{1}{3}(k+1)(k+2)(k+3)-\tfrac{1}{3}(k-n+1)(k-n+2)(k-n+3)-2-2(k+1)b+q+4t,$$

viz. this is

$$=k^2n+k(-n^2+4b-2b)+\tfrac{1}{3}(n-1)(n-2)(n-3)-2b+q+4t.$$

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Hence comparing the two expressions in question we have

$$2(n-4)\,b = \tfrac{1}{3}(n-1)(n-2)(n-3) - 2b + q + 4t,$$

that is,

$$0 = \tfrac{1}{3}(n-1)(n-2)(n-3) - 2(n-3)\,b + q + 4t,$$

or, as I prefer to write it,

$$0 = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)\,b + \tfrac{1}{2}q + 2t;$$

which agrees with a more general formula in my "Memoir on the theory of Reciprocal Surfaces," *Phil. Trans.* vol. CLIX. (1869), [411], see p. 227, [*Coll. Math. Papers*, vol. VI. p. 356]. I consider any two residues, a k-thic residue and a l-thic residue; to each intersection of these there corresponds an intersection of their projections: or the number of intersections of the two residues must be equal to that of the two projections. Now the projections being (as above)

order $(k-n+4)\,n'-3$ passing $(k-n+4)\,r-1$ times through each point α_r,

" $(l-n+4)\,n'-3$ " $(l-n+4)\,r-1$ " " "

the number of the intersections in question is

$$= [(k-n+4)\,n'-3]\,[(l-n+4)\,n'-3] - \Sigma\,[(k-n+4)\,r-1]\,[(l-n+4)\,r-1]\,\alpha_r + \omega,$$

where for a reason which will be afterwards explained I have added the term ω: this is

$$= (k-n+4)(l-n+4)(n'^2 - \Sigma r^2\alpha_r) + (k+l-2n+8)(-3n' + \Sigma r\alpha_r) + 9 - (\Sigma\alpha_r - \omega),$$

viz. it is

$$= (k-n+4)(l-n+4)\,n + (k+l-2n+8)(n-6-2\Theta) + 9 - (\Sigma\alpha_r - \omega),$$

viz. substituting for Θ its value, $= -\tfrac{1}{2}(n-2)(n-3) + b$, and reducing, the number is

$$= kln - 2(k+l)\,b - n^3 + 8n^2 - 16n + 9 + 4(n-4)\,b - (\Sigma\alpha_r - \omega).$$

But the surfaces n, k, l, having in common the curve b which is a nodal curve on n, besides intersect in

$$kln - b\,(n + 2k + 2l - 4) + 2q + qt$$

points (Salmon's *Geometry of three Dimensions*, 2nd Ed. p. 283, except that in the formula as there given the singularity t is not taken account of); that is, the number of intersections of the two residues is

$$= kln - 2(k+l)\,b - (n-4)\,b + 2q + 9t,$$

which is equal to the number of intersections of the two projections[1]: or comparing the numbers in question we have

$$-n^3 + 8n^2 - 16n + 9 + 4(n-4)\,b - (\Sigma\alpha_r - \omega) = -(n-4)\,b + 2q + 9t,$$

that is,

$$2q + 9t = 5(n-4)\,b - n^3 + 8n^2 - 16n + 9 - (\Sigma\alpha_r - \omega).$$

[1] I remark that $n+\lambda$ being positive or not less than $n-3$, two $(n+\lambda)$-thic residues meet in $n(\lambda+4)(\lambda+6) - 12\lambda - 39 - 4(\lambda+4) - (\Sigma\alpha_r - \omega)$ points: in particular, two $(n-3)$-thic residues meet in $3n - 3 - 4\Theta - (\Sigma\alpha_r - \omega)$ points; and two $(n-2)$-thic residues meet in $8n - 15 - 8\Theta - (\Sigma\alpha_r - \omega)$ points.

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But we have already found

$$2q + 8t = 4(n-3)b - \tfrac{2}{3}n^3 + 4n^2 - \tfrac{22}{3}n + 4,$$

and we have therefore

$$t = (n-8)b - \tfrac{1}{3}n^3 + 4n^2 - \tfrac{26}{3}n + 5 - (\Sigma\alpha_r - \omega),$$

and

$$q = -2(n-13)b + n^3 - 14n^2 + 31n - 18 + 4(\Sigma\alpha_r - \omega).$$

I obtain these results in a different manner by investigating expressions for the deficiency (Geschlecht) of the nodal residue $nk - 2b$ and for that of its projection.

First for the projection, we have

$$\begin{aligned}\text{Twice Deficiency} &= [(k-n+4)n' - 4][(k-n+4)n' - 5] \\ &\quad - \Sigma[(k-n+4)r - 1][(k-n+4)r - 2]\alpha_r + 2\omega,\end{aligned}$$

where I have added the term 2ω, as afterwards explained: this is

$$= (k-n+4)^2(n'^2 - \Sigma r^2\alpha_r) + (k-n+4)(-9n' + 3\Sigma r\alpha_r) + 20 - 2(\Sigma\alpha_r - \omega),$$

viz. it is

$$= (k-n+4)^2 n + (k-n+4)(3n - 18 - 6\Theta) + 20 - 2(\Sigma\alpha_r - \omega),$$

or substituting for Θ its value $-\tfrac{1}{2}(n-2)(n-3) + b$ and reducing, it is

$$= k^2n + k(n^2 - 4n - 6b) - 2n^3 + 16n^2 - 32n + 20 + 6(n-4)b - 2(\Sigma\alpha_r - \omega).$$

Next as regards the residue, the number h' of its apparent double points is obtained in terms of h and t by the formula

$$8h + 6t - 2h' = (kn - 4b)(k-1)(n-1) - 2b(k-1),$$

(Salmon, l. c., p. 284, except that the singularity t is not there taken account of); and we thence have

$$\begin{aligned}\text{Twice Deficiency} &= (kn-1)(kn-2) - 2h' \\ &= kn(k+n-4) + 4b^2 + b(-4n - 6k + 12) + 2 - 8h - 6t,\end{aligned}$$

or introducing q instead of h by the formula $4b^2 - 8h = 4q + 4b + 24t$, this is

$$= kn(k+n-4) + b(-4n - 6k + 12) + 4q + 4b + 2 + 18t,$$

viz. it is

$$= k^2n + k(n^2 - 4n - 6b) - 4(n-4)b + 4q + 18t.$$

So that comparing with the deficiency of the projection we have

$$-2n^3 + 16n^2 - 32n + 20 + 6(n-4)b - 2(\Sigma\alpha_r - \omega) = -4(n-4)b + 4q + 2 + 18t,$$

that is,

$$2q + 9t = 5(n-4)b - n^3 + 8n^2 - 16n + 9 - (\Sigma\alpha_r - \omega),$$

the same result as before.

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The necessity for the term ω appears by the consideration that if we apply to the plane figure a Cremona-transformation, thus obtaining a new transformation of the surface, the value of $\Sigma\alpha_r$ will in general be altered; whereas the expressions for q, t should it is clear remain unaltered; and it arises as follows, viz. for certain transformations of the surface the curve of the order $(k-n+4)n'-3$, passing $(k-n+4)r-1$ times through each point α_r and assumed to be the projection of the residue, is not an indecomposable curve but contains a certain number ω of factors (each belonging to a unicursal curve definable by means of the number of its passages through the several points α_r), which factors are to be rejected in order to obtain the equation of the proper residue. Thus reverting to the transformation

$$x : y : z : w = x'^2 : y'^2 : z'^2 : (x'+y'+z')^2$$

of Steiner's surface, the projection of the quadric residue was (as already remarked) a line; applying to the plane figure the ordinary quadric (or inverse) transformation we introduce three fixed points, $(\alpha_2 = 3)$, say these are A, B, C; viz. in the new transformation of the surface the projection of any plane section is a quartic curve having a node at each of the fixed points: the projection of the residue ought clearly to be a conic through the three points; but according to the general formula it is a quintic having at each of these points a triple point: the quintic is in fact made up of the lines BC, CA, AB and of the conic which is the proper residue; viz. in the case in question there are 3 factors thrown out, or we have $\omega = 3$. To apply this to the second investigation of $2q+9t$, by comparison of the two deficiencies, observe that in general if a curve is made up of $\omega+1$ indecomposable curves, the deficiency of the compound curve is equal to the sum of the deficiencies of the component curves $-\omega$; hence if ω of the curves are unicursal, the deficiency of the compound curve is equal to that of the remaining curve $-\omega$; or, what is the same thing, the deficiency of the remaining curve is $=$ that of the compound curve $+\omega$; and the addition of the term $+\omega$ to the expression for the deficiency is thus accounted for. It is easy to see that a like explanation applies to the first investigation of $2q+9t$.

I further remark, reverting to the equations

$$x : y : z : w = X' : Y' : Z' : W'$$

of the transformation, that the product of the ω factors is given as the common factor (if any) of the Jacobians

$$J(Y', Z', W'),\quad J(Z', W', X'),\quad J(W', X', Y') \text{ and } J(X', Y', Z').$$

Such common factor exists whenever we can by a Cremona-transformation of the plane figure reduce the number of the points α_r upon which the transformation of the surface depends; viz. for any given transformation of the surface, ω is equal to the excess of $\Sigma\alpha_r$ above the minimum value of $\Sigma\alpha_r$, or, what is the same thing, $\Sigma\alpha_r - \omega$ is equal to the minimum value of $\Sigma\alpha_r$, and is thus independent of the particular transformation. And of course if $\Sigma\alpha_r$ has this minimum value, viz. if the transformation is such that the number of the points α_r cannot be reduced by any Cremona-trans-

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formation of the plane figure, then we have $\omega = 0$. I presume that for the most simple transformation, that is, when n' has its least value, $\Sigma\alpha_r$ has also its least value, and consequently that ω is $= 0$.

Recapitulating, the results obtained are

$$q = -2(n-13)\,b + n^3 - 14n^2 + 31n - 18 + 4(\Sigma\alpha_r - \omega),$$
$$t = (n-8)\,b - \tfrac{1}{3}n^3 + 4n^2 - \tfrac{26}{3}n + 5 - (\Sigma\alpha_r - \omega),$$

where it will be recollected that

$$b = \tfrac{1}{2}(n-2)(n-3) + \Theta;$$

the formulæ are verified in the several cases:

n'	a_1	a_2'	n	Θ	ω	b	q	t	
2	2	0	2	0	1	0	0	0	Quadric surface
2	1	0	3	1	0	1	0	0	Cubic scroll
2	0	0	4	2	0	3	0	1	Steiner's quartic surface
3	6	0	3	0	0	0	0	0	Cubic surface
3	5	0	4	1	0	2	2	0	Quartic with nodal conic
2	8	1	4	0	0	1	0	0	Do. with nodal line
2	7	1	5	1	0	4	8	0	Quintic with nodal quadriquadric
2	11	0	5	0	0	3	4	0	Do. with nodal skew cubic
2	12	2	5	−1	0	2	0	0	Do. with two non-intersecting nodal lines

which are the transformations chiefly as yet examined: but the first-mentioned case (quadric surface, generalised stereographic projection), although as stated the formulæ are verified with the value $\omega = 1$, does not really come under the foregoing theory. It is interesting to see that they are verified in the last-mentioned case, belonging to a negative value of Θ, that is, to a special system of fixed points.

Cambridge, Dec. 5, 1870.

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