521.

ON DR. WIENER'S MODEL OF A CUBIC SURFACE WITH 27 REAL LINES; AND ON THE CONSTRUCTION OF A DOUBLE-SIXER.

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I.

I CALL to mind that a cubic surface has upon it in general 27 lines which may be all of them real. We may out of the 27 lines (and that in 36 different ways) select 12 lines forming a "double-sixer," viz. denoting such a system of lines by

$$\begin{matrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6, \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6; \end{matrix}$$

then no two lines a meet each other, nor any two lines b, but each line a meets each line b, except that the two lines of a pair (a_1, b_1), (a_2, b_2), ... (a_6, b_6) do *not* meet each other. And such a system of twelve lines leads at once to the remaining fifteen lines; viz. we have a line c_{12}, the intersection of the planes which contain the pairs of lines (a_1, b_2) and (a_2, b_1) respectively.

The model is formed of plaster, and is contained within a cube, the edge of which is $= 18{\cdot}2$ inches: the lines a, b, c are coloured blue, yellow, and red respectively; the lines a_1, b_2, b_5 being at right angles to each other, in such wise that taking the origin at the centre of the cube, the axes parallel to the edges, and the unit of length $= 1{\cdot}6$ inches, the equations of these three lines are

$$\begin{array}{llll} a_1, & x = 0, & y = 0, & \\ b_2, & x = 0, & & z = 1, \\ b_5, & & y = 0, & z = -1. \end{array}$$

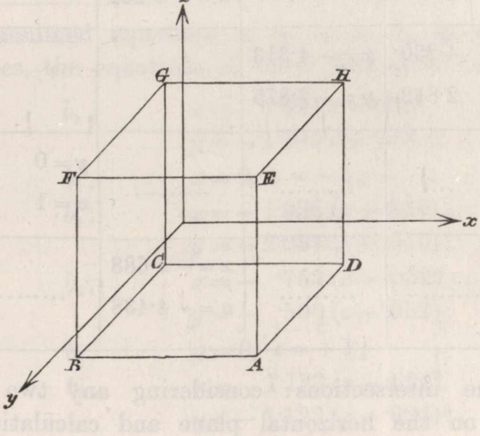
The model is a solid figure bounded by portions of the faces of the cube, and by a portion of the cubic surface, being a surface with three apertures, the collocation of which is not easily explained.

To determine the construction I measured, on the faces of the cube, the coordinates of the two extremities of each of the twelve lines; these were measured in tenths of an inch (taking account of the half division, or twentieth of an inch), and the resulting numbers divided by 16 to reduce them to the before-mentioned unit of 1·6 inches. These reduced values are shewn in the table: knowing then the coordinates of two points on each line, the equations of the several lines became calculable; the true theoretical form of these results—(viz. the form which, but for errors of the model, or of the measurement, they would have assumed)—is

$$
\begin{array}{llll}
b_1, & x = B_1 z + D, & y = B_1' z + D', & \\
b_2, & x = 0, & & z = 1, \\
b_3, & x = B_3(z+\beta_3), & y = B_3'(z+\beta_3), & \\
b_4, & x = B_4(z+\beta_4), & y = B_4'(z+\beta_4), & \\
b_5, & & y = 0, & z = -1, \\
b_6, & x = B_6(z+\beta_6), & y = B_6'(z+\beta_6). & \\
a_1, & x = 0, & y = 0, & \\
a_2, & x = A_2 z + C_2, & y = A_2'(z-1), & \\
a_3, & x = A_3(z+1), & y = A_3'(z-1), & \\
a_4, & x = A_4(z+1), & y = A_4'(z-1), & \\
a_5, & x = A_5(z+1), & y = A_5' z + C_5', & \\
a_6, & x = A_6(z+1), & y = A_6'(z-1); &
\end{array}
$$

but in consequence of such errors, the results are not accurately of the form in question.

The faces of the cube being as in the diagram:



the Table is

Equations calculated from the measurements of the model.		*ABCD* $z=-5{\cdot}688$	*EFGH* $z=+5{\cdot}688$	*AEBF* $y=+5{\cdot}688$	*BCFG* $x=-5{\cdot}688$	*CDGH* $y=-5{\cdot}688$	*AEDH* $x=+5{\cdot}688$
$x=0$ $y=0$	a_1	$x=0$ $y=0$	$x=0$ $y=0$				
$x=-\ {\cdot}780z-\ {\cdot}187$ $y=\ {\cdot}423z+\ {\cdot}406$	a_2	$x=4{\cdot}250$ $y=2{\cdot}812$	$x=-4{\cdot}625$ $y=-2{\cdot}000$				
$x=-\ {\cdot}654z-\ {\cdot}656$ $y=-\ {\cdot}588z+\ {\cdot}531$	a_3	$x=3{\cdot}062$ $y=3{\cdot}875$	$x=4{\cdot}375$ $y=-2{\cdot}812$				
$x=-2{\cdot}912z-2{\cdot}959$ $y=-\ {\cdot}736z+\ {\cdot}752$	a_4	....	....	....	$y={\cdot}0625$ $z={\cdot}9375$	....	$y=2{\cdot}937$ $z=2{\cdot}969$
$x=1{\cdot}024z+1{\cdot}014$ $y=-1{\cdot}049z-\ {\cdot}277$	a_5	$x=-4{\cdot}812$ $y=5{\cdot}688$	....	....	....	....	$y=-5{\cdot}063$ $z=4{\cdot}562$
$x=\ {\cdot}264z+\ {\cdot}187$ $y=-\ {\cdot}104z+\ {\cdot}219$	a_6	$x=-1{\cdot}313$ $y={\cdot}8125$	$x=1{\cdot}687$ $y=-\ {\cdot}375$				
$x=-1{\cdot}611z+\ {\cdot}151$ $y=-1{\cdot}438z+\ {\cdot}288$	b_1				$y=5{\cdot}500$ $z=-3{\cdot}625$	....	$y=-4{\cdot}656$ $z=3{\cdot}437$
$x=0$ $z=-1$	b_2	....	....	$z=-1$	....	$x=0$ $z=-1$	
$x=-1{\cdot}352z-\ {\cdot}685$ $y=-2{\cdot}034z-\ {\cdot}984$	b_3	....	....	$x=3{\cdot}750$ $z=-3{\cdot}281$		$x=-3{\cdot}812$ $z=2{\cdot}313$	
$x=-\ {\cdot}753z-\ {\cdot}0315$ $y=-\ {\cdot}500z-\ {\cdot}0315$	b_4	$x=4{\cdot}250$ $y=2{\cdot}812$	$x=-4{\cdot}313$ $y=-2{\cdot}875$				
$y=0$ $z=+1$	b_5	....	....	....	$y=0$ $z=1$	....	$y=0$ $z=1$
$x=1{\cdot}123z-\ {\cdot}702$ $y=-1{\cdot}123z+\ {\cdot}702$	b_6	....	....	$x=-5{\cdot}688$ $z=-4{\cdot}438$	....	....	$y=-5{\cdot}688$ $z=5{\cdot}688$

I hence calculate the intersections: considering any two lines which ought to intersect, then projecting on the horizontal plane and calculating x, y the coordinates of the point of intersection of the two projections, these values of x, y substituted in the equations should give the same value of z; but if the lines do not accurately intersect, then the values of z will be different.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	*	0 0 − 1	0 0 − ·495 ± ·011	0 0 − ·052 ∓ ·010	0 0 + 1	0 0 + ·625
a_2	− ·077 + ·455 − ·129 ∓ ·013	*	+ ·381 + ·771 − ·796 ± ·067	+ 4·292 + 2·803 − 5·704 ± ·036	− ·967 − ·008 ∓ ·008 + 1	− ·398 + ·227 + ·346 ± ·076
a_3	− ·423 + ·699 − ·264 ± ·021	− ·001 ∓ ·001 + 1·119 − 1	*	− 4·782 − 3·227 + 6·350 ± ·042	− 1·310 − ·028 ∓ ·028 + 1	− ·673 + ·344 + ·162 ± ·146
a_4	− ·957 + 1·238 − ·674 ± ·014	− ·023 ∓ ·023 + 1·488 − 1	+ 1·286 + 1·736 − 1·398 ± ·060	*	− 5·871 + ·008 ± ·008 + 1	− 1·330 + ·847 − ·344 ± ·215
a_5	+ 2·519 − 1·801 + 1·462 ± ·008	− ·005 ∓ ·005 + ·772 − 1	+ ·282 + ·476 − ·717 ± ·001	+ ·412 + ·194 − ·518 ± ·070	*	+ 18·764 − 14·155 + 15·282 ± 4·052
a_6	+ ·194 + ·214 + ·040 ± ·022	− ·038 ∓ ·038 + ·323 − 1	+ ·045 + ·283 − ·582 ± ·042	− ·131 + ·284 − ·423 ± ·208	+ ·451 + ·057 ± ·057 − 1	*

Starting from the assumed equations of b_2, b_3, b_4, b_5, b_6, a_1, and calculating by the theory the remaining lines, the equations of the b-lines (those of b_1 being calculated) are

$$
\begin{array}{ll}
b_1, & x = 1{\cdot}321\,z - {\cdot}310, \\
 & y = -1{\cdot}295\,z + {\cdot}581\,; \\
b_2, & x = 0,\ z = -1\,; \\
b_3, & x = -1{\cdot}352\,(z + {\cdot}510), \\
 & y = -2{\cdot}034\,(z + {\cdot}510)\,; \\
b_4, & x = -\ \ {\cdot}753\,(z + {\cdot}052), \\
 & y = -\ \ {\cdot}500\,(z + {\cdot}052)\,; \\
b_5, & y = 0,\ z = +1\,; \\
b_6, & x = 1{\cdot}123\,(z - {\cdot}624), \\
 & y = -1{\cdot}123\,(z - {\cdot}624)\,;
\end{array}
$$

and the equations of the a-lines (those of all but a_1 being calculated) are

$$
\begin{array}{lll}
a_1, & x=0,\ y=0\,; & \\
a_2, & x=- & \cdot 753\,z-\cdot 091, \\
 & y=- & \cdot 498\,(z-1)\,; \\
a_3, & x=- & \cdot 609\,(z+1), \\
 & y=- & \cdot 677\,(z-1)\,; \\
a_4, & x=- & 2\cdot 506\,(z+1), \\
 & y=- & \cdot 841\,(z-1)\,; \\
a_5, & x= & \cdot 874\,(z+1), \\
 & y=- & \cdot 967\,z-\cdot 288\,; \\
a_6, & x= & \cdot 170\,(z+1), \\
 & y=- & \cdot 071\,(z-1)\,;
\end{array}
$$

and thence for the points of intersection the coordinates are

	b_1	b_2	b_3	b_4	b_5	b_6
		0	0	0	0	0
a_1	*	0	0	0	0	0
		−1	− ·510	− ·052	+1	+ ·624
	− ·170		+ ·662	+ 197·	− ·844	− ·336
a_2	+ ·446	*	+ ·996	+ 131· i.e. lines a_2, b_4 nearly parallel.	0	+ ·336
	+ ·105		− 1	− 262·	+1	+ ·325
	− ·515	0		− 1·805	−1·218	− ·641
a_3	+ ·782	+1·354	*	− 2·007	0	− ·641
	− ·155	−1		+ 3·964	+1	+ ·053
	−1·071	0	+1·438		−5·012	−1·259
a_4	+1·323	+1·682	+2·164	*	0	+1·259
	− ·573	−1	−1·574		+1	− ·497
	+3·189	0	+ ·259	+ ·383		+6·410
a_5	−2·849	+ ·679	+ ·291	+ ·255	*	−6·410
	+2·649	−1	− ·702	− ·561		+6·333
	+ ·241	0	− ·074	+ ·131	+ ·340	
a_6	+ ·041	+ ·142	+ ·112	+ ·087	0	*
	+ ·417	− 1	− ·565	− ·226	+1	

II.

I have in a paper "On the double-sixers of a cubic surface," *Quart. Math. Journal*, t. x. (1870), pp. 58—71, [459], obtained analytical expressions for the twelve lines of a double-sixer, and also calculated numerical values, which however (as there remarked) did not come out convenient ones for the construction of a figure. A different mode of treatment since occurred to me, by means of the following equation of the cubic surface

$$\left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\left(\frac{xz}{\alpha\gamma}-\frac{yw}{\beta\delta}\right)-k\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\left(\frac{xz}{\alpha'\gamma'}-\frac{yw}{\beta'\delta'}\right)=0,$$

which as will appear is a very convenient one for the purpose. We in fact obtain at once eight lines of the double-sixer; viz. these are

1. $x=0,\ w=0,$

3. $y=0,\ z=0,$

5. $\frac{x}{\alpha}-\frac{y}{\beta}=0,\ \frac{z}{\gamma}-\frac{w}{\delta}=0,$

6. $\frac{x}{\alpha'}-\frac{y}{\beta'}=0,\ \frac{z}{\gamma'}-\frac{w}{\delta'}=0,$

2′. $x=0,\ y=0,$

4′. $z=0,\ w=0,$

5′. $\frac{x}{\alpha'}-\frac{w}{\delta'}=0,\ \frac{y}{\beta'}-\frac{z}{\gamma'}=0,$

6′. $\frac{x}{\alpha}-\frac{w}{\delta}=0,\ \frac{y}{\beta}-\frac{z}{\gamma}=0;$

and also five lines not belonging to the double-sixer, viz.

12. $x=0,\quad \left(-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$

23. $y=0,\quad \left(\frac{x}{\alpha'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$

34. $z=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$

41. $w=0,\quad \left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\right)\frac{1}{\alpha'\gamma'}=0,$

56. $\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}=0,\ \frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}=0.$

The remaining lines of the double-sixer are then easily determined; viz. the lines 3, 5, 6, and 12 are met by the line 2′, and by a second line 1′; this, as a line meeting 3, 5, 6, will be given by equations of the form

$$x-\frac{\alpha}{\beta}y=\phi\left(\frac{\delta}{\gamma}z-w\right),\quad x-\frac{\alpha'}{\beta'}y=\phi\left(\frac{\delta'}{\gamma'}z-w\right),$$

and observing that these equations, writing therein $x=0$, give

$$\frac{z}{\gamma}-\frac{w}{\delta}=-\frac{\alpha}{\beta\delta\phi}y,\quad \frac{z}{\gamma'}-\frac{w}{\delta'}=-\frac{\alpha'}{\beta'\delta'\phi}y,$$

the condition of intersection with the line 12 gives

$$\phi = -\frac{\alpha' - k\alpha}{\delta' - k\delta},$$

which is the value of ϕ in the foregoing equations: and to these we may join the resulting equation

$$y\gamma\gamma'(\alpha\beta' - \alpha'\beta) = z\phi\beta\beta'(\gamma\delta' - \gamma'\delta).$$

Proceeding in like manner for the lines 3′, 2, 4, the equations for the remaining four lines of the double-sixer are

2. $$\phi = \frac{\alpha' - k\alpha}{\beta' - k\beta},$$

$$x - w\frac{\alpha'}{\delta'} = \phi\left(y - z\frac{\beta'}{\gamma'}\right),$$

$$x - w\frac{\alpha}{\delta} = \phi\left(y - z\frac{\beta}{\gamma}\right),$$

$$w\gamma\gamma'(\alpha\delta' - \alpha'\delta) = z\phi\delta\delta'(\beta\gamma' - \beta'\gamma).$$

1′. $$\phi = \frac{\alpha' - k\alpha}{\delta' - k\delta},$$

$$x - y\frac{\alpha'}{\beta'} = \phi\left(z\frac{\delta'}{\gamma'} - w\right),$$

$$x - y\frac{\alpha}{\beta} = \phi\left(z\frac{\delta}{\gamma} - w\right),$$

$$y\gamma\gamma'(\alpha\beta' - \alpha'\beta) = z\phi\beta\beta'(\gamma\delta' - \gamma'\delta).$$

4. $$\phi = \frac{\gamma' - k\gamma}{\delta' - k\delta},$$

$$\phi\left(x\frac{\delta'}{\alpha'} - w\right) = y\frac{\gamma'}{\beta'} - z,$$

$$\phi\left(x\frac{\delta}{\alpha} - w\right) = y\frac{\gamma}{\beta} - z,$$

$$x\phi\beta\beta'(\alpha\delta' - \alpha'\delta) = y\alpha\alpha'(\beta\gamma' - \beta'\gamma).$$

3′. $$\phi = -\frac{\gamma' - k\gamma}{\beta' - k\beta},$$

$$\phi\left(x\frac{\beta'}{\alpha'} - y\right) = z - w\frac{\gamma'}{\delta'},$$

$$\phi\left(x\frac{\beta}{\alpha} - y\right) = z - w\frac{\gamma}{\delta},$$

$$x\phi\delta\delta'(\alpha\beta' - \alpha'\beta) = w\alpha\alpha'(\gamma\delta' - \gamma'\delta).$$

It may be added that:—

In plane $x = 0$,

intersection of 1′ lies on line $z : w = (\alpha\beta' - \alpha'\beta)\gamma\gamma' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta$,

" 2 " $y : z = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\delta' - \alpha'\delta)\gamma\gamma'$,

and that the line joining these intersections is the line 12.

In plane $y = 0$,

intersection of 2 lies on line $x : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : -(\beta\gamma' - \beta'\gamma)\delta\delta'$,

" 3′ " $z : w = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\alpha\beta' - \alpha'\beta)\delta\delta'$,

and that the line joining these intersections is the line 23.

In plane $z=0$,

$$\text{intersection of } 3' \text{ lies on line } x : y = \quad (\gamma\delta' - \gamma'\delta)\,\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$

$$\text{,, } \quad 4 \quad \text{,, } \quad x : w = -(\beta\gamma' - \beta'\gamma)\,\alpha\alpha' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$

and that the line joining these intersections is the line 34.

And in plane $w=0$,

$$\text{intersection of } 4 \text{ is on line } y : z = (\alpha\delta' - \alpha'\delta)\,\beta\beta' : \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta,$$

$$\text{,, } \quad 1' \quad \text{,, } \quad x : y = \alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta : (\gamma\delta' - \gamma'\delta)\,\beta\beta',$$

and that the line joining these intersections is the line 14.

The equations of the remaining ten lines of the surface may be obtained without difficulty, and also the forty-five triple planes, but I do not stop to effect this; the planes $x=0$, $y=0$, $z=0$, $w=0$, are, it is clear, triple planes, containing the lines 1, 2′, 12; 2′, 3, 23; 3, 4′, 34; and 4′, 1, 41 respectively.

If, to fix the ideas, the planes $x=0$, $y=0$, $z=0$, $w=0$ are taken to be those of the tetrahedron $ABCD$ ($x=BCD$ &c., as usual), then the edges AB, BC, CD, DA (but not the remaining opposite edges AC, BD) will be lines on the surface. Each plane of the tetrahedron, for instance ABC ($w=0$), is met by the ten lines not contained therein in two vertices A, C, three points on the edge BA, three points on the edge BC, and two other points, viz. these are the intersections of the plane ABC by the lines 4 and 1′. For the construction of a model it is sufficient to determine the three points on each edge, and the two points, say in the plane ABC and in the plane DBC ($x=0$) respectively; for then each of the remaining eight lines will be determined as a line joining two points in these two planes respectively. If in the first instance k is considered as a variable parameter, then the two points in the plane $w=0$ are given as the intersections of two fixed lines by a variable line (14) rotating round the fixed point $\frac{x}{\alpha} - \frac{y}{\beta} + \frac{z}{\gamma} = 0$, $\frac{x}{\alpha'} - \frac{y}{\beta'} + \frac{z}{\gamma'} = 0$; and the like as regards the two points in the plane $x=0$. By making (with assumed values of the other parameters) the proper drawings for the two planes $w=0$, $x=0$, it is easy to fix upon a convenient value of the parameter k; and I have in this manner succeeded in making a string model of the double-sixer; viz. the coordinates x, y, z, w are taken to be as the perpendicular distances of the current point from the faces of a regular tetrahedron (the coordinates being positive for an interior point); the values of α, β, γ, δ were put $=3$, 4, 5, 6 and those of α', β', γ', $\delta' = 1$, 1, 1, 1; the value of k fixed upon as above was $k = -\frac{1}{8}$; this however brings the lines 2 and 4 too close together (viz. the shortest distance between them is not great enough), and also their apparent intersection too close to their intersections with the line 6′; and it is probable that a slightly different value of k would be better.

The results just obtained may be exhibited in a compendious form as follows:

	x :	y :	z :	w
1 is line BC	0			0
2′ „ CD	0	0		
3 „ DA		0	0	
4′ „ AB			0	0
5 meets CD			γ	δ
„ AB	α	β		
6 „ CD			γ'	δ'
„ AB	α'	β'		
6′ „ BC		β	γ	
„ AD	α			δ
5′ „ BC		β'	γ	
„ AD	α'			δ'
1′ „ AD	$\alpha'-k\alpha$			$\delta'-k\delta$
„ BCD		$-(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
„ ABC	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\alpha'-k\alpha)(\gamma\delta'-\gamma'\delta)\beta\beta'$	$-(\delta'-k\delta)(\alpha\beta'-\alpha'\beta)\gamma\gamma'$	
3′ „ BC		$\beta'-k\beta$	$\gamma'-k\gamma$	
„ ACD	$(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$		$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta'$
„ ABD	$(\beta'-k\beta)(\gamma\delta'-\gamma'\delta)\alpha\alpha'$	$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\gamma'-k\gamma)(\alpha\beta'-\alpha'\beta)\delta\delta'$
2 „ AB	$\alpha'-k\alpha$	$\beta'-k\beta$		
„ BCD		$(\beta'-k\beta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	$(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
„ ACD	$(\alpha'-k\alpha)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$		$-(\beta'-k\beta)(\alpha\delta'-\alpha'\delta)\gamma\gamma'$	$-(\alpha'-k\alpha)(\beta\gamma'-\beta'\gamma)\delta\delta'$
4 „ CD			$\gamma'-k\gamma$	$\delta'-k\delta$
„ ABD	$-(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$-(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$		$(\delta'-k\delta)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$
„ ABC	$(\delta'-k\delta)(\beta\gamma'-\beta'\gamma)\alpha\alpha'$	$(\gamma'-k\gamma)(\alpha\delta'-\alpha'\delta)\beta\beta'$	$(\gamma'-k\gamma)(\alpha\gamma\beta'\delta'-\alpha'\gamma'\beta\delta)$	

$$1' \text{ and } 2 \text{ meet } BCD \text{ on line } \left(-\frac{y}{\beta'}+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(-\frac{y}{\beta}+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$2 \text{ and } 3' \;\; ,, \;\; CDA \;\; ,, \;\; \left(\frac{x}{\alpha'}\quad+\frac{z}{\gamma'}-\frac{w}{\delta'}\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}\quad+\frac{z}{\gamma}-\frac{w}{\delta}\right)\frac{1}{\alpha'\gamma'}=0,$$

$$3' \text{ and } 4 \;\; ,, \;\; DAB \;\; ,, \;\; \left(\frac{x}{\alpha'}-\frac{y}{\beta'}\quad-\frac{w}{\delta'}\right)\frac{1}{\beta\delta}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}\quad-\frac{w}{\delta}\right)\frac{1}{\beta'\delta'}=0,$$

$$4 \text{ and } 1' \;\; ,, \;\; ABC \;\; ,, \;\; \left(\frac{x}{\alpha'}-\frac{y}{\beta'}+\frac{z}{\gamma'}\quad\right)\frac{1}{\alpha\gamma}-k\left(\frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}\quad\right)\frac{1}{\alpha'\gamma'}=0;$$

or calculating the numerical values from the foregoing assumed data, say

	x	$: y$	$: z$	$: w$	
1 is line BC	0			0	
2′ ... CD	0	0			
3 ... DA		0	0		
4′ ... AB			0	0	
5 meets CD			5	6	$z = 45{\cdot}5,\ w = 54{\cdot}5.$
... AB	3	4			$x = 42{\cdot}9,\ y = 57{\cdot}1.$
6 meets CD			1	1	$z = 50,\quad w = 50.$
... AB	1	1			$x = 50,\quad y = 50.$
6′ meets BC		4	5		$y = 44{\cdot}4,\ z = 55{\cdot}6.$
... AD	3			6	$x = 33{\cdot}3,\ w = 66{\cdot}7.$
5′ meets BC		1	1		$y = 50,\quad z = 50.$
... AD	1			1	$x = 50,\quad w = 50.$
1′ meets AD	11			14	$x = 44,\quad z = 56.$
... BCD		−44	70	126	$y = -25{\cdot}1,\ z = 39{\cdot}9,\ w = 71{\cdot}8.$
... ABC	99	44	−70		Not required.
3′ meets BC		12	13		$y = 48,\ z = 52.$
... ACD	−36		117	78	Not required.
... ABD	36	108		−78	$x = 47{\cdot}2,\ y = 141{\cdot}7,\ w = -102{\cdot}3.$
2 meets AB	11	12			$x = 47{\cdot}8,\ y = 52{\cdot}2.$
... BCD		100	180	66	$y = 26{\cdot}4,\ z = 44,\ w = 16{\cdot}2.$
... ACD	−99		189	66	Not required.
4 meets CD			13	14	$z = 48{\cdot}1,\ w = 51{\cdot}9.$
... ABD	42	156		−126	$x = 50{\cdot}5,\ y = 187{\cdot}6,\ w = -151{\cdot}5.$
... ABC	42	156	117		Not required.

$1'$ and 2 meet BCD on line $35y - 32z + 30w = 0$,

2 and $3'$ ... CDA ...... $26x + 22z - 21w = 0$,

$3'$ and 4 ... DAB ...... $8x - 7y - 6w = 0$,

$4'$ and 1 ... ABC ...... $52x - 47y - 44z = 0$,

which last four equations serve as a verification.

The outside numerical values are given in the manner most convenient for the construction of a drawing; viz. when the coordinates refer to a point on an edge of the tetrahedron, or say on the side of an equilateral triangle, then taking the length of this edge (or side) to be $= 100$, the numerical values are fixed so that the sum of the two coordinates may be $= 100$, and the two coordinates thus denote the distances from the extremities of the edge or side: but when the three coordinates belong to a point in the face of the tetrahedron, or say in the plane of an equilateral triangle, then the sum of the coordinates is made $= 86{\cdot}6$, and the three coordinates thus denote the perpendicular distances from the sides of the triangle.

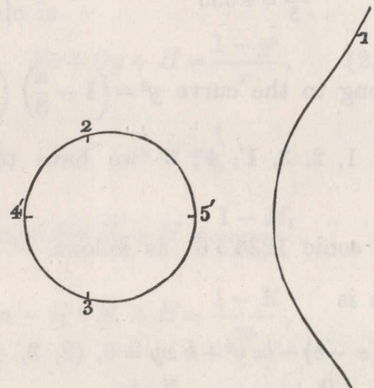
III.

It is possible to find on a cubic curve a double-sixer of points 1, 2, 3, 4, 5, 6 and $1', 2', 3', 4', 5', 6'$ such that any six points such as $1, 2, 3, 4', 5', 6'$ lie in a conic. In fact considering a cubic surface having upon it the double-sixer of lines 1, 2, 3, 4, 5, 6 and $1', 2', 3', 4', 5', 6'$, the section by any plane is a cubic curve meeting the lines, say in the points $1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6'$: each of the lines 1, 2, 3 meets each of the lines $4', 5', 6'$, and consequently the six lines lie in a quadric surface: therefore the points $1, 2, 3, 4', 5', 6'$ lie in a conic: and so in the other cases; the number of the conics is of course $= 60$.

The cubic curve may be a given curve, and six of the points upon it (not being points on a conic) may also be taken to be given; for instance the points $1, 2, 3, 1', 4', 5'$. For take through the points 2, 3 respectively any two lines 1, 2; through $1', 4', 5'$ respectively the lines $1', 4', 5'$ each meeting each of the lines 2, 3: and through 1 a line meeting each of the lines $4', 5'$. It is easy to see that a cubic surface may be drawn through the cubic curve and the lines $1, 2, 3, 1', 4', 5'$: for the passage through the cubic curve requires 9 conditions; the surface then passes through the point 2 and to make it pass through the line 2 requires 3 conditions; similarly the surface passes through the point 3, and to make it pass through the line 3 requires 3 conditions. The surface now passes through $1'$ and through the points of intersection of the line $1'$ with the lines 2, 3: to make it pass through the line $1'$ requires 1 condition; similarly to make it pass through the lines $4', 5', 1$ requires in each case 1 condition; or there are in all 19 conditions, so that the cubic surface is completely determined. Take now through the points $1, 2, 3, 4', 5'$, a conic meeting the cubic in the point $6'$: then through the lines $1, 2, 3, 4', 5'$ we have a quadric surface passing through this conic, and therefore through 6′: hence through 6′ we may draw a line 6′ meeting each of the lines 1, 2, 3; and since the cubic surface passes through the point 6′ and also through the intersections of the line 6′ with the lines 1, 2, 3, it passes through the line 6′. We complete in this manner by constructions in the plane of the cubic the system of the twelve points, viz. each new point is given as the intersection of the cubic curve by a conic drawn through five points of the cubic curve. It is then shown as for the point 6′ and the line 6′ through it, that through each new point there can be drawn a line denoted by the same number and meeting each of the lines which it ought to meet, and hence lying on the cubic surface: the twelve points are thus the intersections of the plane of the cubic curve by the twelve lines of the double-sixer; and it follows that the six points which ought to lie in a conic (in every case where such conic has not been used in the plane construction) do actually lie in a conic.

I was anxious to construct such a double-sixer of points on a cubic curve; for this purpose I take the equation of the curve to be $y^2 = \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right)\left(1 - \frac{x}{c}\right)$, or say for shortness $y^2 = X$; where, to fix the ideas, a, b are supposed to be positive, a greater than b; and c to be negative.

The cubic curve is thus a parabola symmetrical in regard to the axis of x, and consisting of a loop and infinite branch; and I take upon it the points 1, 2, 3, 1′, 4′, 5′



as shown in the figure, viz. the coordinates of these points are as stated in the Table, where m is the x coordinate, and $\sqrt{M} = \sqrt{\left(1 - \frac{m}{a}\right)\left(1 - \frac{m}{b}\right)\left(1 - \frac{m}{c}\right)}$ and so in other cases, $\sqrt{14} = 3{\cdot}74165$.

	x	y	x	y
1	m	$\sqrt{M}$	6	$\sqrt{14} = 3{\cdot}742$
2	0	1	0	1
3	0	-1	0	-1
4	θ	$\sqrt{\Theta}$	$2 - \dfrac{75}{1369} = 1{\cdot}945$	$-\dfrac{2280}{(37)^3}\sqrt{14} = -{\cdot}168$
5	ϕ	$\sqrt{\Phi}$	$-1 + \dfrac{75}{361} = 0{\cdot}792$	$\dfrac{1110}{(19)^3}\sqrt{14} = {\cdot}606$
6	m_1	$-\sqrt{M_1}$	$\dfrac{13}{3} = 4{\cdot}333$	$-\dfrac{4}{9}\sqrt{14} = -1{\cdot}641$
1′	m	$-\sqrt{M}$	6	$-\sqrt{14} = -3{\cdot}742$
2′	σ	$\sqrt{\Sigma}$	$\dfrac{1560\,(14 - \sqrt{14})}{(31\sqrt{14} - 5)^2} = 1{\cdot}299$	$\cdot\;\cdot\; = +{\cdot}676$
3′	τ	$\sqrt{\mathrm{T}}$	$-\dfrac{1560\,(14 + \sqrt{14})}{(31\sqrt{14} + 5)^2} = 1{\cdot}887$	$\cdot\;\cdot\; = +{\cdot}247$
4′	c	0	-1	0
5′	b	0	2	0
6′	m_1	$\sqrt{M_1}$	$\dfrac{13}{3} = 4{\cdot}333$	$\dfrac{4}{9}\sqrt{14} = 1{\cdot}641.$

The numerical values belong to the curve $y^2 = \left(1 - \dfrac{x}{3}\right)\left(1 - \dfrac{x}{2}\right)\left(1 + x\right)$ and to $m = 6$.

Starting with the points 1, 2, 3, 1′, 4′, 5′ we have to find the remaining points 6′, 6, 4, 5, 2′, 3′.

Point 6′ by means of the conic 1234′5′6′, as follows.

The equation of the conic is

$$(x-b)(x-c) - bc\,y^2 + k\,xy = 0, \quad (2,\ 3,\ 4',\ 5'),$$

and making this pass through the point 1 ($x = m$, $y = \sqrt{M}$) we find

$$(m-b)(m-c) + ka\sqrt{M} = 0. \quad (1).$$

Hence taking the coordinates of 6′ to be m_1, $\sqrt{M_1}$, we have

$$(m_1 - b)(m_1 - c) + ka\sqrt{M_1} = 0, \quad (6'),$$

and thence

$$\frac{\sqrt{M_1}}{\sqrt{M}} = \frac{(m_1-b)(m_1-c)}{(m-b)(m-c)} = \frac{M_1\,(m-a)}{M\,(m_1-a)},$$

that is,

$$\frac{\sqrt{M_1}}{\sqrt{M}} = \frac{(m_1 - b)(m_1 - c)}{(m - b)(m - c)} = \frac{m_1 - a}{m - a}.$$

We have thus for m_1 a quadric equation satisfied by $m = m_1$, so that throwing out the factor $m - m_1$, the equation is a linear one, viz. we find

$$m_1 = \frac{ma - ab - ac + bc}{m - a},$$

or, what is the same thing,

$$m_1 - a = \frac{(a - b)(a - c)}{m - a},$$

and thence also

$$\sqrt{M_1} = \frac{(a - b)(a - c)}{(m - a)^2} \sqrt{M},$$

viz. $\sqrt{M_1}$ is determined rationally in terms of m, $\sqrt{M}$; this is of course as it should be, since the point 6′ is uniquely determinate.

Point 6 by means of the conic 2361′4′5′.

In precisely the same manner the coordinates are m_1, $-\sqrt{M_1}$, where m_1, $\sqrt{M_1}$, denote the same quantities as before.

Point 4 by means of the conic 2341′5′6′.

The equation of the conic is

$$Fx + Gy + H = \frac{1 - y^2}{x}, \quad (2, 3),$$

where

$$Fb \qquad + H = \frac{1}{b}, \quad (5')$$

$$Fm_1 + G\sqrt{M_1} + H = \frac{1 - M_1}{m_1}, \quad (6')$$

$$Fm - G\sqrt{M} + H = \frac{1 - M}{m}, \quad (1')$$

which give without difficulty

$$abc\,F = -a - c + P,$$

$$\sqrt{M}\,abc\,G = (m - b)(-m + P),$$

$$abc\,H = ab + ac + bc - bP,$$

where $P = 2a - c - \dfrac{2(a - c)(b - c)}{m + m_1 - 2c}$, a quantity which will presently be expressed in terms of m only.

And then

$$F\theta + G\sqrt{\Theta} + H = \frac{1 - \Theta}{\theta},$$

or say

$$F(\theta-b)+G\sqrt{\Theta}=\frac{1-\Theta}{\theta}-\frac{1}{b}$$

$$=-(\theta-b)\left(\frac{1}{ab}+\frac{1}{bc}-\frac{\theta}{abc}\right),$$

that is,

$$(\theta-b)(abc\,F+a+c-\theta)+G\,abc\sqrt{\Theta}=0,$$

viz. that is,

$$(\theta-b)(P-\theta)+(m-b)\frac{\sqrt{\Theta}}{\sqrt{M}}(P-m)=0$$

or, rationalising and throwing out the factor $\theta-b$, this is

$$(\theta-b)(\theta-P)^2-(m-b)(m-P)^2\frac{(\theta-a)(\theta-b)}{(m-a)(m-b)}=0,$$

which is a cubic equation satisfied by $\theta=m$ and $\theta=m_1$; so that throwing out the factors $\theta-m$, $\theta-m_1$ we have for θ a linear equation.

Putting for shortness

$$A=(m-a)^2-(a-b)(a-c),$$
$$B=(m-b)^2-(b-c)(b-a),$$
$$C=(m-c)^2-(c-a)(c-b),$$

the value of θ may be expressed in the forms

$$\theta-a=\frac{B^2}{C^2}(c-a),\quad \theta-b=\frac{A^2}{C^2}(c-b),\quad \theta-c=\frac{4(m-a)(m-b)(m-c)(b-c)(a-c)}{C^2}.$$

We have moreover

$$P-c=\frac{2(a-c)(m-b)(m-c)}{C},\quad P-m=-\frac{(m-c)A}{C},$$

equations which express P in terms of m only; also

$$\theta-P=\frac{-2(a-c)(m-b)(m-c)B}{C^2},$$

and then

$$\sqrt{\Theta}=-\sqrt{M}\,\frac{\theta-b}{m-b}\,\frac{P-\theta}{P-m},$$

whence

$$\sqrt{\Theta}=2\sqrt{M}(b-c)(c-a)\frac{AB}{C^3},$$

so that θ, $\sqrt{\Theta}$ are now determined.

Point 5 by means of the conic $2351'4'6'$.

The conic is

$$Fx + Gy + H = \frac{1-y^2}{x}, \quad (2, 3)$$

where

$$Fc. \qquad + H = \frac{1}{c}, \quad (4')$$

$$Fm_1 + G\sqrt{M_1} + H = \frac{1-M_1}{m_1}, \quad (6')$$

$$Fm - G\sqrt{M} + H = \frac{1-M}{m}, \quad (1').$$

Everything is the same as for the point 4 except that b, c are interchanged: hence writing Q instead of P, and using A, B, C to denote as before, we have

$$abc\,F = -a-b+Q,$$

$$\sqrt{M}\,abc\,G = (m-c)(-m+Q),$$

$$abc\,H = ab+ac+bc-cQ,$$

and

$$\phi - a = \frac{C^2(b-a)}{B^2},$$

$$\phi - b = \frac{4(m-a)(m-b)(m-c)(c-b)(a-b)}{B^2},$$

$$\phi - c = \frac{A^2(b-c)}{B^2},$$

$$Q - b = \frac{2(m-b)(m-c)(a-b)}{B},$$

$$Q - m = -\frac{A(m-b)}{B},$$

$$\phi - Q = -\frac{2(m-b)(m-c)C(a-b)}{B^2},$$

and

$$\sqrt{\Phi} = 2\sqrt{M}(c-b)(b-a)\frac{AC}{B^3},$$

which determine ϕ, $\sqrt{\Phi}$.

Point $3'$ by means of the conic $1263'4'5'$.

The conic is

$$Fx + Gy + H = \frac{(x-b)(x-c)}{y}, \quad (4', 5')$$

and we have

$$G + H = bc, \qquad (2)$$

$$Fm + G\sqrt{M} + H = \frac{(m-b)(m-c)}{\sqrt{M}}, \qquad (1)$$

$$Fm_1 - G\sqrt{M_1} + H = \frac{(m_1-b)(m_1-c)}{-\sqrt{M_1}}, \qquad (6).$$

Eliminating F, we have

$$G(m_1\sqrt{M} + m\sqrt{M_1}) + H(m - m_1) = \frac{m_1(m-b)(m-c)}{\sqrt{M}} + \frac{m(m_1-b)(m_1-c)}{\sqrt{M_1}},$$

which is easily reduced first to

$$G\frac{2mm_1 - a(m+m_1)}{(m-a)\sqrt{M}} + H(m-m_1) = (m+m_1)\frac{(m-a)(m-b)}{\sqrt{M}},$$

and then to

$$G\{aA + 2m(a-b)(a-c)\} - H\frac{(m-a)A}{\sqrt{M}} + abc\{-A + 2m(m-a)\} = 0;$$

and combining herewith $G + H = bc$, we have

$$H = \frac{2bc\,m\,[a(m-a) + (a-b)(a-c)]}{aA + 2m(a-b)(a-c) + \dfrac{(m-a)A}{\sqrt{M}}};$$

$$G = bc - H;$$

and we have then

$$F(m+m_1) + G(\sqrt{M} - \sqrt{M_1}) + 2H = 0,$$

that is,

$$F\{2m(m-a) - A\} + G\frac{A\sqrt{M}}{m-a} + 2H(m-a) = 0,$$

or, what is the same thing,

$$F\{2m(m-a) - A\} = -bc\frac{A\sqrt{M}}{m-a} - H\left\{2(m-a) - \frac{A\sqrt{M}}{m-a}\right\}.$$

We then have

$$Fx + H = y\left(-G + \frac{(x-b)(x-c)}{y^2}\right)$$

$$= y\left(-G - \frac{abc}{x-a}\right) = -\frac{y(Ha+Gx)}{x-a},$$

that is,

$$(Fx+H)^2 = -\frac{(x-b)(x-c)}{abc(x-a)}(Ha+Gx)^2,$$

or

$$abc\,(x-a)\,(Fx+H)^2+(x-b)\,(x-c)\,(Gx+Ha)^2=0.$$

Developing and throwing out the factor x, this is

$$\begin{aligned}&G^2x^3\\ &+\{2a\,GH-(b+c)\,G^2+abc\,F^2\}\,x^2\\ &+\{a^2H^2-2a\,(b+c)\,GH+bc\,G^2+abc\,(2FH-aF^2)\}\,x\\ &+\{-(b+c)\,a^2H^2+2abc\,GH+abc\,(H^2-2aFH)\}=0.\end{aligned}$$

This must be satisfied by $x=m$, $x=m_1$; hence the left hand must be $=G^2(x-m)(x-m_1)(x-\sigma)$, or equating the constant terms we have

$$G^2\,mm_1\,\sigma=aH\,\{-2abc\,F+2bc\,G+(bc-ab-ac)\,H\},$$

which gives σ; and we then have

$$\sqrt{\Sigma}=-\frac{\sigma-a}{G\sigma+Ha}\,(F\sigma+H),$$

but I have not attempted the further reduction of these expressions.

The numerical values for the example are

$$3F=\frac{-140+62\sqrt{14}}{5+21\sqrt{14}},\quad G=\frac{-10+62\sqrt{14}}{5+21\sqrt{14}},\quad H=\frac{-104\sqrt{14}}{5+\sqrt{14}},$$

whence σ as in the Table.

Point 2′ by means of the conic 1362′4′5′.

The equation of the conic is

$$Fx+Gy+H=\frac{(x-b)\,(x-c)}{y},\qquad (4',\ 5')$$

where

$$-G+H=-bc,\qquad (3)$$

$$Fm+G\sqrt{M}+H=\frac{(m-b)\,(m-c)}{\sqrt{M}},\qquad (1)$$

$$Fm_1-G\sqrt{M_1}+H=\frac{(m_1-b)\,(m_1-c)}{-\sqrt{M_1}},\qquad (6),$$

which are the same as for point 3′, if only we reverse the signs of F, H and $\sqrt{M}$, $\sqrt{M_1}$.

Hence the formulæ are

$$H = -\frac{2bc\,m\,[a(m-a)+(a-b)(a-c)]}{aA+2m(a-b)(a-c)-\dfrac{(m-a)A}{\sqrt{M}}},$$

$$G = \quad bc + H,$$

$$F\{2m(m-a)-A\} = -bc\frac{A\sqrt{M}}{m-a} - H\left\{2(m-a)+\frac{A\sqrt{M}}{m-a}\right\},$$

$$G^2 mm_1\,\tau = aH\{-2abc\,F - 2bc\,G + (bc-ab-ac)H\},$$

which gives τ; and then

$$\sqrt{T} = \frac{(\tau-a)}{G\tau - Ha}(F\tau + H),$$

which are also unreduced.

The numerical values are

$$3F = \frac{140+62\sqrt{14}}{5-21\sqrt{14}},\quad G = \frac{-10-62\sqrt{14}}{5-21\sqrt{14}},\quad H = \frac{-104\sqrt{14}}{5-21\sqrt{14}},$$

whence τ as in the Table.