# 520.

## ON THE CENTRO-SURFACE OF AN ELLIPSOID.

[From the *Transactions of the Cambridge Philosophical Society*, vol. XII. Part I. (1873), pp. 319—365. Read March 7, 1870.]

THE Centro-surface of any given surface is the locus of the centres of curvature of the given surface, or say it is the locus of the intersections of consecutive normals, (the normals which intersect the normal at any particular point of the surface being those at the consecutive points along the two curves of curvature respectively which pass through the point on the surface). The terms, *normal, centre of curvature, curve of curvature,* may be understood in their ordinary sense, or in the generalised sense referring to the case where the Absolute (instead of being the imaginary circle at infinity) is any quadric surface whatever; viz. the normal at any point of a surface is here the line joining that point with the pole of the tangent plane in respect of the quadric surface called the Absolute: and of course the centre of curvature and curve of curvature refer to the normal as just defined.

The question of the centro-surface of a quadric surface has been considered in the two points of view, viz. 1°, when the terms "normal," &c. are used in the ordinary sense, and the equation of the quadric surface (assumed to be an ellipsoid) is taken to be $\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$; 2°, when the Absolute is the surface $X^2+Y^2+Z^2+W^2=0$, and the equation of the quadric surface is taken to be $\alpha X^2+\beta Y^2+\gamma Z^2+\delta W^2=0$: in the first of them by Salmon, *Quart. Math. Jour.* t. II. pp. 217—222 (1858), and in the second by Clebsch, *Crelle,* t. LXII. pp. 64—107 (1863): see also Salmon's *Solid Geometry*, 2nd Ed. 1865, pp. 143, 402, &c. In the present Memoir, as shown by the title, the quadric surface is taken to be an Ellipsoid; and the question is considered exclusively from the first point of view: the theory is further developed in various respects, and in particular as regards the nodal curve upon the centro-surface: the distinction of real and imaginary is of course attended to. The new results suitably modified would be applicable to the theory treated from the second point of view; but I do not on the present occasion attempt so to present them.

*The Ellipsoid; Parameters* $\xi$, $\eta$, &c. Art. Nos. 1—6.

1. The position of a point $(X, Y, Z)$ on the ellipsoid

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1$$

may be determined by means of the parameters, or elliptic coordinates, $\xi$, $\eta$; viz. these are such that we have

$$\frac{X^2}{a^2+\xi}+\frac{Y^2}{b^2+\xi}+\frac{Z^2}{c^2+\xi}=1,$$

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1;$$

or, what is the same thing, $\xi$, $\eta$ are the roots of the quadric equation

$$\frac{X^2}{a^2+\upsilon}+\frac{Y^2}{b^2+\upsilon}+\frac{Z^2}{c^2+\upsilon}=1.$$

(In its actual form this is a cubic equation, but there is a root $\upsilon=0$, which is to be thrown out, and the quadric equation is thus

$$\begin{aligned}&\upsilon^2\\&+\upsilon\,(a^2+b^2+c^2-X^2-Y^2-Z^2)\\&+\{b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)\,X^2-(c^2+a^2)\,Y^2-(a^2+b^2)\,Z^2\}=0,\end{aligned}$$

or putting

$$P=a^2+b^2+c^2,$$

$$Q=b^2c^2+c^2a^2+a^2b^2,$$

$$R=a^2b^2c^2,$$

the equation is

$$\upsilon^2+\upsilon\,(P-X^2-Y^2-Z^2)+Q-(b^2+c^2)\,X^2-(c^2+a^2)\,Y^2-(a^2+b^2)\,Z^2=0.)$$

2. It is convenient to write throughout

$$b^2-c^2=\alpha,$$

$$c^2-a^2=\beta,$$

$$a^2-b^2=\gamma,$$

(whence $\alpha+\beta+\gamma=0$).

As usual, $a$ is taken to be the greatest and $c$ the least of the semi-axes; we have thus $\alpha$, $\gamma$ each of them positive, and $\beta$ negative, $=-\beta'$ where $\beta'$ is a positive quantity $=\alpha+\gamma$. A distinction arises in the sequel between the two cases $a^2+c^2>2b^2$ and $a^2+c^2<2b^2$, but the two cases are not essentially different, and it is convenient to assume $a^2+c^2>2b^2$, that is, $a^2-b^2>b^2-c^2$ or $\gamma>\alpha$, say $\gamma-\alpha$ positive. The limiting case $a^2+c^2=2b^2$ or $\gamma=\alpha$ requires special consideration.

3. We have

$$-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta),$$
$$-\gamma\alpha\ Y^2 = b^2(b^2+\xi)(b^2+\eta),$$
$$-\alpha\beta\ Z^2 = c^2(c^2+\xi)(c^2+\eta).$$

It is in fact easy to verify that these values satisfy as well the equation of the ellipsoid as the assumed equations defining the elliptic coordinates $\xi$, $\eta$. We may also obtain the relations

$$X^2+Y^2+Z^2 = a^2+b^2+c^2+\xi+\eta,$$
$$a^2X^2+b^2Y^2+c^2Z^2 = a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2+(a^2+b^2+c^2)(\xi+\eta)+\xi\eta.$$

These, however, are obtained more readily from the equation in $v$, viz. the roots thereof being $\xi$, $\eta$, we have

$$-\xi-\eta = a^2+b^2+c^2-X^2-Y^2-Z^2,$$
$$\xi\eta = b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)X^2-(c^2+a^2)Y^2-(a^2+b^2)Z^2,$$

which lead at once to the relations in question.

4. Considering $\xi$ as constant, the locus of the point $(X, Y, Z)$ is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2+\xi}+\frac{Y^2}{b^2+\xi}+\frac{Z^2}{c^2+\xi}=1;$$

viz. this is one of the curves of curvature through the point; and similarly considering $\eta$ as constant, the locus of the point is the intersection of the ellipsoid with the confocal ellipsoid

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1;$$

viz. this is the other of the curves of curvature through the point.

5. If instead of $\xi$ and $\eta$ we write $h$ and $k$, we may consider $h$ as extending between the values $-a^2$, $-b^2$, and $k$ as extending between the values $-b^2$, $-c^2$.

$h=$ const. will thus give the series of curves of curvature one of which is the section by the plane $X=0$, or ellipse semi-axes $b$, $c$; say this is the *minor-mean* series. In particular $h=-a^2$ gives the ellipse just referred to; and $h=-b^2$, or say $h=-b^2-\epsilon$, gives two detached portions of the ellipse semi-axes $a$, $c$; viz. each of these portions extends from an umbilicus above the plane of $xy$, through the extremity of the semi-axis $a$, to an umbilicus below the plane of $xy$.

And in like manner $k=\text{const.}$ gives the series of curves of curvature one of which is the section by the plane $Z=0$, or ellipse semi-axes $a, b$; say this is the *major-mean* series. In particular $k=-c^2$ gives the ellipse just referred to; and $k=-b^2$, or say $k=-b^2+\epsilon$, gives the remaining portions of the ellipse semi-axes $a, c$; viz. these are two portions each extending from an umbilicus above the plane of $xy$, through the extremity of the semi-axis $c$, to an umbilicus above the plane of $xy$.

The ellipse last referred to may be called the umbilicar section, the other two principal sections being the major-mean section and the minor-mean section respectively.

In the limiting case $h=k=-b^2$, we have the umbilici, viz. these are given by

$$\frac{X^2}{a^2}=-\frac{\gamma}{\beta},\quad Y=0,\quad \frac{Z^2}{c^2}=-\frac{\alpha}{\beta}.$$

The two series of curves of curvature cover the whole real surface of the ellipsoid; so that at any real point thereof we have $\xi=h$, $\eta=k$, or else $\xi=k$, $\eta=h$, where $h, k$ are negative real values lying within the foregoing limits $-a^2$, $-b^2$ and $-b^2$, $-c^2$ respectively. But observe that $\xi$, $\eta$ taken separately may each extend between the limits $-a^2$, $-c^2$.

6. Suppose $\xi=\eta$, the equation in $v$ will have equal roots, or the condition is

$$(P-X^2-Y^2-Z^2)^2=4\{Q-(b^2+c^2)X^2-(c^2+a^2)Y^2-(a^2+b^2)Z^2\},$$

viz. this surface by its intersection with the ellipsoid determines the envelope of the curves of curvature. This envelope is in fact a system of eight imaginary lines, four of them belonging to one of the systems of right lines on the ellipsoid, the other four to the other of the systems of right lines. For in the values of $X^2$, $Y^2$, $Z^2$ writing $\eta=\xi$, we find

$$\pm\sqrt{-\beta\gamma}\,\frac{X}{a}=a^2+\xi,$$

$$\pm\sqrt{-\gamma\alpha}\,\frac{Y}{b}=b^2+\xi,$$

$$\pm\sqrt{-\alpha\beta}\,\frac{Z}{c}=c^2+\xi,$$

or representing for shortness the left-hand functions by $\pm X'$, $\pm Y'$, $\pm Z'$, the eight lines are

| | | | | |
|---|---|---|---|---|
| $a^2+\xi=$ | $X'$ | $=X'$ | $=-X'$ | $=-X'$ |
| $b^2+\xi=$ | $Y'$ | $=-Y'$ | $=Y'$ | $=-Y'$ |
| $c^2+\xi=$ | $Z'$ | $=-Z'$ | $=-Z'$ | $=Z'$ |
| $a^2+\xi=$ | $-X'$ | $=X'$ | $=X'$ | $=-X'$ |
| $b^2+\xi=$ | $Y'$ | $=-Y'$ | $=Y'$ | $=-Y'$ |
| $c^2+\xi=$ | $Z'$ | $=Z'$ | $=-Z'$ | $=-Z'$, |

so that in the two tetrads each line intersects the four lines of the other tetrad, but it does not intersect the remaining three lines of its own tetrad. The intersections are four points corresponding to $\xi = -a^2$, being the imaginary umbilici in the plane $X = 0$: four to $\xi = -b^2$, being the real umbilici in the plane $Y = 0$: four to $\xi = -c^2$, being the imaginary umbilici in the plane $Z = 0$: and four corresponding to $\xi = \infty$, which may be called the umbilici at infinity (1).

### *Sequential and Concomitant Centro-curves.* Art. No. 7.

7. Consider any particular curve of curvature; the normals at the several points thereof successively intersect each other in a series of points forming a curve; and we have thus, corresponding to the particular curve of curvature, a curve on the centro-surface, which curve may be called the *sequential centro-curve.* Again the same normals, viz. those at the several points of the particular curve of curvature, are intersected, the normal at each point by the consecutive normal belonging to the other curve of curvature through that point; and we have thus, corresponding to the particular curve of curvature, a curve on the centro-surface, which curve may be called the *concomitant centro-curve.* If instead of a single curve of curvature we consider the whole series, say of the major-mean curves of curvature, we have a series of major-mean sequential centro-curves, and also a series of major-mean concomitant centro-curves; and similarly considering the series of the minor-mean curves of curvature we have a series of minor-mean sequential centro-curves and also a series of minor-mean concomitant curves; the configuration of the several curves will be discussed further on, but it may be convenient to remark here that the centro-surface may be considered as consisting of two portions, say,

(A) locus of the major-mean sequential centro-curves; and also of the minor-mean concomitant centro-curves;

(B) locus of the minor-mean sequential centro-curves, and also of the major-mean concomitant centro-curves.

### *Investigation of expressions for the Coordinates of a point on the Centro-surface.* Art. Nos. 8 to 13.

8. Consider the normal at the point $(X, Y, Z)$. Taking in the first instance $(x, y, z)$ as current coordinates, the equations are

$$\frac{x - X}{\frac{X}{a^2}} = \frac{y - Y}{\frac{Y}{b^2}} = \frac{z - Z}{\frac{Z}{c^2}}, = \lambda \text{ suppose,}$$

[1] According to Salmon, *Solid Geometry*, [2nd Ed. 1865], p. 229, the number of umbilici for a surface of the $n^{\text{th}}$ order is $= n(10n^2 - 25n + 16)$; viz. for $n = 2$, this is $= 12$, as in the ordinary theory, not recognizing the umbilici at infinity. But whether properly umbilici or not, the 4 points which I call the umbilici at infinity do in the present theory present themselves in like manner with the 12 umbilici.

or, what is the same thing,

$$x = X\left(1+\frac{\lambda}{a^2}\right),\quad y = Y\left(1+\frac{\lambda}{b^2}\right),\quad z = Z\left(1+\frac{\lambda}{c^2}\right).$$

Suppose now that the normal meets the consecutive normal, or normal at the point $X+dX$, $Y+dY$, $Z+dZ$; and let $x$, $y$, $z$ belong to the point of intersection of the two normals; we must have

$$0 = dX\left(1+\frac{\lambda}{a^2}\right)+\frac{X}{a^2}\,d\lambda,$$

$$0 = dY\left(1+\frac{\lambda}{b^2}\right)+\frac{Y}{b^2}\,d\lambda,$$

$$0 = dZ\left(1+\frac{\lambda}{c^2}\right)+\frac{Z}{c^2}\,d\lambda,$$

which determine the direction of the consecutive point; the equations in fact give

$$0 = \begin{vmatrix} dX, & \dfrac{dX}{a^2}, & \dfrac{X}{a^2} \\ dY, & \dfrac{dY}{b^2}, & \dfrac{Y}{b^2} \\ dZ, & \dfrac{dZ}{c^2}, & \dfrac{Z}{c^2} \end{vmatrix},$$

or, what is the same thing,

$$0 = \begin{vmatrix} a^2dX, & dX, & X \\ b^2dY, & dY, & Y \\ c^2dZ, & dZ, & Z \end{vmatrix},$$

which is the differential equation of the curve of curvature. This equation must therefore be satisfied by taking for $X+dX$, $Y+dY$, $Z+dZ$, the coordinates of the consecutive point along either of the curves of curvature,—say along that which is the intersection with the surface

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1.$$

9. To verify this, observe that we then have

$$\frac{XdX}{a^2}+\frac{YdY}{b^2}+\frac{ZdZ}{c^2}=0,$$

$$\frac{XdX}{a^2+\eta}+\frac{YdY}{b^2+\eta}+\frac{ZdZ}{c^2+\eta}=0;$$

or, what is the same thing,

$$XdX : YdY : ZdZ = a^2(a^2+\eta)\,\alpha : b^2(b^2+\eta)\,\beta : c^2(c^2+\eta)\,\gamma.$$

But from the equations $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$ &c., these become

$$XdX : YdY : ZdZ = \frac{X^2}{a^2+\xi} : \frac{Y^2}{b^2+\xi} : \frac{Z^2}{c^2+\xi},$$

or, what is the same thing,

$$dX : dY : dZ = \frac{X}{a^2+\xi} : \frac{Y}{b^2+\xi} : \frac{Z}{c^2+\xi};$$

and, substituting these values in the determinant equation, it becomes

$$0 = \frac{XYZ}{(a^2+\xi)(b^2+\xi)(c^2+\xi)} \begin{vmatrix} a^2, & 1, & a^2+\xi \\ b^2, & 1, & b^2+\xi \\ c^2, & 1, & c^2+\xi \end{vmatrix},$$

which is identically true, since evidently the determinant vanishes.

10. Proceeding with the solution, we have from the three equations

$$XdX + YdY + ZdZ + \lambda\left(\frac{XdX}{a^2} + \frac{YdY}{b^2} + \frac{ZdZ}{c^2}\right) + d\lambda\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2}\right) = 0,$$

and observing that from the equation

$$X^2 + Y^2 + Z^2 = a^2 + b^2 + c^2 + \xi + \eta,$$

considering therein $\eta$ as constant, we have

$$XdX + YdY + ZdZ = \tfrac{1}{2}d\xi,$$

the equation becomes

$$\tfrac{1}{2}d\xi + d\lambda = 0;$$

and the three equations then are

$$0 = dX\left(1 + \frac{\lambda}{a^2}\right) - \tfrac{1}{2}\frac{X}{a^2}d\xi, \text{ \&c.},$$

or say

$$0 = dX(a^2+\lambda) - \tfrac{1}{2}Xd\xi, \text{ \&c.}$$

But from the equation $-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta)$, considering therein $\eta$ as a constant, we have

$$\frac{dX}{X} = \frac{\tfrac{1}{2}d\xi}{a^2+\xi},$$

and the equations thus become

$$0 = \frac{a^2+\lambda}{a^2+\xi} - 1, \text{ \&c.},$$

viz. these are all satisfied if only $\lambda = \xi$.

11. The coordinates of the point of intersection of the two normals thus are

$$x = X\left(1 + \frac{\xi}{a^2}\right), \quad y = Y\left(1 + \frac{\xi}{b^2}\right), \quad z = Z\left(1 + \frac{\xi}{c^2}\right),$$

or squaring, and substituting for $X^2$, &c., their values as given by

$$-\beta\gamma X^2 = a^2(a^2+\xi)(a^2+\eta), \text{ \&c.},$$

the equations become

$$-\beta\gamma\, a^2x^2 = (a^2+\xi)^3(a^2+\eta),$$
$$-\gamma\alpha\, b^2y^2 = (b^2+\xi)^3(b^2+\eta),$$
$$-\alpha\beta\, c^2z^2 = (c^2+\xi)^3(c^2+\eta),$$

viz. these equations give $(x, y, z)$ the coordinates of a point on the centro-surface, the intersection of the normal at the point $(X, Y, Z)$ of the ellipsoid, (determined by the parameters $\xi$, $\eta$), by the normal at the consecutive point along the curve of curvature

$$\frac{X^2}{a^2+\eta}+\frac{Y^2}{b^2+\eta}+\frac{Z^2}{c^2+\eta}=1,$$

or say $\eta$ is the sequential parameter[1].

Of course by interchanging $\xi$ and $\eta$ we should obtain the coordinates of the point of intersection of the normal at the same point $(X, Y, Z)$ by the normal at the consecutive point along the other curve of curvature: $\xi$ being in this case the sequential parameter.

12. I stop for a moment to consider the foregoing two equations

$$\lambda=\xi, \quad d\lambda=-\tfrac{1}{2}d\xi,$$

which at first sight appear inconsistent. But observe that in the foregoing solution $\lambda$ is the parameter of the point $(x, y, z)$ of the centro-surface considered as a point on the normal at $(X, Y, Z)$; $\lambda+d\lambda$ is the parameter of the *same point* considered as a point on the normal at the consecutive point $(X+dX,\ Y+dY,\ Z+dZ)$: the value $\lambda+d\lambda=\xi+d\xi$ would belong to a different point, viz. the consecutive point of the centro-surface considered as a point on the consecutive normal—wherefore the $d\lambda$ of the solution ought not to be $=d\xi$. In further explanation, observe that the equations

$$x=X\left(1+\frac{\lambda}{a^2}\right), \text{ \&c. where } \lambda=\xi,$$

if we pass from $(x, y, z)$ to the consecutive point on the centro-surface, give

$$dx=dX\left(1+\frac{\lambda}{a^2}\right)+\frac{X}{a^2}d\xi;$$

but since by what precedes,

$$0=dX\left(1+\frac{\lambda}{a^2}\right)-\tfrac{1}{2}\frac{X}{a^2}d\xi,$$

this is

$$dx=\tfrac{3}{2}\frac{X}{a^2}d\xi.$$

[1] The expressions are given in effect, but not explicitly, Salmon, p. 143.

Or since

$$a^2x = X(a^2+\xi),$$

this is

$$\frac{dx}{x} = \tfrac{3}{2}\frac{d\xi}{a^2+\xi};$$

and similarly

$$\frac{dy}{y} = \tfrac{3}{2}\frac{d\xi}{b^2+\xi},$$

$$\frac{dz}{z} = \tfrac{3}{2}\frac{d\xi}{c^2+\xi},$$

which are the correct values of $dx$, $dy$, $dz$ as derived from the equations

$$-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta), \text{ \&c.}$$

13. The equations $-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta)$, &c. give expressions for the coordinates $(x, y, z)$ of a point on the centro-surface in terms of the two parameters $(\xi, \eta)$: the elimination of $(\xi, \eta)$ from these equations will therefore lead to the equation of the surface; but the discussion of the surface may also be effected by means of these expressions for the coordinates in terms of the two parameters.

*Discussion by means of the equations* $-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta)$, &c.; *Principal Sections, &c.*
Art. Nos. 14 to 24 (*several subheadings*).

14. To fix the ideas consider the section of the surface by the plane $z=0$; we have in the surface $z=0$, that is, $\xi=-c^2$, or else $\eta=-c^2$, values which give respectively

$$\begin{array}{ll|ll} -\beta\gamma a^2x^2 = -\beta^3(a^2+\eta), & & -\beta\gamma a^2x^2 = -\beta(a^2+\xi)^3, & \\ -\gamma\alpha b^2y^2 = \alpha^3(b^2+\eta); & & -\gamma\alpha b^2y^2 = \alpha(b^2+\xi)^3; & \end{array}$$

or, what is the same thing,

$$\begin{array}{l|l} \dfrac{\gamma}{\beta^2}a^2x^2 = a^2+\eta, & \gamma a^2x^2 = (a^2+\xi)^3, \\ \dfrac{\gamma}{\alpha^2}b^2y^2 = -b^2-\eta; & \gamma b^2y^2 = -(b^2+\xi)^3. \end{array}$$

The first set of equations gives

$$\frac{a^2x^2}{\beta^2} + \frac{b^2y^2}{\alpha^2} = 1,$$

which is the equation of an ellipse.

The second set gives

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \gamma^{\frac{2}{3}},$$

or in a rationalised form

$$(a^2x^2+b^2y^2-\gamma^2)^3 + 27a^2b^2\gamma^2x^2y^2 = 0,$$

which is the equation of an evolute of an ellipse.

15. The ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$ is a cuspidal curve on the surface, and the section by the plane $z=0$ is consequently made up of this ellipse counting three times, and of the evolute; it is therefore of the twelfth order; and the order of the surface is in fact $=12$.

It is clear that the section of the centro-surface arises from the section $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, viz. the normal at any point of this ellipse lies in the plane $Z=0$, and its intersection by a normal at the consecutive point of the ellipse gives a point of the evolute; the evolute being thus the sequential centro-curve of this section: the intersection by the normal at the consecutive point on the other curve of curvature gives a point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$, which ellipse is therefore the concomitant centro-curve. Observe that this other curve of curvature cuts the ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$ at right angles, and that the normals at the consecutive points above and below the point on the ellipse will meet each other and also the normal at the point of the same ellipse at the point on the ellipse $\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1$: this shows that the last-mentioned ellipse is a cuspidal curve on the centro-surface.

16. The three principal sections of the centro-surface are consequently

$$x=0,\quad \frac{b^2y^2}{\gamma^2}+\frac{c^2z^2}{\beta^2}=1,\quad \text{and } (by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=\alpha^{\frac{2}{3}};$$

$$y=0,\quad \frac{c^2z^2}{\alpha^2}+\frac{a^2x^2}{\gamma^2}=1,\quad \text{and } (cz)^{\frac{2}{3}}+(ax)^{\frac{2}{3}}=\beta^{\frac{2}{3}};$$

$$z=0,\quad \frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad \text{and } (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=\gamma^{\frac{2}{3}};$$

viz. each section is made up of an ellipse counting three times and of an evolute (of an ellipse). I have for shortness represented the three evolutes by their irrational equations. It will presently appear that the section (imaginary) by the plane infinity is of the like character.

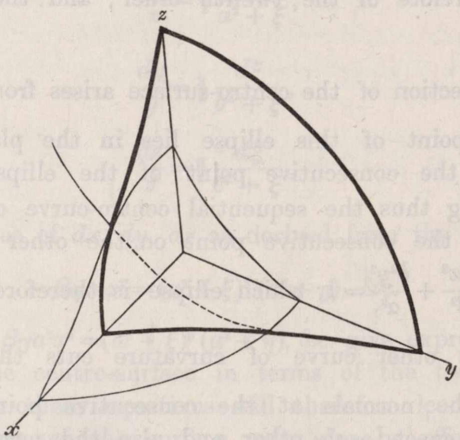
17. Considering only the positive directions of the axes, we have on each axis two points, viz.

$$\text{axis of } x,\quad x=\frac{\gamma}{a}\quad x=-\frac{\beta}{a};$$

$$\text{axis of } y,\quad y=\frac{\alpha}{b},\quad y=\frac{\gamma}{b};$$

$$\text{axis of } z,\quad z=-\frac{\beta}{c},\quad z=\frac{\alpha}{c};$$

through each of which, in the two different planes through the axis respectively, there passes an ellipse and an evolute. In the assumed case $a^2 + c^2 > 2b^2$, the disposition of the points is as shown in the figure.




Plane of $xz$, evolute is outside ellipse,

$yz$, „ inside „

$xy$, „ cuts „ ;

but in the contrary case $a^2 + c^2 < 2b^2$, the disposition is

Plane of $xz$, evolute is outside ellipse,

$yz$, „ cuts „

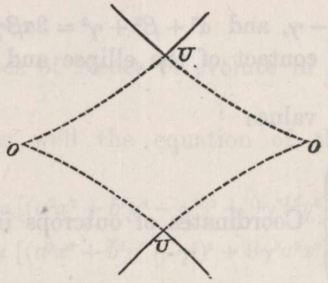
$xy$, „ is inside „ ;

there is no real difference, and to fix the ideas I attend exclusively to the first-mentioned case

$$a^2 + c^2 > 2b^2.$$

18. In each of the principal planes, the evolute and ellipse, quà curves of the orders 6 and 2 respectively, intersect in twelve points, 3 in each quadrant; viz. of the 3 points two unite together into a twofold point or point of contact, and the third is a point of simple intersection; assuming for the moment that this is so, the figure at once shows that in the plane of $xz$ or umbilicar plane the contact is real, the intersection imaginary; in the plane of $xy$, or major-mean plane, the contact is imaginary, the intersection real; but in the plane of $yz$ or minor-mean plane the contact and intersection are each imaginary. The contacts arise, as will appear, from the umbilici of the ellipsoid, and may be termed "umbilicar centres," or "omphaloi;" the simple intersections "points of outcrop," or simply "outcrops." By what precedes there are in the umbilicar plane, four real umbilicar centres (in each quadrant one); and in the major-mean plane four real outcrops (in each quadrant one); the other umbilicar centres and outcrops are respectively imaginary.

19. The surface consists of two sheets intersecting in a *nodal curve* connecting the outcrop with the umbilicar centre. As to the form of this curve there is a cusp at the outcrop; and the curve does not terminate at the umbilicar centre but, on passing it, from crunodal becomes acnodal (viz. there is no longer through the curve any real sheet of the surface): moreover the curve is not at the umbilicar centre




perpendicular to the plane of $xz$, and there is consequently on the opposite side of the plane a symmetrically situate branch of the curve, viz. the umbilicar centre is a node on the nodal curve. Completing the curve, the nodal curve consists of two distinct portions, one on the positive side of the plane of $yz$ or minor-mean plane consisting of two cuspidal branches as shown in the figure; the other a symmetrically situate portion on the negative side of the minor-mean plane.

*Intersections of Evolute and Ellipse.*

20. Consider in the plane of $xy$ the ellipse and evolute,

$$\frac{a^2x^2}{\beta^2}+\frac{b^2y^2}{\alpha^2}=1,\quad (a^2x^2+b^2y^2-\gamma^2)^3+27\gamma^2a^2b^2x^2y^2=0.$$

First, these are satisfied by

$$\left.\begin{aligned} a^2x^2&=-\frac{\beta^3}{\gamma},\\ b^2y^2&=-\frac{\alpha^3}{\gamma},\end{aligned}\right\}\text{ Coordinates of Umbilicar centres in plane of } xy \text{ (imaginary)},$$

viz. the equations respectively become

$$-\frac{\beta}{\gamma}-\frac{\alpha}{\gamma}=1,\quad (-\frac{\beta^3+\alpha^3}{\gamma}-\gamma^2)^3+27\alpha^3\beta^3=0,$$

the first of which is $\alpha+\beta+\gamma=0$, and the second is $(\alpha^3+\beta^3+\gamma^3)^3-27\alpha^3\beta^3\gamma^3=0$. But the equation $\alpha+\beta+\gamma=0$ gives $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, and the two equations are thus identically satisfied. Moreover the condition for a contact is at once found to be

$$\beta^2\left[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2\right]=\alpha^2\left[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2\right],$$

or, what is the same thing,

$$(\alpha^2-\beta^2)(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2(\alpha^2a^2x^2-\beta^2b^2y^2)=0;$$

and substituting the foregoing values, this is

$$(\alpha^2-\beta^2)\left(-\frac{\alpha^3+\beta^3}{\gamma}-\gamma^2\right)^2+9\gamma^2\,\frac{-\alpha^2\beta^3+\alpha^3\beta^2}{\gamma}=0,$$

that is,

$$\frac{\alpha^2-\beta^2}{\gamma^2}(\alpha^3+\beta^3+\gamma^3)^2+9\gamma\alpha^2\beta^2(\alpha-\beta)=0,$$

which, putting therein $\alpha+\beta=-\gamma$, and $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$, is also satisfied; that is, the points in question are points of contact of the ellipse and evolute.

21. Secondly, consider the values

$$\left.\begin{aligned} a^2x^2&=-\frac{\beta^3}{\gamma}\left(\frac{\gamma-\alpha}{\alpha-\beta}\right)^3,\\ b^2y^2&=-\frac{\alpha^3}{\gamma}\left(\frac{\beta-\gamma}{\alpha-\beta}\right)^3,\end{aligned}\right\}\text{ Coordinates of outcrops in plane of } xy \text{ (real).}$$

Substituting in the equation of the ellipse, we have

$$\alpha(\beta-\gamma)^3+\beta(\gamma-\alpha)^3+\gamma(\alpha-\beta)^3=0,$$

which is

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)=0,$$

or the equation is satisfied identically: and substituting in the equation of the evolute, we have first

$$a^2x^2+b^2y^2-\gamma^2=-\frac{\alpha^3(\beta-\gamma)^3+\beta^3(\gamma-\alpha)^3+\gamma^3(\alpha-\beta)^3}{\gamma(\alpha-\beta)^3};$$

which in virtue of $\alpha(\beta-\gamma)+\beta(\gamma-\alpha)+\gamma(\alpha-\beta)=0$ becomes

$$a^2x^2+b^2y^2-\gamma^2=-\frac{3\alpha\beta\gamma(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{\gamma(\alpha-\beta)^3},$$

$$=-\frac{3\alpha\beta(\beta-\gamma)(\gamma-\alpha)}{(\alpha-\beta)^2},$$

and then, completing the substitution, it is seen that the equation of the evolute is also satisfied. The points last considered are simple intersections, and we have thus the complete number $(8+4,=12)$ of the intersections of the evolute and ellipse.

22. We have $\alpha$, $\gamma$ positive, $\beta$ negative; whence $\alpha-\beta$ is positive, $\beta-\gamma$ negative; $\gamma-\alpha\,(=a^2+c^2-2b^2)$ is positive, and hence, the outcrops in the plane of $xy$ are real; the umbilicar centres are imaginary for this plane, but real for the plane of $zx$, the coordinates being

$$\left.\begin{aligned} c^2z^2&=-\frac{\alpha^3}{\beta},\\ a^2x^2&=-\frac{\gamma^3}{\beta},\end{aligned}\right\}\text{ Coordinates of Umbilicar centres in plane of } xz \text{ (real).}$$

*Nodes of the Evolute.*

23. The Evolute is a curve with four nodes, all of them imaginary; viz. for the evolute in the plane of $xy$, the equation of which is

$$(a^2x^2+b^2y^2-\gamma^2)^3+27\gamma^2a^2b^2x^2y^2=0,$$

these are

$$\left.\begin{array}{l} a^2x^2=-\gamma^2, \\ b^2y^2=-\gamma^2, \end{array}\right\} \text{ Coordinates of Nodes of evolute in plane of } xy \text{ (imaginary)},$$

in fact these values satisfy as well the equation of the evolute, as the two derived equations

$$6a^2x\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2b^2y^2]=0,$$
$$6b^2y\,[(a^2x^2+b^2y^2-\gamma^2)^2+9\gamma^2a^2x^2]=0,$$

or the points in question are nodes of the evolute.

The evolute has the four cusps on the axes and two cusps at infinity, in all 6 cusps as just mentioned; it has 4 nodes: and the order being 6, the class is

$$30-2\,.\,4-3\,.\,6,\ =4.$$

*Section by the plane infinity.*

24. The surface itself is finite, and the section by the plane infinity is therefore imaginary; but by what precedes the nodal curve must have real points at infinity, viz. there must be real acnodal points on this imaginary section. The section by the plane infinity resembles in fact the principal sections; viz. writing successively $\xi=\infty$, and $\eta=\infty$, we have

$$-\beta\gamma a^2x^2\ :\ -\gamma\alpha b^2y^2\ :\ -\alpha\beta c^2z^2=a^2+\eta\ :\ b^2+\eta\ :\ c^2+\eta$$

or

$$=(a^2+\xi)^3\ :\ (b^2+\xi)^3\ :\ (c^2+\xi)^3,$$

giving respectively

$$a^2x^2+b^2y^2+c^2z^2=0,\quad \text{and}\quad (a\alpha x)^{\frac{2}{3}}+(b\beta y)^{\frac{2}{3}}+(c\gamma z)^{\frac{2}{3}}=0,$$

where the first equation represents an imaginary conic which counts three times; and the second equation, the rationalised form of which is

$$(a^2\alpha^2x^2+b^2\beta^2y^2+c^2\gamma^2z^2)^3-27a^2b^2c^2\alpha^2\beta^2\gamma^2x^2y^2z^2=0,$$

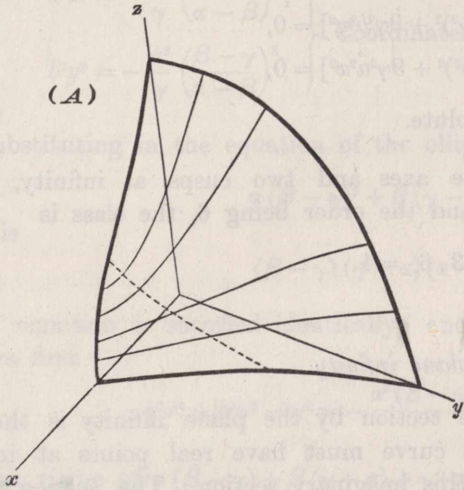
an imaginary evolute. The conic and evolute have four contacts and four simple intersections (in all $4\,.\,2+4=12$ intersections) which are all of them imaginary. But the evolute has four real nodes (acnodes) $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$; or, what is the same thing, there are four real lines $a^2\alpha^2x^2=b^2\beta^2y^2=c^2\gamma^2z^2$, which are respectively asymptotes of the nodal curve: viz. inasmuch as the equation of the surface contains only the squares

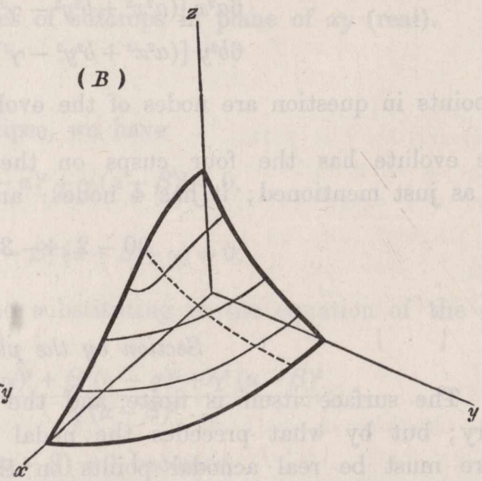
$x^2$, $y^2$, $z^2$, the lines in question will be not merely parallel to, but will be, the asymptotes of the nodal curve.

The plane infinity may be reckoned as a principal plane, and we may say that in each of the four principal planes there are four umbilicar centres, four outcrops, and four evolute-nodes.

*The generation of the surface considered geometrically.* Art. Nos. 25 to 28.

25. I have deferred until this point the discussion of the generation of the centro-surface by means of the centro-curves, for the reason that it can be carried on more precisely now that we know the forms of the principal sections and of the nodal

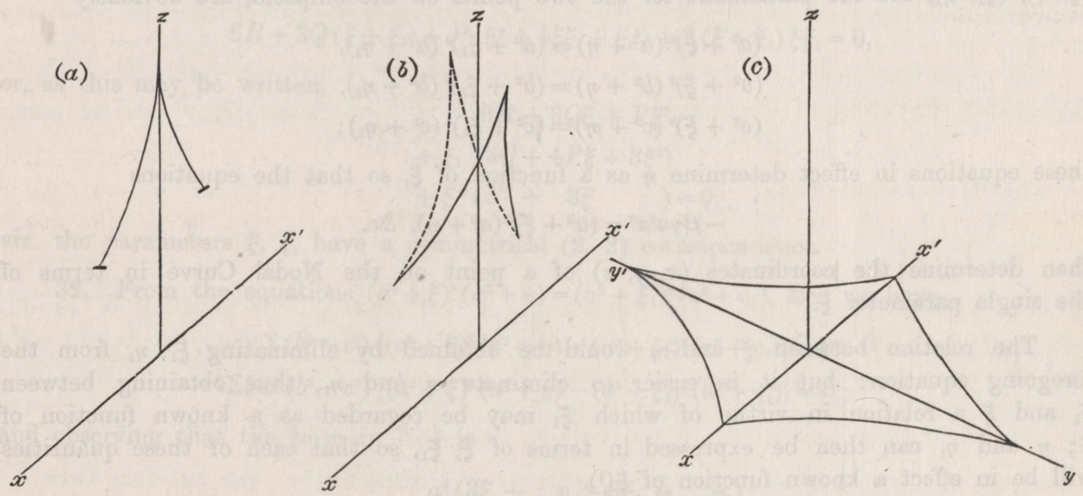



curve. The two figures exhibit (as regards one octant of the surface) the portions already distinguished as (A), and (B): they intersect each other in the nodal curve, shown in each of the figures.

26. Consider first the generation of the portion (A) by means of the major-mean sequential centro-curves. The major-mean curves of curvature (attending to those below the plane of $xy$) commence with a portion (extending from umbilicus to umbilicus) of the ellipse $\frac{X^2}{a^2}+\frac{Z^2}{c^2}=1$, this may be termed the vertical curve, and they end with the whole ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, which may be termed the horizontal curve. The normals at the several points of the vertical curve successively intersect along a portion (terminated each way at an umbilicar centre) of the evolute in the plane of $xz$ or umbilicar plane; viz. this portion of the evolute, shown fig. ($a$), is the sequential centro-curve belonging to the vertical curve of curvature. The curve of curvature is at first a narrow oval surrounding the vertical curve; the corresponding form of the sequential centro-curve is at once seen to be a four-cusped curve as in fig. ($b$), and which we may imagine as derived from the curve ($a$) by first doubling this curve and then

opening out the two component parts thereof: the two upper cusps of the curve (*b*) are situate on the *yz*-ellipse of the centro-surface, and the two lower cusps upon two detached portions respectively of the *xz*-ellipse of the centro-surface. And as the curve




of curvature gradually broadens out and ultimately coincides with the *XY*-section of the ellipsoid, the four-cusped curve continues to open itself out, and ultimately coincides as shown figure (*c*) with the *xy*-evolute of the centro-surface, viz. this evolute is the sequential centro-curve belonging to the horizontal curve of curvature or *XY*-section of the ellipsoid. The successive sequential curves are also shown (so far as regards an octant of the surface) in the figure (A).

27. We consider next the generation of the portion (B) by means of the major-mean concomitant centro-curves. Starting as before with the vertical curve of curvature, the concomitant centro-curve is a finite portion (terminated each way at an umbilicar centre) of the *xz*-ellipse of the centro-surface. As the curve of curvature opens itself out into an oval, the concomitant centro-curve in like manner opens itself out into an oval, the two further vertices thereof situate on two detached portions of the *xz*-evolute of the centro-surface, and the two nearer vertices on the *yz*-evolute of the central surface. And as the curve of curvature continues to open itself out, and ultimately coincides with the horizontal curve or *XY*-section of the ellipsoid, so the concomitant centro-curve continues to open itself out and ultimately coincides with the *xy*-ellipse of the centro-surface. The successive forms (so far as relates to an octant of the surface) are shown in the figure (B). We have in each case attended only to the curves of curvature below the plane of *xy*, and the corresponding centro-curves above the plane of *xy*, but of course everything is symmetrical as regards the two sides of the plane.

28. There is a precisely similar generation of the portion (A) by the minor-mean concomitant centro-curves, and of the portion (B) by means of the minor-mean sequential centro-curves.

*The Nodal Curve.* Art. Nos. 29 to 60.

29. If two different points on the ellipsoid correspond to the same point on the centro-surface, this will be a point on the Nodal Curve: the conditions for this if $(\xi, \eta)$, $(\xi_1, \eta_1)$ are the parameters for the two points on the ellipsoid, are obviously

$$\begin{aligned}(a^2+\xi)^3(a^2+\eta)&=(a^2+\xi_1)^3(a^2+\eta_1),\\(b^2+\xi)^3(b^2+\eta)&=(b^2+\xi_1)^3(b^2+\eta_1),\\(c^2+\xi)^3(c^2+\eta)&=(c^2+\xi_1)^3(c^2+\eta_1);\end{aligned}$$

these equations in effect determine $\eta$ as a function of $\xi$, so that the equations

$$-\beta\gamma a^2x^2=(a^2+\xi)^3(a^2+\eta),\ \&c.$$

then determine the coordinates $(x, y, z)$ of a point on the Nodal Curve in terms of the single parameter $\xi$.

The relation between $\xi$ and $\eta$ would be obtained by eliminating $\xi_1$, $\eta_1$ from the foregoing equation: but it is easier to eliminate $\eta$ and $\eta_1$, thus obtaining between $\xi_1$ and $\xi$ a relation in virtue of which $\xi_1$ may be regarded as a known function of $\xi$; $\eta$ and $\eta_1$ can then be expressed in terms of $\xi$, $\xi_1$, so that each of these quantities will be in effect a known function of $\xi$(1).

30. The relation between $\xi$, $\xi_1$ is in the first instance given in the form

$$\begin{vmatrix} a^2[(a^2+\xi)^3-(a^2+\xi_1)^3], & (a^2+\xi)^3, & (a^2+\xi_1)^3\\ b^2[(b^2+\xi)^3-(b^2+\xi_1)^3], & (b^2+\xi)^3, & (b^2+\xi_1)^3\\ c^2[(c^2+\xi)^3-(c^2+\xi_1)^3], & (c^2+\xi)^3, & (c^2+\xi_1)^3\end{vmatrix}=0.$$

Throwing out a factor $(\xi-\xi_1)^2$, this becomes

$$\Sigma[a^2\{3a^4+3a^2(\xi+\xi_1)+\xi^2+\xi\xi_1+\xi_1^2\}$$
$$\times(b^2-c^2).(1,\ 1,\ 1\between(b^2+\xi)(c^2+\xi_1),\ (b^2+\xi_1)(c^2+\xi))^2]=0,$$

where the left-hand side is a symmetrical function of $\xi$, $\xi_1$ vanishing for $\xi=\xi_1$, and therefore divisible by $(\xi-\xi_1)^2$; it is also divisible by $\Delta, =(b^2-c^2)(c^2-a^2)(a^2-b^2)\ (=\alpha\beta\gamma)$. To work this out, write $\xi+\xi_1=p$, $\xi\xi_1=q$, the equation may be written

$$\Sigma\left\{(b^2-c^2)a^2\left|\begin{matrix}3a^4\\+3a^2p\\+p^2-q\end{matrix}\right|\ \left|\begin{matrix}3b^4c^4\\+3b^2c^2(b^2+c^2)p\\+(b^4+c^4)(p^2-q)\\+b^2c^2(p^2+8q)\\+3(b^2+c^2)pq\\+3q^2\end{matrix}\right|\right\}=0,$$

where the left-hand side divides by $\Delta(p^2-4q)$.

1 This was my first method of solution; and I have thought the results quite interesting enough to retain them—but it will appear in the sequel that I have succeeded in expressing $\xi$, $\eta$, $\xi_1$, $\eta_1$, in terms of a single parameter $\sigma$.

31. Developing and reducing, and omitting this factor, the final result is

$$6R + 3Qp + P(p^2 + 2q) + 3pq = 0,$$

where as before $P$, $Q$, $R$ denote $a^2+b^2+c^2$, $b^2c^2+c^2a^2+a^2b^2$, $a^2b^2c^2$, respectively; that is,

$$6R + 3Q(\xi+\xi_1) + P(\xi^2 + 4\xi\xi_1 + \xi_1^2) + 3(\xi+\xi_1)\xi\xi_1 = 0,$$

or, as this may be written,

$$\begin{aligned} & 6R + 3Q\xi + P\xi^2 \\ & + \xi_1\,(3Q + 4P\xi + 3\xi^2) \\ & + \xi_1^2\,(P + 3\xi) = 0, \end{aligned}$$

viz. the parameters $\xi$, $\xi_1$ have a symmetrical (2, 2) correspondence.

32. From the equations $(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1)$, &c., we have

$$\Sigma(b^2-c^2)\{(a^2+\xi)^3(a^2+\eta) - (a^2+\xi_1)^3(a^2+\eta_1)\} = 0,$$
$$\Sigma b^2c^2(b^2-c^2)\{(a^2+\xi)^3(a^2+\eta) - (a^2+\xi_1)^3(a^2+\eta_1)\} = 0;$$

and observing that the term in { } is

$$\begin{aligned} & a^6(3\xi + \eta - 3\xi_1 - \eta_1) \\ & + a^4(3\xi^2 + 3\xi\eta - 3\xi_1^2 - 3\xi_1\eta_1) \\ & + a^2(\xi^3 + 3\xi^2\eta - \xi_1^3 - 3\xi_1^2\eta_1) \\ & + (\xi^3\eta - \xi_1^3\eta_1), \end{aligned}$$

these are readily reduced to

$$(3\xi+\eta-3\xi_1-\eta_1)P + (3\xi^2+3\xi\eta-3\xi_1^2-3\xi_1\eta_1) = 0,$$
$$(3\xi+\eta-3\xi_1-\eta_1)R + \xi^3\eta - \xi_1^3\eta_1 = 0,$$

or, what is the same thing,

$$3(\xi-\xi_1)(P+\xi+\xi_1) + \eta(P+3\xi) - \eta_1(P+3\xi_1) = 0,$$
$$3(\xi-\xi_1)R + \eta(R+\xi^3) - \eta_1(R+\xi_1^3) = 0,$$

and if we hence determine the ratios $3(\xi-\xi_1) : \eta : \eta_1$, the first of the resulting terms divides by $\xi-\xi_1$, and we have

$$\begin{aligned} 3 : \eta : \eta_1 = & -P(\xi^2+\xi\xi_1+\xi_1^2) + 3R - 3\xi\xi_1(\xi+\xi_1) \\ & : R(2\xi_1-\xi) - \xi_1^3(P+\xi+\xi_1) \\ & : R(2\xi-\xi_1) - \xi^3(P+\xi+\xi_1). \end{aligned}$$

Hence observing that by the relation between $\xi$, $\xi_1$ the first term is

$$= 3\{P\xi\xi_1 + Q(\xi+\xi_1) + 3R\},$$

the equations become

$$\begin{aligned} 1 : \eta : \eta_1 = & P\xi\xi_1 + Q(\xi+\xi_1) + 3R \\ & : R(2\xi_1-\xi) - \xi_1^3(P+\xi+\xi_1) \\ & : R(2\xi-\xi_1) - \xi^3(P+\xi+\xi_1); \end{aligned}$$

and we thus have

$$\eta = \frac{R(2\xi_1 - \xi) - \xi_1^3(P + \xi + \xi_1)}{P\xi\xi_1 + Q(\xi + \xi_1) + 3R},$$

which, considering $\xi_1$ as a given function of $\xi$, gives $\eta$ as a function of $\xi$.

33. I write $\xi + \xi_1 = 2\mathrm{x}$, $\xi - \xi_1 = 2\mathrm{y}$, so that $p = 2\mathrm{x}$, $q = \mathrm{x}^2 - \mathrm{y}^2$. The relation between $\xi$, $\xi_1$ takes the form

$$6(R + Q\mathrm{x} + P\mathrm{x}^2 - \mathrm{x}^3) - (6\mathrm{x} + 2P)\,\mathrm{y}^2 = 0,$$

or, what is the same thing,

$$\mathrm{y}^2 = \frac{(\mathrm{x} + a^2)(\mathrm{x} + b^2)(\mathrm{x} + c^2)}{\mathrm{x} + \frac{1}{3}(a^2 + b^2 + c^2)};$$

so that taking x at pleasure and considering y as denoting this function of x, the values of $\xi$, $\xi_1$ belonging to a point on the nodal curve are $\xi = (\mathrm{x} + \mathrm{y})$, $\xi_1 = (\mathrm{x} - \mathrm{y})$; and the value of $\eta$ is then given as before.

34. The form just given is analytically the most convenient, but there is some advantage in writing $\frac{1}{\sqrt{2}}\mathrm{x}$, $\frac{1}{\sqrt{2}}\mathrm{y}$, in the place of x, y respectively; viz. we then have

$$\mathrm{y}^2 = \frac{(\mathrm{x} + a^2\sqrt{2})(\mathrm{x} + b^2\sqrt{2})(\mathrm{x} + c^2\sqrt{2})}{\mathrm{x} + \frac{1}{3}\sqrt{2}\,(a^2 + b^2 + c^2)},$$

where $\xi = \frac{1}{\sqrt{2}}(\mathrm{x} + \mathrm{y})$, $\xi_1 = \frac{1}{\sqrt{2}}(\mathrm{x} - \mathrm{y})$, so that if $(\xi, \xi_1)$ be taken as rectangular coordinates of a point in a plane, (x, y) will be the rectangular coordinates of the same point referred to axes inclined at angles of 45° to the first-mentioned axes respectively.

35. The curve is a cubic curve symmetrical in regard to the axis of x, and having the three asymptotes,

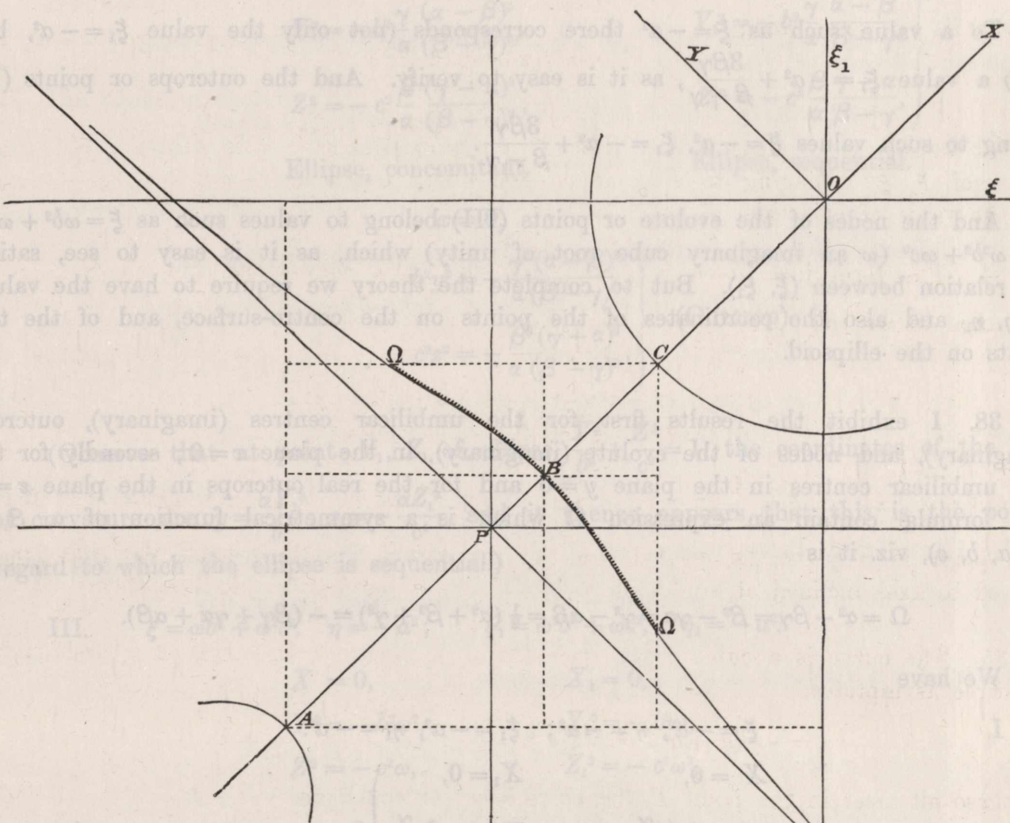
$$\mathrm{x} = -\tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}, \quad \mathrm{y} = \pm\{\mathrm{x} + \tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}\},$$

viz. these all meet in the point $P$ the coordinates of which are

$$\mathrm{x} = -\tfrac{1}{3}(a^2 + b^2 + c^2)\sqrt{2}, \quad \mathrm{y} = 0:$$

moreover we have $\mathrm{y} = 0$ for the values $\mathrm{x} = -a^2\sqrt{2}$, $-b^2\sqrt{2}$, $-c^2\sqrt{2}$, that is, the curve meets the axis of x in the points $A$, $B$, $C$; the order in the direction of $-\mathrm{x}$ being $C$, $B$, $P$, $A$ as shown in the figure: and with these data it is easy to draw the curve: the portion which gives the crunodal part of the nodal curve is that extending from $B$ to the points $\Omega$; viz. at $B$ we have $\xi = \xi_1 = -b^2$, corresponding to the umbilicar centre; and at $\Omega$, $\Omega$ we have $\xi$ or $\xi_1 = -c^2$, $\xi_1$ or $\xi = -c^2 + \frac{3\alpha\beta}{\alpha - \beta}$, corresponding to the outcrop.

36. The nodal curve passes through (I) the umbilicar centres, (II) the outcrops, (III) the nodes of the evolute. The geometrical construction led to the conclusion that the real umbilicar centre was a node on the nodal curve, and that the real outcrop was a cusp (the tangent lying in the principal plane). It will presently appear generally, as regards the several points real or imaginary, that the umbilicar centre is a node on the nodal curve, and the outcrop a cusp—the tangent at the outcrop being in the principal plane: as regards the node on the evolute this is a simple point on the nodal curve, and by reason of the symmetry in regard to the principal




plane, the nodal curve will at this (imaginary) point cut the principal planes at right angles. Hence considering the intersections of the nodal curve by a principal plane, the umbilicar centre, outcrop and node of the evolute count respectively as 2 points, 3 points and 1 point, and as for each kind the number is 4, the whole number of intersections is $4(2+3+1), =24$. It may be shown that these are the only intersections of the nodal curve with the principal plane; and this being so, it follows that the order of the nodal curve is $=24$; which agrees with the result of a subsequent analytical investigation.

37. The umbilicar centres or points (I) belong to values such as $\xi = \xi_1 = -a^2$ which are the *united values* in the equation between $(\xi, \xi_1)$, viz. writing herein $\xi_1 = \xi$ the equation becomes

$$(\xi + a^2)(\xi + b^2)(\xi + c^2) = 0,$$

so that the united values are $\xi = \xi_1 = -a^2, -b^2$ or $-c^2$. (It may be remarked, that treating this cubic as a degenerate quartic, a united value would be $\xi = \xi_1 = \infty$, corresponding to the umbilicar centres at infinity.)

To a value such as $\xi = -a^2$ there corresponds (not only the value $\xi_1 = -a^2$, but also) a value $\xi_1 = -a^2 + \dfrac{3\beta\gamma}{\beta - \gamma}$, as it is easy to verify. And the outcrops or points (II) belong to such values $\xi = -a^2$, $\xi_1 = -a^2 + \dfrac{3\beta\gamma}{\beta - \gamma}$.

And the nodes of the evolute or points (III) belong to values such as $\xi = \omega b^2 + \omega^2 c^2$, $\xi_1 = \omega^2 b^2 + \omega c^2$ ($\omega$ an imaginary cube root of unity) which, as it is easy to see, satisfy the relation between $(\xi, \xi_1)$. But to complete the theory we require to have the values of $\eta$, $\eta_1$ and also the coordinates of the points on the centro-surface, and of the two points on the ellipsoid.

38. I exhibit the results first for the umbilicar centres (imaginary), outcrops (imaginary), and nodes of the evolute (imaginary), in the plane $x = 0$; secondly for the real umbilicar centres in the plane $y = 0$ and for the real outcrops in the plane $z = 0$. The formulæ contain an expression $\Omega$ which is a symmetrical function of $\alpha, \beta, \gamma$ (or $a, b, c$), viz. it is

$$\Omega = \alpha^2 - \beta\gamma = \beta^2 - \gamma\alpha = \gamma^2 - \alpha\beta = \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2) = -(\beta\gamma + \gamma\alpha + \alpha\beta).$$

We have

I. $\qquad \xi = -a^2,\ \eta = -a^2;\quad \xi_1 = -a^2,\ \eta_1 = -a^2.$

$$\left.\begin{aligned} X &= 0, & X_1 &= 0, \\ Y^2 &= -b^2\frac{\gamma}{\alpha}, & Y_1^2 &= -b^2\frac{\gamma}{\alpha}, \\ Z^2 &= -c^2\frac{\beta}{\alpha}, & Z_1^2 &= -c^2\frac{\beta}{\alpha}, \end{aligned}\right\} \text{(Umbilicus).}$$

$$\left.\begin{aligned} x &= 0, \\ b^2y^2 &= -\frac{\gamma^3}{\alpha}, \\ c^2z^2 &= -\frac{\beta^3}{\alpha}, \end{aligned}\right\} \text{(Umbilicar centre).}$$

II. $\xi = -a^2, \qquad \eta = -a^2 + \dfrac{9\beta\gamma\Omega}{(\beta-\gamma)^3},$

$$\left(\text{or } \eta + b^2 = -\gamma\frac{(\alpha-\beta)^3}{(\beta-\gamma)^3}, \quad \eta + c^2 = \frac{\beta(\gamma-\alpha)^3}{(\beta-\gamma)^3}\right).$$

$$\xi_1 = -a^2 + \frac{3\beta\gamma}{\beta-\gamma}, \quad \eta_1 = -a^2.$$

$$\begin{array}{ll} X = 0, & X_1 = 0, \\ Y^2 = -b^2\dfrac{\gamma}{\alpha}\dfrac{(\alpha-\beta)^3}{(\beta-\gamma)^3}, & Y_1^2 = -b^2\dfrac{\gamma}{\alpha}\dfrac{\alpha-\beta}{\beta-\gamma}, \\ Z^2 = -c^2\dfrac{\beta}{\alpha}\dfrac{(\gamma-\alpha)^3}{(\beta-\gamma)^3}, & Z_1^2 = -c^2\dfrac{\beta}{\alpha}\dfrac{\gamma-\alpha}{\beta-\gamma}, \end{array}$$

Ellipse, concomitant. Ellipse, sequential.

$$\left.\begin{array}{l} x = 0, \\ b^2y^2 = -\dfrac{\gamma^3(\alpha-\beta)^3}{\alpha(\beta-\gamma)^3}, \\ c^2z^2 = -\dfrac{\beta^3(\gamma-\alpha)^3}{\alpha(\beta-\gamma)^3}, \end{array}\right\} \text{(Outcrop).}$$

(Observe that at point $Y_1$, $Z_1$ of ellipse $\dfrac{Y^2}{b^2} + \dfrac{Z^2}{c^2} = 1$, the coordinates of the centre of curvature are $y = \dfrac{\alpha Y_1^3}{b^4}$, $z = -\dfrac{\alpha Z_1^3}{c^4}$, and it thence appears that this is the point in regard to which the ellipse is sequential.)

III. $\xi = \omega b^2 + \omega^2 c^2, \quad \eta = -a^2; \quad \xi_1 = \omega^2 b^2 + \omega c^2, \quad \eta_1 = -a^2.$

$$\begin{array}{ll} X = 0, & X_1 = 0, \\ Y^2 = -b^2\omega^2, & Y_1^2 = -b^2\omega, \\ Z^2 = -c^2\omega, & Z_1^2 = -c^2\omega^2, \end{array}$$

$$\left.\begin{array}{l} x = 0, \\ b^2y^2 = -\alpha^2, \\ c^2z^2 = -\alpha^2, \end{array}\right\} \text{(Node of evolute).}$$

39. Observe that these are the only ways in which it is possible to satisfy the equations

$$0 = (a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1),$$

viz. starting from this equation we have

I. $\qquad a^2 + \xi = 0, \quad a^2 + \xi_1 = 0,$

whence in the equations for $\eta$, $\eta_1$, substituting the values $\xi = \xi_1 = -a^2$, we have

$$\begin{aligned} 1 : \eta : \eta_1 = & \; a^4P - 2a^2Q + 3R, \\ & : -a^2R + a^6(P - 2a^2), \\ & : -a^2R + a^6(P - 2a^2), \end{aligned}$$

that is,

$$1 : \eta : \eta_1 = -a^2\beta\gamma : a^4\beta\gamma : a^4\beta\gamma,$$

or

$$\eta = \eta_1 = -a^2.$$

40. II. $a^2 + \xi = 0$ without $a^2 + \xi_1 = 0$, consequently $a^2 + \eta_1 = 0$; writing $\xi = -a^2$, in the relation between $(\xi, \xi_1)$, this is

$$6R + 3Q(\xi_1 - a^2) + P(\xi_1^2 - 4a^2\xi_1 + a^4) - 3a^2\xi_1(\xi_1 + a^2) = 0,$$

viz. this is

$$\xi_1^2(b^2 + c^2 - 2a^2) + \xi_1(-a^4 - a^2b^2 - a^2c^2 + 3b^2c^2) + a^2(a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2) = 0,$$

where the left-hand side should divide by $\xi_1 + a^2$; the equation in fact is

$$(\xi_1 + a^2)\{\xi_1(b^2 + c^2 - 2a^2) + a^4 - 2a^2b^2 - 2a^2c^2 + 3b^2c^2\} = 0;$$

or, what is the same thing,

$$(\xi_1 + a^2)\{(\xi_1 + a^2)(\beta - \gamma) - 3\beta\gamma\} = 0,$$

whence

$$\xi_1 = -a^2 + \frac{3\beta\gamma}{\beta - \gamma}.$$

41. Considering these values of $\xi$, $\xi_1$ as given, the verification of the value $\eta_1 = -a^2$, and determination of $\eta = -a^2 + \dfrac{9\beta\gamma\Omega}{(\beta - \gamma)^3}$ is somewhat complex.

Writing for a moment $\Lambda = -\dfrac{3\beta\gamma}{\beta - \gamma}$, we have

$$\begin{aligned} 1 : \eta : \eta_1 = & \;\; P(a^4 + a^2\Lambda) - Q(2a^2 + \Lambda) + 3R \\ & : -R(a^2 + 2\Lambda) - (a^2 + \Lambda)^3(2a^2 - P + \Lambda \\ & : -R(a^2 - \;\; \Lambda) - a^6(2a^2 - P + \Lambda). \end{aligned}$$

The first term is

$$a^4P - 2a^2Q + 3R + \Lambda(a^2P - Q),$$

which is

$$= -a^2\beta\gamma + \Lambda(a^4 - b^2c^2);$$

and for the value of $\eta_1$, proceeding to the third term, this is

$$-a^2R - a^6(2a^2 - P) + \Lambda(R - a^6),$$

which is

$$= a^4\beta\gamma - a^2\Lambda(a^4 - b^2c^2),$$

so that without any further reduction $\eta_1 = -a^2$.

42. We have then

$$\eta = \frac{-R(a^2+2\Lambda)-(a^2+\Lambda)^3(a^2-b^2-c^2+\Lambda)}{-\alpha^2\beta\gamma+\Lambda(a^4-b^2c^2)},$$

and I assume

$$\eta = -a^2 + \frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega,$$

and investigate the value of $\Omega$.

We have

$$-R(a^2+2\Lambda)-(a^2+\Lambda)^3(a^2-b^2-c^2+\Lambda), = \alpha^4\beta\gamma+\Lambda\odot, \text{ suppose.}$$

The equation therefore is

$$\frac{\alpha^4\beta\gamma+\Lambda\odot}{-\alpha^2\beta\gamma+\Lambda(a^4-b^2c^2)} = -a^2+\frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega,$$

that is,

$$\Lambda\odot = -a^2\Lambda(a^4-b^2c^2)+\frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega\{-\alpha^2\beta\gamma+\Lambda(a^4-b^2c^2)\}=0,$$

or writing $\frac{9\beta\gamma}{(\beta-\gamma)^3} = -\frac{3\Lambda}{(\beta-\gamma)^2}$, omitting the factor $\Lambda$, and multiplying by $(\beta-\gamma)^2$, this is

$$(\beta-\gamma)^2\{\odot+a^2(a^4-b^2c^2)\}+3\Omega\{-\alpha^2\beta\gamma+\Lambda(a^4b^2c^2)\}=0,$$

in which equation

$$\odot = -2R-a^6-(3a^4+3a^2\Lambda+\Lambda^2)(a^2-b^2-c^2+\Lambda),$$

and thence

$$\begin{aligned}\odot+a^2(a^4-b^2c^2) &= \text{same } + a^2(a^4-b^2c^2),\\ &= -3a^6+3a^4(b^2+c^2)-3a^2b^2c^2\\ &\quad +\Lambda\{-6a^4+3a^2(b^2+c^2)\}\\ &\quad +\Lambda^2(-4a^2+\quad b^2+c^2)\\ &\quad -\Lambda^3\\ &= 3a^2\beta\gamma+3a^2\Lambda(\beta-\gamma)+\Lambda^2(\beta-\gamma-2a^2)-\Lambda^3.\end{aligned}$$

43. Hence, substituting for $\Lambda$ its value and multiplying by $(\beta-\gamma)^3$, we have

$$\begin{aligned}&(\beta-\gamma)^3\{\odot+a^2(a^4-b^2c^2)\}\\ &=3\alpha^2\beta\gamma(\beta-\gamma)^3-9\alpha^2\beta\gamma(\beta-\gamma)^3+9\beta^2\gamma^2(\beta-\gamma-2a^2)(\beta-\gamma)+27\beta^3\gamma^3,\end{aligned}$$

which is

$$=-6a^2\beta\gamma(\beta-\gamma)^3+9\beta^2\gamma^2(\beta-\gamma)^2-18\alpha^2\beta^2\gamma^2(\beta-\gamma)+27\beta^3\gamma^3;$$

viz. this is

$$\begin{aligned}&=\{-6a^2(\beta-\gamma)+9\beta\gamma\}\{(\beta-\gamma)^2+3\beta\gamma\}\beta\gamma,\\ &=\{-6a^2(\beta-\gamma)+9\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma,\end{aligned}$$

and the equation thus is

$$\{-2a^2(\beta-\gamma)+3\beta\gamma\}(\beta^2+\beta\gamma+\gamma^2)\beta\gamma+\Omega\left\{-a^2\beta\gamma-\frac{3\beta\gamma}{\beta-\gamma}(a^4-b^2c^2)\right\}(\beta-\gamma)=0,$$

or finally

$$\Omega\{a^2(\beta-\gamma)+3(a^4-b^2c^2)\}=(-2a^2(\beta-\gamma)+3\beta\gamma)(\beta^2+\beta\gamma+\gamma^2).$$

But $c^2=a^2+\beta$, $b^2=a^2-\gamma$, and hence $a^4-b^2c^2=-a^2(\beta-\gamma)+\beta\gamma$, and therefore

$$a^2(\beta-\gamma)+3(a^4-b^2c^2)=-2a^2(\beta-\gamma)+3\beta\gamma;$$

the equation thus divides by $-2a^2(\beta-\gamma)+3\beta\gamma$ and we have

$$\Omega=\beta^2+\beta\gamma+\gamma^2,$$

or, as this may also be written, $\Omega=\alpha^2-\beta\gamma, =\beta^2-\gamma\alpha, =\gamma^2-\alpha\beta$. So that $\Omega$ has the value originally so denoted, and we have then

$$\eta=-a^2+\frac{9\beta\gamma}{(\beta-\gamma)^3}\Omega.$$

44. III. Lastly the equation $0=(a^2+\xi)^3(a^2+\eta)=(a^2+\xi_1)^3(a^2+\eta_1)$ is satisfied if $a^2+\eta=0$, $a^2+\eta_1=0$: the equations

$$(b^2+\xi)^3(b^2+\eta)=(b^2+\xi_1)^3(b^2+\eta_1),$$
$$(c^2+\xi)^3(c^2+\eta)=(c^2+\xi_1)^3(c^2+\eta_1),$$

then give

$$(b^2+\xi)^3=(b^2+\xi_1)^3,$$
$$(c^2+\xi)^3=(c^2+\xi_1)^3,$$

which can be satisfied by $\xi=\xi_1$, leading to $\xi=\xi_1=-a^2$, which is the case I., or else by

$$b^2+\xi=\omega(b^2+\xi_1),$$
$$c^2+\xi=\omega^2(c^2+\xi_1),$$

that is,

$$\xi=\omega b^2+\omega^2c^2,\quad \xi_1=\omega^2b^2+\omega c^2.$$

To show that these values satisfy the relation between $\xi$, $\xi_1$, observe that they give

$$\xi+\xi_1=-b^2-c^2,\quad \xi\xi_1=b^4-b^2c^2+c^4,$$

whence also

$$\xi^2+4\xi\xi_1+\xi_1{}^2=3(b^4+c^4),$$

and the relation becomes

$$6a^2b^2c^2-3[a^2(b^2+c^2)+b^2c^2](b^2+c^2)$$
$$+[a^2+(b^2+c^2)].3(b^4+c^4)-3(b^2+c^2)(b^4-b^2c^2+c^4)=0,$$

which is an identity

45. I will show that these values of $\xi$, $\xi_1$ give the foregoing values $\eta = \eta_1 = -a^2$. We have

$$\begin{aligned} 1 : \eta - \eta_1 : \eta + \eta_1 = & P\xi\xi_1 + Q(\xi + \xi_1) + 3R \\ & : (\xi_1 - \xi)\{3R - (\xi^2 + \xi\xi_1 + \xi_1^2)(P + \xi + \xi_1)\} \\ & : (\xi_1 + \xi)\{R - (\xi^2 - \xi\xi_1 + \xi_1^2)(P + \xi + \xi_1)\}, \end{aligned}$$

this is

$$1 : \eta - \eta_1 : \eta + \eta_1 = a^2(b^2 + c^2) : 0(\xi_1 - \xi) : -2a^2a^2(b^2 + c^2),$$

or

$$\eta - \eta_1 = 0,\ \eta + \eta_1 = -2a^2;\ \text{that is, } \eta = \eta_1 = -a^2.$$

46. For the real umbilicar centres and outcrops we have

I. $$\xi = -b^2,\ \eta = -b^2,\ \xi_1 = -b^2,\ \eta_1 = -b^2.$$

$$X^2 = -a^2\frac{\gamma}{\beta},\qquad X_1^2 = -a^2\frac{\gamma}{\beta},$$

$$Y = 0,\qquad Y_1 = 0,$$

$$Z^2 = -c^2\frac{\alpha}{\beta},\qquad Z_1^2 = -c^2\frac{\alpha}{\beta},$$

$$\left.\begin{aligned} a^2x^2 &= -\frac{\gamma^3}{\beta}, \\ y &= 0, \\ c^2z^2 &= -\frac{\alpha^3}{\beta}, \end{aligned}\right\}\ \text{(real umbilicar centre).}$$

II. $$\xi = -c^2,\qquad \eta = -c^2 + \frac{9\alpha\beta}{(\alpha - \beta)^3}$$

$$\left(\text{or } \eta + a^2 = -\beta\frac{(\gamma - \alpha)^3}{(\alpha - \beta)^3},\ \eta + b^2 = \frac{\alpha(\beta - \gamma)^3}{(\alpha - \beta)^3}\right).$$

$$\xi_1 = -c^2 + \frac{3\alpha\beta}{\alpha - \beta},\ \eta_1 = -c^2,$$

$$\left.\begin{aligned} X^2 &= -a^2\frac{\beta}{\gamma}\frac{(\gamma - \alpha)^3}{(\alpha - \beta)^3}, & X_1^2 &= -a^2\frac{\beta}{\gamma}\frac{\gamma - \alpha}{\alpha - \beta}, \\ Y^2 &= -b^2\frac{\alpha}{\gamma}\frac{(\beta - \gamma)^3}{(\alpha - \beta)^3}, & Y_1^2 &= -b^2\frac{\alpha}{\gamma}\frac{\beta - \gamma}{\alpha - \beta}, \\ Z &= 0, & Z_1 &= 0, \end{aligned}\right\}$$

ellipse concomitant. ellipse sequential.

$$\left.\begin{aligned} a^2x^2 &= -\frac{\beta^3}{\gamma}\frac{(\gamma - \alpha)^3}{(\alpha - \beta)^3} \\ b^2y^2 &= -\frac{\alpha^3}{\gamma}\frac{(\beta - \gamma)^3}{(\alpha - \beta)^3} \\ z &= 0. \end{aligned}\right\}\ \text{(real outcrop).}$$

*Nodal curve in vicinity of umbilicar centre,* $a^2x^2 = -\frac{\gamma^3}{\beta}$, $y=0$, $c^2z^2 = -\frac{\alpha^3}{\beta}$. Art. Nos. 47 to 49.

47. Write

$$\xi = -b^2 + q, \quad \eta = -b^2 + r,$$

$$\xi_1 = -b^2 + q_1, \quad \eta_1 = -b^2 + r_1,$$

we have to find the relation between $q$, $q_1$, $r$, $r_1$; first for $q$, $q_1$, the equation of correspondence gives

$$\begin{aligned} &6R \\ &+ 3Q(-2b^2 + q + q_1) \\ &+ \ P\{6b^4 - 6b^2(q+q_1) + q^2 + 4qq_1 + q_1^2\} \\ &+ 3\ \ \{-2b^6 + 3b^4(q+q_1) - b^2(q^2 + 4qq_1 + q_1^2) + qq_1(q+q_1)\} = 0, \end{aligned}$$

that is,

$$\begin{aligned} &3(q+q_1)(3b^4 - 2b^2P + Q) \\ &+ (q^2 + qq_1 + q_1^2)(-3b^2 + P) \\ &+ 3qq_1(q+q_1) = 0, \end{aligned}$$

viz. this is

$$\begin{aligned} &-3(q+q_1)\alpha\gamma \\ &+ (q^2 + 4qq_1 + q_1^2)(\gamma - \alpha) \\ &+ 3qq_1(q+q_1) = 0, \end{aligned}$$

whence approximately $q + q_1 = 0$; but it will appear that the value is required to the second order; we have therefore

$$q + q_1 = \tfrac{1}{3}\frac{\gamma - \alpha}{\gamma\alpha}(q^2 + 4qq_1 + q_1^2)$$

$$= -\tfrac{2}{3}\frac{\gamma - \alpha}{\gamma\alpha}q^2.$$

48. Now the equations

$$(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)^3(a^2+\eta_1), \text{ and } (c^2+\xi)^3(c^2+\eta) = (c^2+\xi_1)^3(c^2+\eta_1),$$

putting therein for $\xi$, $\eta$, $\xi_1$, $\eta_1$, their values, give the first of them

$$\log\left(1+\frac{r}{\gamma}\right) + 3\log\left(1+\frac{q}{\gamma}\right) = \log\left(1+\frac{r_1}{\gamma}\right) + 3\log\left(1+\frac{q_1}{\gamma}\right),$$

that is,

$$r + 3q - \frac{1}{2\gamma}(r^2 + 3q^2) + \frac{1}{3\gamma^2}(r^3 + 3q^3) = r_1 + 3q_1 - \frac{1}{2\gamma}(r_1^2 + 3q_1^2) + \frac{1}{3\gamma^2}(r_1^3 + 3q_1^3);$$

and similarly the second equation

$$r + 3q + \frac{1}{2\alpha}(r^2 + 3q^2) + \frac{1}{3\alpha^2}(r^3 + 3q^3) = r_1 + 3q_1 + \frac{1}{2\alpha}(r_1^2 + 3q_1^2) + \frac{1}{3\alpha^2}(r_1^3 + 3q_1^3);$$

whence multiplying by $\gamma$, $\alpha$, and adding,

$$(\gamma+\alpha)\left\{r+3q+\frac{1}{3\alpha\gamma}(r^3+3q^3)\right\}=(\gamma+\alpha)\left\{r_1+3q_1+\frac{1}{3\alpha\gamma}(r_1^3+3q_1^3)\right\},$$

which, neglecting terms of the third order, is

$$r+3q=r_1+3q_1.$$

Subtracting the two equations we have

$$\tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)(r^2+3q^2)+\tfrac{1}{3}\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)(r^3+3q^3)=\tfrac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)(r_1^2+3q_1^2)+\tfrac{1}{3}\left(\frac{1}{\alpha^2}-\frac{1}{\gamma^2}\right)(r_1^3+3q_1^3),$$

viz. this is

$$r^2+3q^2+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(r^3+3q^3)=r_1^2+3q_1^2+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(r_1^3+3q_1^3),$$

or, what is the same thing,

$$r^2-r_1^2+3(q^2-q_1^2)+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}\{r^3-r_1^3+3(q^3-q_1^3)\}=0,$$

which, putting therein $r-r_1=-3(q-q_1)$, is

$$-r-r_1+q+q_1+\tfrac{2}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-r^2-rr_1-r_1^2+q^2+qq_1+q_1^2)=0,$$

say this is

$$-r-r_1+q+q_1+2\Delta=0;$$

combining herewith

$$r-r_1+3q-3q_1\quad=0,$$

we have

$$r+q-2q_1-\Delta=0,$$

and

$$r_1-2q+q_1-\Delta=0,$$

where

$$\Delta=\tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-r^2-rr_1-r_1^2+q^2+qq_1+q_1^2).$$

But substituting herein the values $r=-q+2q_1$, $r_1=2q-q_1$, this becomes

$$\Delta=\tfrac{1}{3}\frac{\gamma-\alpha}{\gamma\alpha}(-2q^2+4qq_1-2q_1^2),\ =-\tfrac{8}{3}\frac{\gamma-\alpha}{\gamma\alpha}q^2,$$

and then

$$r=-q+2q_1+\Delta,$$

that is,

$$r+3q=2(q+q_1)+\Delta,\ =-\frac{4(\gamma-\alpha)}{\gamma\alpha}q^2.$$

49. We have then

$$a^2x^2 = -\frac{\gamma^3}{\beta}\left(1+\frac{r}{\gamma}\right)\left(1+\frac{q}{\gamma}\right)^3,$$

$$= -\frac{\gamma^3}{\beta}\left(1+\frac{r+3q}{\gamma}+\frac{3q(r+q)}{\gamma^2}\right),$$

$$= -\frac{\gamma^3}{\beta}\left(1+\frac{r+3q}{\gamma^2}-\frac{6}{\gamma^2}q^2\right)$$

$$= -\frac{\gamma^3}{\beta}\left\{1+q^2\left(-\frac{4(\gamma-\alpha)}{\gamma^2\alpha}-\frac{6\alpha}{\gamma^2\alpha}\right)\right\}$$

$$= -\frac{\gamma^3}{\beta}\left\{1+q^2\frac{2(\beta-\gamma)}{\gamma^2\alpha}\right\};$$

and in the same way from $c^2z^2 = -\frac{\alpha^3}{\beta}\left(1-\frac{r}{\alpha}\right)\left(1-\frac{r}{\alpha}\right)^3$, we have

$$c^2z^2 = -\frac{\alpha^3}{\beta}\left\{1-q^2\frac{2(\alpha-\beta)}{\gamma\alpha^2}\right\}:$$

moreover we have at once

$$b^2y^2 = -\frac{q^3r}{\gamma\alpha} = \frac{3q^4}{\gamma\alpha}.$$

Hence, writing $x+\delta x$, $0+\delta y$, $z+\delta z$ for $x$, $y$, $z$, we find

$$\delta x = \tfrac{1}{2}x\,.\,\frac{2(\beta-\gamma)}{\gamma^2\alpha}\,.\,q^2,$$

$$\delta y = \pm\frac{1}{b}\sqrt{\frac{3}{\gamma\alpha}}\,.\,q^2,$$

$$\delta z = \tfrac{1}{2}z\,.\,\frac{-2(\alpha-\beta)}{\gamma\alpha^2}\,.\,q^2,$$

or, what is the same thing,

$$\delta x : \delta y : \delta z = x\frac{2(\beta-\gamma)}{\gamma^2\alpha} : \pm\frac{2}{b}\sqrt{\frac{3}{\gamma\alpha}} : z\frac{-2(\alpha-\beta)}{\gamma\alpha^2},$$

where $x$, $z$ denote the values at the umbilicar centre.

*Nodal curve in vicinity of real outcrop, viz.*

$$a^2x^2 = -\frac{\beta^3}{\gamma}\frac{(\gamma-\alpha)^3}{(\alpha-\beta)^3},\quad b^2y^2 = -\frac{\alpha^3}{\gamma}\frac{(\alpha-\gamma)^3}{(\alpha-\beta)^3},\quad z=0.\quad \text{Art. Nos. 50 to 52.}$$

50. Write

$$\xi = -c^2+q,\qquad\qquad \eta = -c^2+\frac{q\alpha\beta\Omega}{(\alpha-\beta)^3}+\theta,$$

$$\xi_1 = -c^2+\frac{3\alpha\beta}{\alpha-\beta}+q_1,\qquad\qquad \eta_1 = -c^2+\theta_1;$$

and first for the relation between $q$ and $q_1$, writing for a moment $\dfrac{3\alpha\beta}{\alpha-\beta}+q_1=Q_1$, and therefore $\xi_1=-c^2+Q_1$, the equation of correspondence gives

$$-3\alpha\beta(q+Q_1)+(q^2+4qQ_1+Q_1^2)(\alpha-\beta)+3qQ_1(q+Q_1)=0,$$

which, putting for $Q_1$ its value, is

$$\begin{aligned}&-3\alpha\beta\left(q+q_1+\frac{3\alpha\beta}{\alpha-\beta}\right)\\&+(\alpha-\beta)\left(q^2+4qq_1+q_1^2+(4q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right)\\&+3q\left\{q(q+q_1)+(q+2q_1)\frac{3\alpha\beta}{\alpha-\beta}+\frac{9\alpha^2\beta^2}{(\alpha-\beta)^2}\right\}=0;\end{aligned}$$

that is,

$$\begin{aligned}&-3\alpha\beta(q+q_1)\\&+3\alpha\beta(4q+2q_1)+(\alpha-\beta)(q^2+4qq_1+q_1^2)\\&+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}q+\frac{9\alpha\beta}{\alpha-\beta}(q^2+2qq_1)+3qq_1(q+q_1)=0,\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}&\left(9\alpha\beta+\frac{27\alpha^2\beta^2}{(\alpha-\beta)^2}\right)q+3\alpha\beta q_1\\&+q^2\left\{\frac{\alpha^2+7\alpha\beta+\beta^2}{\alpha-\beta}+qq_1\frac{\alpha^2+16\alpha\beta+\beta^2}{\alpha-\beta}+q_1^2(\alpha-\beta)\right\}\\&+3qq_1(q+q_1)=0,\end{aligned}$$

or, for small values,

$$\left(3+\frac{9\alpha\beta}{(\alpha-\beta)^2}\right)q+q_1=0,\ \text{that is,}\ \frac{3\Omega}{(\alpha-\beta)^2}q+q_1=0.$$

51. Moreover, from the equation $(c^2+\xi)^3(c^2+\eta)=(c^2+\xi_1)^3(c^2+\eta_1)$, we have

$$q^3\frac{9\alpha\beta\Omega}{(\alpha-\beta)^3}=\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3.\theta_1,\ \text{that is,}\ \theta_1=\tfrac{1}{3}\frac{\Omega}{\alpha^2\beta^2}\cdot q^3,$$

or, since $q$ and $q_1$ are of the same order, $\theta_1$ is of the order $q_1^3$. Hence, starting from the equations $-\beta\gamma a^2x^2=(a^2+\xi_1)^3(a^2+\eta_1)$ &c., the terms of $x$, $y$ arising from the variation of $\eta_1$ are indefinitely small in regard to those arising from the variation of $\xi_1$; and we have

$$\frac{2\delta x}{x}=\frac{3q_1}{-\beta+\dfrac{3\alpha\beta}{\alpha-\beta}},=-3q_1\frac{(\alpha-\beta)}{\beta(\gamma-\alpha)},$$

$$\frac{2\delta y}{y}=\frac{3q_1}{\alpha+\dfrac{3\alpha\beta}{\alpha-\beta}},=3q_1\frac{\alpha-\beta}{\alpha(\beta-\gamma)},$$

and for $\delta z$ $(=z)$ we have

$$c^2(\delta z)^2 = -\frac{1}{\alpha\beta}\left(\frac{3\alpha\beta}{\alpha-\beta}\right)^3\theta_1, = \frac{-9\Omega q^3}{(\alpha-\beta)^3}, = \frac{(\alpha-\beta)^3}{3\Omega^2}q_1^3,$$

so that writing for greater simplicity, $(\alpha-\beta)q_1 = -\alpha\beta\varpi$, the formulæ become

$$\frac{2\delta x}{x} = \frac{3\alpha}{\gamma-\alpha}\varpi,$$

$$\frac{2\delta y}{y} = -\frac{3\beta}{\beta-\gamma}\varpi,$$

$$c\delta z = \frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{\Omega\sqrt{3}}.$$

52. This shows that there is at the outcrop a cusp, the cuspidal tangent being in the plane of $xy$. It appears moreover that this tangent coincides with the tangent of the evolute. In fact, from the equation $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}-\gamma^2=0$ of the evolute we have

$$\frac{(ax)^{\frac{2}{3}}.\,dx}{x}+\frac{(by)^{\frac{2}{3}}.\,dy}{y}=0,$$

or substituting for $(x, y)$ their values at the outcrop,

$$\frac{\beta(\gamma-\alpha)}{\gamma^{\frac{1}{3}}(\alpha-\beta)}\frac{dx}{x}+\frac{\alpha(\beta-\gamma)}{\gamma^{\frac{1}{3}}(\alpha-\beta)}\frac{dy}{y}=0;$$

that is,

$$\beta(\gamma-\alpha)\frac{dx}{x}+\alpha(\beta-\gamma)\frac{dy}{y}=0,$$

which is satisfied by the foregoing values of $\frac{\delta x}{x}$, and $\frac{\delta y}{y}$, and the two tangents therefore coincide.

We have

$$4\{(\delta x)^2+(\delta y)^2\} = \frac{-9\varpi^2\alpha^2\beta^2}{\gamma(\alpha-\beta)^3}.\left\{\frac{\beta(\gamma-\alpha)}{a^2}+\frac{\alpha(\beta-\gamma)}{b^2}\right\},$$

which in virtue of

$$a^2\alpha(\beta-\gamma)+b^2\beta(\gamma-\alpha)+c^2\gamma(\alpha-\beta)=3\alpha\beta\gamma,$$

is

$$4\{(\delta x)^2+(\delta y)^2\} = \frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2(\alpha-\beta)^3}\{3\alpha\beta-c^2(\alpha-\beta)\}$$

(observe $3\alpha\beta-c^2(\alpha-\beta)$, $=-c^2(\gamma-\alpha)-3a^2\alpha$, is negative)

$$= \frac{-9\varpi^2\alpha^2\beta^2}{a^2b^2(\alpha-\beta)^2}\xi_1,$$

if $\xi_1$ be the value at the outcrop. Writing $\delta s$ for the element of the arc we have

$$\delta s = -\tfrac{3}{2}\frac{\alpha\beta}{ab(\alpha-\beta)}\sqrt{-\xi_1}\,.\,\varpi,$$

$$\delta z = \frac{(-\alpha\beta\varpi)^{\frac{3}{2}}}{c\,\Omega\sqrt{3}},$$

which exhibit the form at the outcrop.

*The Nodal Curve; expressions for the coordinates in terms of a single parameter* $\sigma$.
Art. Nos. 53 to 60.

53. After the foregoing investigation of the nodal curve, I was led to perceive that it is possible to express $\xi$, $\eta$, $\xi_1$, $\eta_1$ in terms of a single variable $\sigma$, and thus to obtain expressions for the coordinates of a point of the nodal curve in terms of the single variable $\sigma$. The result was obtained by the consideration that the acnodal portion of the nodal curve could only arise from imaginary values of $\xi$, $\eta$; the question thus was, what imaginary values of these quantities give real values for the coordinates $x$, $y$, $z$. To make $y$ real we may assume

$$\xi = -b^2 - p(\theta - \phi i)^3,$$
$$\eta = -b^2 + p(\theta + \phi i)^3,$$

($i = \sqrt{-1}$ as usual): this being so, if $\Delta$ denote one or other of the quantities

$$\gamma,\ -\alpha\ (= a^2 - b^2,\ c^2 - b^2),$$

the expressions for $-\beta\gamma a^2x^2$, $-\gamma\alpha b^2y^2$ will be

$$= \{\Delta - p(\theta - \phi i)\}^3\,\{\Delta + p(\theta + \phi i)^3\},$$

and we have therefore the condition that this shall be real (for the two values $\Delta = \gamma$, $\Delta = -\alpha$): being real, it will in certain cases be positive, and we shall then have real values for the remaining coordinates $x$, $z$.

54. The condition of reality is easily found to be

$$\Delta^2(3\theta^2 - \phi^2 + 3) - 6\theta p\Delta(\theta^2 + \phi^2 + 1) + p^2\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\} = 0,$$

viz. this equation in $\Delta$ must have the roots $\gamma$, $-\alpha$, or the expression on the left hand must be

$$= (3\theta^2 - \phi^2 + 3)\{\Delta^2 - (\gamma - \alpha)\Delta - \alpha\gamma\}:$$

we have therefore

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha} = \frac{36(\theta^2 + \phi^2 + 1)^2}{(3\theta^2 - \phi^2 + 3)\{3(\theta^2 + \phi^2)^2 + 3\theta^2 - \phi^2\}};$$

$$\gamma - \alpha = \frac{6\theta p(\theta^2 + \phi^2 + 1)}{3\theta^2 - \phi^2 + 3};$$

and writing $\theta^2+\phi^2=X$, $3\theta^2-\phi^2=Y$, the first of these is

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha}=\frac{9(X+Y)(X+1)^2}{(Y+3)\{3(X^2-1)+Y+3\}},$$

which regarding $X$, $Y$ as the coordinates of a point in a plane is a cubic curve having the point $X+1=0$, $Y+3=0$ as a node: hence writing $Y+3=3\sigma(X+1)$, $X$ and $Y$ will be each of them a rational function of $\sigma$. The second equation is

$$\frac{6\theta p(X+1)}{Y+3}=\gamma-\alpha, \text{ that is, } p=\frac{(\gamma-\alpha)\sigma}{2\theta}, =\frac{(\gamma-\alpha)\sigma}{\sqrt{X+Y}};$$

and we have also

$$2\theta=\sqrt{X+Y},\ 2\phi=\sqrt{3X-Y};$$

the equations thus become

$$\xi=-b^2-\frac{(\gamma-\alpha)\sigma}{2}\left\{1-i\sqrt{\frac{3X-Y}{X+Y}}\right\},$$

$$\eta=-b^2+\frac{(\gamma-\alpha)\sigma(X+Y)}{8}\left\{1+i\sqrt{\frac{3X-Y}{X+Y}}\right\}^3,$$

which are better written in the form

$$\xi=-b^2-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{-3X+Y}{X+Y}}\right\},$$

$$\eta=-b^2+\tfrac{1}{8}(\gamma-\alpha)\sigma(X+Y)\left\{1+\sqrt{\frac{-3X+Y}{X+Y}}\right\}^3,$$

where $X$, $Y$ are given functions of $\sigma$. We in fact thus obtain an analytical expression of the nodal curve, quite independent of the considerations as to real and imaginary which suggested the process: the foregoing values substituted for $\xi$, $\eta$ will give $-\beta\gamma a^2x^2$, &c. equal to rational functions of $\sigma$, so that taking for $\xi_1$, $\eta_1$ the same expressions, only changing therein the sign of the radical $\sqrt{\frac{-3X+Y}{X+Y}}$, these values of $\xi_1$, $\eta_1$ give the very same values of $-\beta\gamma a^2x^2$, &c., or the values of $\xi$, $\eta$, $\xi_1$, $\eta_1$ satisfy the conditions

$$(a^2+\xi)^3(a^2+\eta)=(a^2+\xi_1)^3(a^2+\eta_1), \text{ \&c.}$$

for a point on the nodal curve.

55. To complete the investigation, writing as above $Y+3=3\sigma(X+1)$, we obtain

$$\frac{(\gamma-\alpha)^2}{-\gamma\alpha}=\frac{(3\sigma+1)X+3\sigma-3}{\sigma(X+\sigma-1)};$$

or putting for a moment

$$\frac{(\gamma-\alpha)^2\sigma}{-\gamma\alpha}=K,$$

we have

$$X = \frac{(K-3)(\sigma-1)}{3\sigma+1-K}, \quad X+1 = \frac{K(\sigma-2)+4}{3\sigma+1-K};$$

$$Y+3 = \frac{3K\sigma(\sigma-2)+12\sigma}{3\sigma+1-K}, \quad Y = \frac{3(\sigma-1)\{K(\sigma-1)+1\}}{3\sigma+1-K};$$

$$X+Y = \frac{(\sigma-1)(3\sigma-2)K}{3\sigma+1-K}, \quad -3X+Y = \frac{3(\sigma-1)\{K(\sigma-2)+4\}}{3\sigma+1-K};$$

or substituting for $K$ its value we have

$$K(\sigma-2)+4 = -\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left\{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}$$

$$= -\frac{(\gamma-\alpha)^2}{\gamma\alpha}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right),$$

$$3\sigma+1-K = \frac{1}{\gamma\alpha}\{(3\sigma+1)\gamma\alpha+(\gamma-\alpha)^2\sigma\}, = \frac{1}{\gamma\alpha}(\Omega\sigma+\gamma\alpha),$$

if as before $\Omega = \beta^2-\gamma\alpha$; and the result is

$$\xi = -b^2-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\},$$

$$\eta = -b^2-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3,$$

and changing the sign of the radical we have the values of $\xi_1$, $\eta_1$.

56. Write for a moment

$$\left[\Delta-\tfrac{1}{2}(\gamma-\alpha)\sigma\left\{1-\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}\right] = (\Delta-\mathrm{a}+\mathrm{a}\sqrt{S})^3 = A+B\sqrt{S},$$

$$\Delta-\tfrac{1}{8}(\gamma-\alpha)^3\frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma+\gamma\alpha}\left\{1+\sqrt{\frac{3\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)}{\sigma(3\sigma-2)}}\right\}^3 = A'+B'\sqrt{S};$$

then in the product of these two expressions the rational part is $= AA'+BB'S$; but from the manner in which they were arrived at we have $0 = AB'+A'B$, and the rational part is thus

$$= -\frac{B'}{B}(A^2-B^2S).$$

We have

$$A^2 - B^2S = \{(\Delta - \mathrm{a})^2 - \mathrm{a}^2S\}^3,$$

$$B' = -\tfrac{1}{8}(\gamma - \alpha)^3 \frac{\sigma^2(\sigma-1)(3\sigma-2)}{\Omega\sigma + \gamma\alpha}(3+S),$$

$$B = \tfrac{1}{2}(\gamma - \alpha)\{3(\Delta - \mathrm{a})^2 + \mathrm{a}^2S\};$$

hence the rational part in question is

$$= \tfrac{1}{4}\frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\,\frac{\sigma(\sigma-1)(3\sigma-2)(3+S)}{3(\Delta-\mathrm{a})^2+\mathrm{a}^2S}\{(\Delta-\mathrm{a})^2+\mathrm{a}^2S\}^3,$$

which putting therein $\Delta = 0$ gives the value of $-\gamma\alpha b^2y^2$; and putting $\Delta = \gamma$, or $\Delta = -\alpha$, gives the value of $-\beta\gamma a^2x^2$ or $-\alpha\beta c^2z^2$.

57. We have

$$1 - S = \frac{1}{\sigma(3\sigma-2)}\left[3\sigma^2 - 2\sigma - 3\left\{\sigma^2 - 2\sigma - \frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}\right]$$

$$= \frac{4\left\{\sigma + \dfrac{3\alpha\gamma}{(\gamma-\alpha)^2}\right\}}{\sigma(3\sigma-2)},$$

$$3 + S = \frac{3}{\sigma(3\sigma-2)}\left[3\sigma^2 - 2\sigma + \left\{\sigma^2 - 2\sigma - \frac{4\alpha\gamma}{(\gamma-\alpha)^2}\right\}\right]$$

$$= \frac{12}{\sigma(3\sigma-2)}\left(\sigma + \frac{\alpha}{\gamma-\alpha}\right)\left(\sigma - \frac{\gamma}{\gamma-\alpha}\right).$$

Hence we have at once the value of

$$-\gamma\alpha b^2y^2, = \tfrac{1}{4}\frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\cdot\frac{\sigma(\sigma-1)(3\sigma-2)}{\mathrm{a}^2}\,\mathrm{a}^6(1-S)^3,$$

where

$$\mathrm{a} = \tfrac{1}{2}(\beta - \gamma)\sigma.$$

58. Moreover

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2S = \Delta^2 - 2\mathrm{a}\Delta + \mathrm{a}^2(1-S)$$

$$= \frac{1}{3\sigma-2}\left[(3\sigma-2)\{-(\gamma-\alpha)\Delta\sigma + \Delta^2\} + (\gamma-\alpha)^2\sigma^2 + 3\alpha\gamma\right],$$

where the term in [ ] is

$$\sigma^2(\gamma-\alpha)(\gamma-\alpha-3\Delta) + \sigma\{3\Delta^2 + 2(\gamma-\alpha)\Delta + 3\alpha\gamma\} - 2\Delta^2,$$

and since $\Delta = \gamma$ or $-\alpha$, that is, $\Delta^2 - (\gamma-\alpha)\Delta - \alpha\gamma = 0$, the coefficient of $\sigma$ is

$$= \Delta\{6\Delta - (\gamma - \alpha)\},$$

or the term is the product of two linear functions of $\sigma$; and we have

$$(\Delta - \mathrm{a})^2 - \mathrm{a}^2S = \frac{1}{3\sigma-2}\{\sigma(\gamma-\alpha) - 2\Delta\}\{\sigma(\gamma-\alpha-3\Delta) + \Delta\}.$$

Similarly

$$3(\Delta - a)^2 + a^2 S = 3(\Delta^2 - 2a\Delta) + a^2(3+S)$$

$$= \frac{3}{3\sigma - 2}\left[(3\sigma-2)\{-(\gamma-\alpha)\sigma\Delta + \Delta^2\} + \sigma\{(\gamma-\alpha)\sigma+\alpha\}\{(\gamma-\alpha)\sigma-\gamma\}\right],$$

where the term in [ ] is

$$(\gamma-\alpha)^2\sigma^3 - (\gamma-\alpha)(\gamma-\alpha+3\Delta)\sigma^2 + \{3\Delta^2 + 2(\gamma-\alpha)\Delta - \alpha\gamma\}\sigma - 2\Delta^2,$$

in which the coefficient of $\sigma$ is $=\Delta\{2\Delta + 3(\gamma-\alpha)\}$, and the term is a product of three linear functions: hence

$$3(\Delta - a)^2 + a^2 S = \frac{3(\sigma-1)}{3\sigma-2}\{(\gamma-\alpha)\sigma - \Delta\}\{(\gamma-\alpha)\sigma - 2\Delta\}.$$

59. Substituting these values we have the expression

$$\frac{1}{\Omega\sigma+\gamma\alpha}\,\frac{\{(\gamma-\alpha)\sigma+\alpha\}\{(\gamma-\alpha)\sigma-\gamma\}\{(\gamma-\alpha)\sigma-2\Delta\}^2\{(\gamma-\alpha-3\Delta)\sigma+\Delta\}^3}{\{(\gamma-\alpha)\sigma-\Delta\}(3\sigma-2)^2};$$

which writing therein $\Delta=\gamma$ gives $-\beta\gamma a^2x^2$, and writing $\Delta=-\alpha$ gives $-\alpha\beta c^2z^2$; we have above an expression for $-\gamma\alpha b^2y^2$ requiring only a simple reduction, and the final results are

$$-\beta\gamma a^2x^2 = \frac{\{(\gamma-\alpha)\sigma+\alpha\}\{(\gamma-\alpha)\sigma-2\gamma\}^2\{(\beta-\gamma)\sigma+\gamma\}^3}{(\Omega\sigma+\gamma\alpha)(3\sigma-2)^2},$$

$$-\gamma\alpha b^2y^2 = \frac{(\sigma-1)\sigma^2\{(\gamma-\alpha)^2\sigma+3\alpha\gamma\}^3}{(\Omega\sigma+\gamma\alpha)(3\sigma-2)^2},$$

$$-\alpha\beta c^2z^2 = \frac{\{(\gamma-\alpha)\sigma-\gamma\}\{(\gamma-\alpha)\sigma+2\alpha\}^2\{(\alpha-\beta)\sigma-\alpha\}^3}{(\Omega\sigma+\gamma\alpha)(3\sigma-2)^2},$$

where it is to be observed that, equating the denominator to 0, we have a triple root $\sigma=\infty$; to indicate this, we may insert in the denominator the factor $(1-0\sigma)^3$.

60. We see here the meaning of all the factors, viz.

| | Planes. | | | |
|---|---|---|---|---|
| | $x=0$ | $y=0$ | $z=0$ | $\infty$ |
| Evolute nodes | $\sigma=-\dfrac{\alpha}{\gamma-\alpha}$ | $\sigma=1$ | $\sigma=\dfrac{\gamma}{\gamma-\alpha}$ | $\sigma=\dfrac{-\gamma\alpha}{\Omega}$ |
| Umbilicar centres | $\sigma=\dfrac{2\gamma}{\gamma-\alpha}$ | $\sigma=0$ | $\sigma=\dfrac{-2\alpha}{\gamma-\alpha}$ | $\sigma=\frac{2}{3}$ |
| Outcrops | $\sigma=\dfrac{-\gamma}{\beta-\gamma}$ | $\sigma=\dfrac{-3\gamma\alpha}{(\gamma-\alpha)^2}$ | $\sigma=\dfrac{\alpha}{\alpha-\beta}$ | $\sigma=\infty$ |

For the real curve $\sigma$ extends from $\dfrac{\alpha}{\alpha-\beta}$ through 0 to $-\dfrac{\gamma\alpha}{\Omega}$, viz.

$$\sigma=\frac{\alpha}{\alpha-\beta} \text{ gives outcrop in plane } z=0,$$

$$\sigma=\quad 0 \quad \text{,, umbilicar centre in plane } y=0,$$

$$\sigma=\frac{-\gamma\alpha}{\Omega} \quad \text{,, evolute-node in plane } \infty.$$

It is to be noticed that the order of magnitude of the terms in the table is

$$\infty,\ \frac{2\gamma}{\gamma-\alpha},\ \frac{\gamma}{\gamma-\alpha},\ 1,\ \tfrac{2}{3},\ \frac{-\gamma}{\beta-\gamma},\ \frac{\alpha}{\alpha-\beta},\ 0,\ \frac{-\gamma\alpha}{\Omega},\ \frac{-\alpha}{\gamma-\alpha},\ \frac{-2\alpha}{\gamma-\alpha},\ \frac{-3\gamma\alpha}{(\gamma-\alpha)^2},\ -\infty,$$

so that the values $\dfrac{\alpha}{\alpha-\beta}$, 0, $\dfrac{-\gamma\alpha}{\Omega}$ which belong to the real curve are contiguous; this is as it should be. Several of the preceding investigations conducted by means of the quantities $\xi$, $\eta$, $\xi_1$, $\eta_1$ might have been conducted more simply by means of the formulæ involving $\sigma$.

*The Eight Cuspidal Conics.* Art. Nos. 61 to 71.

61. The centro-surface is the envelope of the quadric

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0.$$

Hence it has a cuspidal curve given by means of this equation and the first and second derived equations

$$\frac{a^2x^2}{(a^2+\xi)^3}+\frac{b^2y^2}{(b^2+\xi)^3}+\frac{c^2z^2}{(c^2+\xi)^3}=0,$$

$$\frac{a^2x^2}{(a^2+\xi)^4}+\frac{b^2y^2}{(b^2+\xi)^4}+\frac{c^2z^2}{(c^2+\xi)^4}=0,$$

which equations determine $a^2x^2$, $b^2y^2$, $c^2z^2$ in terms of $\xi$, viz. we have

$$-\beta\gamma\, a^2x^2=(a^2+\xi)^4,$$

$$-\gamma\alpha\, b^2y^2=(b^2+\xi)^4,$$

$$-\alpha\beta\, c^2z^2=(c^2+\xi)^4;$$

so that, comparing with the equations $-\beta\gamma\, a^2x^2=(a^2+\xi)^3(a^2+\eta)$ &c. which give the centro-surface, we see that for the cuspidal curve $\xi=\eta$; or the cuspidal curve now in question arises from the eight lines on the ellipsoid, which lines are the envelope of the curves of curvature: it is clear that the curve is imaginary.

62. From the foregoing equations we have:

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \sqrt{-\alpha\beta\gamma},$$

$$\alpha^{\frac{3}{4}} \sqrt{ax} + \beta^{\frac{3}{4}} \sqrt{by} + \gamma^{\frac{3}{4}} \sqrt{cz} = 0,$$

the second of which is best written in the rationalised form

$$(1, 1, 1, -1, -1, -1)\,(\alpha \sqrt{\alpha}\, ax,\ \beta \sqrt{\beta}\, by,\ \gamma \sqrt{\gamma}\, cz)^2 = 0,$$

and combining herewith the equation

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \sqrt{-\alpha\beta\gamma},$$

then for any given signs of $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{-\alpha\beta\gamma}$ the first of these equations represents a quadric surface, the second a plane, or the two equations together represent a conic.

By changing the signs of the radicals (observing that when all the signs are changed simultaneously the curve is unaltered) we obtain in all 8 conics, but only four quadric surfaces; viz. the two conics

$$\sqrt{\alpha}\, ax + \sqrt{\beta}\, by + \sqrt{\gamma}\, cz = \pm \sqrt{-\alpha\beta\gamma}$$

lie on the same quadric surface.

63. The conics form two sets of four, corresponding to the two sets of four lines on the ellipsoid. The analysis seems to establish a correspondence of each conic of the one set to a single conic of the other set; viz. the conics have been obtained in pairs as the intersections of the same quadric surface by a pair of planes: there is a like correspondence of each line of the one set to a single line of the other set, viz. the lines meet in pairs on the umbilici at infinity, but this correspondence is included in a more general property: in fact each line of the one set meets each line of the other set in an umbilicus; and the corresponding conics (not only meet but) touch at the corresponding umbilicar centre; and *quà* touching conics they have two points of intersection, and consequently lie on the same quadric surface. It is to be added that the two conics touch also, at the umbilicar centre, the cuspidal conic of the principal section.

64. The 8 conics form two tetrads, and the principal conics (reckoning as one of them the conic at infinity) another tetrad: the complete cuspidal curve consists therefore of three tetrads of conics: with these we may form (one conic out of each tetrad) 16 triads; viz. each conic of one tetrad is combined with each conic of either of the other tetrads, and with a determinate conic of the third tetrad, to form a triad. And the conics of each triad, not only meet but touch at an umbilicar centre, the common tangent being also by what precedes, the tangent of the evolute at that point, which point is also a node of the nodal curve.

65. In fact consider the two conics

$$\sqrt{\alpha}\,ax \pm \sqrt{\beta}\,by + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(1,\ 1,\ 1,\ -1,\ -1,\ -1)\,(\alpha\sqrt{\alpha}\,ax,\ \pm\beta\sqrt{\beta}\,by,\ \gamma\sqrt{\gamma}\,cz)^2 = 0\,;$$

for the intersections with the plane $y = 0$ we have

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$(\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz)^2 = 0\,;$$

so that the two conics each meet the plane in question in the same two coincident points, that is, they each touch the plane $y = 0$ at the same point, viz. the point given by the equations

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

$$\alpha\sqrt{\alpha}\,ax - \gamma\sqrt{\gamma}\,cz = 0\,;$$

viz. this is the point, $ax - \dfrac{\gamma\sqrt{\gamma}}{\sqrt{-\beta}}$, $cz = \dfrac{\alpha\sqrt{\alpha}}{\sqrt{-\beta}}$, which is one of the umbilicar centres

$$\left(a^2x^2 = -\frac{\gamma^3}{\beta},\quad c^2z^2 = -\frac{\alpha^3}{\beta}\right);$$

and the common tangent at this point is

$$\sqrt{\alpha}\,ax + \sqrt{\gamma}\,cz = \sqrt{-\alpha\beta\gamma},$$

which is also the common tangent of the ellipse and evolute in the plane $y = 0$.

66. It has been seen that the nodal curve meets each principal conic at four outcrops, which points are cusps of the nodal curve: it is to be further shown that the nodal curve meets each of the 8 cuspidal conics in four points (giving 32 new points, which may be called 'outcrops,' the 16 points heretofore so called being distinguished as the principal outcrops or 16 outcrops, and the points now in question as the 32 outcrops), which points are cusps of the nodal curve.

In fact to obtain the intersections of the nodal curve with the 8 cuspidal conics, we must in the equation of the nodal curve, or (what is the same thing) in the expressions of $\xi$, $\eta$ in terms of $\sigma$, write $\eta = \xi$.

67. Putting for shortness,

$$\Theta = \tfrac{1}{4}\,\frac{(\gamma-\alpha)^2\,\sigma\,(\sigma-1)\,(3\sigma-2)}{\Omega\sigma + \gamma\alpha},$$

and as before

$$S = \frac{3\left(\sigma + \dfrac{2\alpha}{\gamma-\alpha}\right)\left(\sigma - \dfrac{2\gamma}{\gamma-\alpha}\right)}{\sigma\,(3\sigma-2)},$$

we have thus

$$\Theta\,(1+\sqrt{S})^3 = 1 - \sqrt{S},$$

or, what is the same thing,

$$\Theta(1+3S)-1+\sqrt{S}\{\Theta(3+S)+1\}=0:$$

we have without difficulty

$$\Theta(3+S)+1=\frac{(\gamma-\alpha)^2}{\Omega\sigma+\gamma\alpha}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\},$$

$$\Theta(1+3\sqrt{S})-1=\frac{(\gamma-\alpha)^2(3\sigma-2)\left(\sigma+\dfrac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\dfrac{2\gamma}{\gamma-\alpha}\right)}{\Omega\sigma+\gamma\alpha},$$

so that the resulting equation contains the factor

$$\sqrt{\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\left(\sigma-\frac{2\gamma}{\gamma-\alpha}\right)},\ =\sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}}.$$

Omitting it, the equation becomes

$$\sqrt{\sigma^2-2\sigma-\frac{4\gamma\alpha}{(\gamma-\alpha)^2}}\,(3\sigma-2)^{\frac{3}{2}}\sqrt{\sigma}+\sqrt{3}\left\{3\sigma^3-6\sigma^2+4\sigma+\frac{4\gamma\alpha}{(\gamma-\alpha)^2}\right\}^2=0,$$

or putting for shortness $\dfrac{4\gamma\alpha}{(\gamma-\alpha)^2}=M$, and rationalising, this is

$$-(\sigma^2-2\sigma-M)(3\sigma-2)^3\sigma+3(3\sigma^3-6\sigma^2+4\sigma+M)^2=0,$$

and, working this out, the terms in $\sigma^6$, $\sigma^5$ disappear, and the result is

$$(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2=0,$$

or, as this may also be written,

$$3M^2+M(27\sigma^4-36\sigma^3+16\sigma)+4(9\sigma^4-16\sigma^3+8\sigma^2)=0,$$

a quartic equation in $\sigma$: to each of the 4 roots there correspond 8 intersections, viz. there will be in all 32 intersections, lying in 4's upon the 8 cuspidal conics.

68. To show that these points are cusps, or stationary points on the nodal curve, starting from the expressions of $-\beta\gamma a^2x^2$ &c. in terms of $\sigma$ we have, first for $dy$,

$$-\frac{2dy}{y}=d\sigma\left\{\frac{1}{\sigma-1}+\frac{2}{\sigma}+\frac{3(\gamma-\alpha)^2}{(\gamma-\alpha)^2\sigma+3\alpha\gamma}-\frac{\Omega}{\Omega\sigma+\gamma\alpha}-\frac{6}{3\sigma-2}\right\},$$

or, as this may be written,

$$-\frac{2dy}{y}=d\sigma\left\{\frac{1}{\sigma-1}+\frac{2}{\sigma}+\frac{12}{4\sigma+3M}-\frac{4+3M}{(4+3M)\sigma+M}-\frac{6}{3\sigma-2}\right\},$$

$$=d\sigma\left\{\frac{3\sigma^2-6\sigma+4}{3\sigma^3-5\sigma^2+2\sigma}+\frac{(32+24M)\sigma-9M^2}{(16+12M)\sigma^2+(16M+9M^2)\sigma+3M^2}\right\},$$

$$=d\sigma\,\frac{4\{(36+27M)\sigma^4-(64+36M)\sigma^3+32\sigma^2+16M\sigma+3M^2\}}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}},$$

viz. the numerator vanishes when $\sigma$ is a root of the quartic equation.

69. We have next

$$-\frac{2dx}{x}=d\sigma\left\{\frac{\gamma-\alpha}{(\gamma-\alpha)\sigma+\alpha}+\frac{2(\gamma-\alpha)}{(\gamma-\alpha)\sigma-2\gamma}+\frac{3(\beta-\gamma)}{(\beta-\gamma)\sigma+\gamma}-\frac{\Omega}{\Omega\sigma+\gamma\alpha}-\frac{6}{3\sigma-2}\right\},$$

which, putting $\frac{\gamma}{\gamma-\alpha}=B$, and therefore $\frac{\alpha}{\gamma-\alpha}=B-1$, and $\frac{\gamma}{\beta-\gamma}=C$, is

$$=d\sigma\left\{\frac{1}{\sigma+B-1}+\frac{2}{\sigma-2B}+\frac{3}{\sigma+C}-\frac{4+3M}{(4+3M)\sigma+M}-\frac{6}{3\sigma-2}\right\},$$

and adding the fractions except $\frac{3}{\sigma+C}$, the numerator is

$$\begin{aligned}&\sigma^2(27MB+36B-4)\\&+\sigma\{54(B^2-B)M+72B^2-80B+8\}\\&+\quad 4M-16B^2+16B,\end{aligned}$$

which, observing that $B^2-B=\tfrac{1}{4}M$, is

$$\begin{aligned}=\;&\sigma^2(27MB+36B-4)\\&+\sigma(\tfrac{27}{2}M^2+18M-8B+8),\end{aligned}$$

and, substituting for $M$ and $B$ their values, this is found to be

$$=\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma^2+\frac{8(2\gamma+\alpha)^3\alpha}{(\gamma-\alpha)^4}\sigma,$$

$$=\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\sigma\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right).$$

70. Hence observing that $C=\frac{\gamma}{\beta-\gamma}=\frac{-\gamma}{2\gamma+\alpha}$, the whole coefficient of $d\sigma$ is

$$\frac{\frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3}\left(\sigma^2+\frac{2\alpha}{\gamma-\alpha}\sigma\right)}{(3\sigma-2)(\sigma+B-1)(\sigma-2B)[(3M+4)\sigma+M]}+\frac{3}{\sigma+C},$$

and the numerator of this expressed as a single fraction is

$$\begin{aligned}&=\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^3}\sigma\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)[(2\gamma+\alpha)\sigma-\gamma]\\&\quad+3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M-B\sigma)\{(3M+4)\sigma+M\},\end{aligned}$$

which is

$$\begin{aligned}&=3(3\sigma-2)(\sigma^2-\sigma-\tfrac{1}{2}M)\{(3M+4)\sigma+M\}\\&\quad+\sigma\left[-3B(3\sigma-2)\{(3M+4)\sigma+M\}\right.\\&\qquad\left.+\frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^2}\left(\sigma+\frac{2\alpha}{\gamma-\alpha}\right)\{(2\gamma+\alpha)\sigma-\gamma\}\right]:\end{aligned}$$

the term in [ ] is

$$= \sigma^2 \left[ -(27M+36)B + \frac{4(2\gamma+\alpha)^3}{(\gamma-\alpha)^3} \right]$$

$$+ \sigma \left[ 3B(3M+8) + \frac{4(2\gamma+\alpha)^2}{(\gamma-\alpha)^3} \left( -\gamma + \frac{2\alpha(2\gamma+\alpha)}{\gamma-\alpha} \right) \right]$$

$$+ \left[ 6BM - \frac{8(2\gamma+\alpha)^2 \gamma\alpha}{(\gamma-\alpha)^4} \right],$$

which is found to be

$$= -4\sigma^2 + \sigma(8 + 15M + \tfrac{27}{2}M^2) - 2M - \tfrac{9}{2}M^2,$$

and the whole numerator is thus

$$3(3\sigma-2)(\sigma^2 - \sigma - \tfrac{1}{2}M)[(3M+4)\sigma + M]$$
$$-4\sigma^3 + \sigma^2(8 + 15M + \tfrac{27}{2}M^2) + \sigma(-2M - \tfrac{9}{2}M^2),$$

which is

$$= (36+27M)\sigma^4 - (64+36M)\sigma^3 + 32\sigma^2 + 16M\sigma + 3M^2.$$

71. We have thus

$$-\frac{2dx}{x} = d\sigma \frac{(36+27M)\sigma^4 - (64+36M)\sigma^3 + 32\sigma^2 + 16M\sigma + 3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma + \dfrac{\gamma}{\beta-\gamma}\right)},$$

and thence also

$$\frac{2dz}{z} = d\sigma \frac{(36+27M)\sigma^4 - (64+36M)\sigma^3 + 32\sigma^2 + 16M\sigma + 3M^2}{\sigma(\sigma-1)(3\sigma-2)(4\sigma+3M)\{(4+3M)\sigma+M\}\left(\sigma - \dfrac{\alpha}{\alpha-\beta}\right)},$$

so that $dx$ and $dz$ also vanish when $\sigma$ is a root of the quartic equation: the points in question are therefore cusps of the nodal curve.

*Centro-surface as the envelope of the quadric* $\Sigma a^2x^2(a^2+\xi)^{-2} = 1$. Art. Nos. 72 to 76.

72. The equations $-\beta\gamma a^2x^2 = (a^2+\xi)^3(a^2+\eta)$, &c. considering therein $\xi$, $\eta$ as variable give the centro-surface; considering $\eta$ as a given constant but $\xi$ as variable they give the sequential centro-curve; and considering $\xi$ as a given constant but $\eta$ as variable they give the concomitant centro-curve.

73. Suppose first that $\eta$ is a given constant; to eliminate $\xi$ we may write the equations in the form

$$-(\beta\gamma)^{\frac{1}{3}}(ax)^{\frac{2}{3}}(a^2+\eta)^{-\frac{1}{3}} = (a^2+\xi), \text{ \&c.},$$

and then multiplying first by $\alpha(a^2+\eta)$, &c. and adding, and secondly by $\alpha$, &c., and adding (observing that $\Sigma\alpha(a^2+\xi)(a^2+\eta) = -\alpha\beta\gamma$, $\Sigma\alpha(a^2+\xi) = 0$); we have

$$\Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{\frac{2}{3}} = (\alpha\beta\gamma)^{\frac{2}{3}},$$

$$\Sigma(\alpha ax)^{\frac{2}{3}}(a^2+\eta)^{-\frac{1}{3}} = 0,$$

which equations, considering therein $\eta$ as a given constant, are the equations of a sequential centro-curve.

If from the two equations we eliminate $\eta$ we should have the equation of the centro-surface; the second equation is the derivative of the first in regard to $\eta$; and it thus appears that the equation of the centro-surface might be obtained by equating to zero the discriminant of the rationalised function

$$\text{norm. } [\{\Sigma (\alpha ux)^{\frac{2}{3}} (a^2+\eta)^{\frac{2}{3}}\} - (\alpha\beta\gamma)^{\frac{2}{3}}];$$

but the form is too inconvenient to be of any use.

74. Taking next $\xi$ as a given constant, and writing the equations in the form

$$-\beta\gamma a^2x^2 (a^2+\xi)^{-3} = (a^2+\eta), \text{ \&c.};$$

then multiplying by $\alpha(a^2+\xi)$, &c. and adding, and again multiplying by $\alpha$, &c. and adding, we have

$$\Sigma a^2x^2 (a^2+\xi)^{-2} = 1,$$
$$\Sigma a^2x^2 (a^2+\xi)^{-3} = 0;$$

or writing these equations at full length,

$$\frac{a^2x^2}{(a^2+\xi)^2} + \frac{b^2y^2}{(b^2+\xi)^2} + \frac{c^2z^2}{(c^2+\xi)^2} - 1 = 0,$$
$$\frac{a^2x^2}{(a^2+\xi)^3} + \frac{b^2y^2}{(b^2+\xi)^3} + \frac{c^2z^2}{(c^2+\xi)^3} = 0,$$

which equations, considering therein $\xi$ as a constant, are the equations of any concomitant centro-curve: since the equations are each of the second order it thus appears that the concomitant centro-curves are quadriquadrics.

75. If from the two equations we eliminate $\xi$, we have the equation of the centro-surface; the second equation is the derivative of the first in regard to $\xi$; and it thus appears that the equation of the centro-surface is obtained by equating to zero the discriminant in regard to $\xi$ of the integralised function

$$(a^2+\xi)^2 (b^2+\xi)^2 (c^2+\xi)^2 \{(\Sigma a^2x^2 (a^2+\xi)^{-2} - 1\},$$

or, what is the same thing, the discriminant of the sextic function

$$(a^2+\xi)^2 (b^2+\xi)^2 (c^2+\xi)^2 - \Sigma a^2x^2 (b^2+\xi)^2 (c^2+\xi)^2.$$

76. If instead hereof we consider the homogeneous function

$$w^2 (a^2+\xi)^2 (b^2+\xi)^2 (c^2+\xi)^2 - \Sigma a^2x^2 (b^2+\xi)^2 (c^2+\xi)^2,$$

then the coefficients are of the second order in $(x, y, z, w)$, and the discriminant, being of the tenth order in the coefficients, is of the order 20 in $(x, y, z, w)$. But the sextic function has a twofold factor $\left(1+\frac{\xi}{\infty}\right)^2$ if $w^2=0$, and it has evidently a twofold factor if $x^2=0$ or $y^2=0$ or $z^2=0$, that is, the discriminant contains the factor $x^2y^2z^2w^2$; or, omitting this factor, it will be of the order 12 in $(x, y, z, w)$; whence writing $w=1$, the centro-surface is of the order 12. I have in this manner actually obtained the equation of the centro-surface: see the memoir "On a certain Sextic Torse," *Camb. Phil. Trans.* t. XI. (1871), pp. 507—523, [436].

*Another generation of the Centro-surface.* Art. Nos. 77 to 83.

77. By what precedes, the equation of the centro-surface is obtained as the condition in order that the equation

$$\{\Sigma a^2x^2(a^2+\xi)^{-2}\}-1=0$$

may have two equal roots. But taking $m$ an arbitrary constant, this is the derived equation of

$$\{\Sigma a^2x^2(a^2+\xi)^{-1}\}+\xi+m=0,$$

and as such it will have two equal roots if the last-mentioned equation has three equal roots; and conversely, we have thus the equation of the centro-surface by expressing that the last-mentioned equation, or, what is the same thing, the quartic equation

$$(\xi+m)(\xi+a^2)(\xi+b^2)(\xi+c^2)-\Sigma a^2x^2(\xi+b^2)(\xi+c^2)=0$$

has three equal roots. The conditions for this are that the quadrinvariant and the cubinvariant shall each of them vanish; the two invariants are respectively a quadric and a cubic function of $m$; viz. the equations are

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c})(m,\ 1)^2=0,\quad (\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{d}')(m,\ 1)^3=0;$$

where the degrees in $(x,\ y,\ z)$ of a, b, c are 0, 2, 4 and those of a′, b′, c′, d′ are 0, 2, 4, 6 respectively: the equation of the centro-surface then is

$$\left|\begin{array}{ccccc} & & \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ & \mathrm{a}, & \mathrm{b}, & \mathrm{c} & \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} & & \\ & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' & \end{array}\right|=0,$$

which is of the right order 12; but it would be difficult to obtain thereby the developed equation.

78. For the nodal curve the cubic equation must be satisfied by each root of the quadric equation, or, what is the same thing, the quadric function must completely divide the cubic function; the conditions are

$$\left\|\begin{array}{cccc} & \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} & \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}' \end{array}\right\|=0,$$

where the degrees may be taken to be

$$\left|\begin{array}{cccc} 0, & 0, & 2, & 4 \\ 0, & 2, & 4, & 0 \\ 0, & 2, & 4, & 6 \end{array}\right|,$$

and the order of the nodal curve is thus $=24$: two of the equations in fact are

$$\begin{vmatrix} & \mathrm{a}, & \mathrm{b} \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}' \end{vmatrix} = 0, \qquad \begin{vmatrix} & \mathrm{b}, & \mathrm{c} \\ \mathrm{a}, & \mathrm{c} & \\ \mathrm{a}', & \mathrm{c}', & \mathrm{d}' \end{vmatrix} = 0,$$

which are surfaces of the orders 4, 6; or the nodal curve is a complete intersection $4 \times 6$. By the results above obtained as to the nodal curve, it appears that the two surfaces must have an ordinary contact at each of the 16 umbilicar centres, and a stationary or singular contact at each of 48 outcrops.

79. The derivation of the centro-surface from the surface

$$\frac{a^2x^2}{a^2+\xi} + \frac{b^2y^2}{b^2+\xi} + \frac{c^2z^2}{c^2+\xi} + \xi - m = 0$$

requires to be further explained. The surface, say $V=0$, is a quadric surface depending on the two parameters $\xi$, $m$; the axes coincide in direction with those of the ellipsoid, and their relative magnitudes are as

$$\frac{1}{a}\sqrt{a^2+\xi} : \frac{1}{b}\sqrt{b^2+\xi} : \frac{1}{c}\sqrt{c^2+\xi},$$

viz. these are as the axes of the confocal surface

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1 = 0$$

divided by $a$, $b$, $c$ respectively; to fix the absolute magnitudes observe that the equation may be written

$$x^2+y^2+z^2-m-\xi\left(\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} - 1\right) = 0,$$

viz. the surface $V=0$ is a surface through the spheroconic which is the intersection of the confocal surface by the arbitrary sphere $x^2+y^2+z^2-m=0$; but, while the surface is hereby and by the preceding condition as to the axes completely determined, the geometrical significance is anything but clear.

80. Considering then the quadric surface $V=0$, depending on the parameters $\xi$, $m$; suppose that $m$ remains constant while $\xi$ alone varies; we have thus three consecutive surfaces $V=0$, $V'=0$, $V''=0$; and these I say intersect in a point of the centro-surface; the point in question will depend on the two parameters ($\xi$, $m$), and if these vary simultaneously we have the whole system of points on the centro-surface; but if only one of them varies, the other being constant, we have a curve on the centro-surface.

The three equations may be replaced by $V=0$, $\delta_\xi V=0$, $\delta_\xi{}^2 V=0$; of which the first alone contains $m$; and it thus appears that if $m$ be the variable parameter, the equations of the curve are $\delta_\xi V=0$, $\delta_\xi{}^2 V=0$, viz. the curve is then the quadriquadric curve which is the concomitant centro-curve of the curve of curvature for the parameter $\xi$. But if the variable parameter be $\xi$, then this is a curve on the 12-thic surface $\Omega = 0$ obtained by the elimination of $\xi$ from the equations $V=0$, $\delta_\xi V = 0$; viz. we have $\Omega = S^3 - T^2 = 0$, where $S = (\text{a, b, c})(m, 1)^2$, $T = (\text{a}', \text{b}', \text{c}', \text{d}')(m, 1)^3$, and the curve in question is the curve $S=0$, $T=0$, which is the cuspidal curve on the surface $\Omega = 0$; the elimination of $m$ from the two equations $S=0$, $T=0$ gives as above the equation of the centro-surface.

81. The surface $\Omega = S^3 - T^2 = 0$ obtained as above by the elimination of $\xi$ from the equations $V=0$, $\delta_\xi V = 0$, (or, what is the same thing, by equating to zero the discriminant of $V$ in regard to $\xi$,) may be termed the sociate-surface: we have then the quartic and sextic surfaces $S=0$, $T=0$ intersecting in the before-mentioned curve, which may be called the sociate-edge; and the locus of these sociate-edges is the centro-surface.

82. We may if we please, changing the parameter in one of the functions, consider the two series of surfaces $S=0$, $T=0$ depending on the parameters $m$, $m'$ respectively; a surface of the first series will correspond to one of the second series when the parameters are equal, $m = m'$, and we have then a sociate-edge. Taking a point anywhere in space, through this point there pass two surfaces $S=0$, and three surfaces $T=0$; but there is no pair of corresponding surfaces, or sociate-edge. If however the point be taken anywhere on the centro-surface, then there is a pair of corresponding surfaces $S=0$, $T=0$; that is, through each point of the centro-surface there passes a single sociate-edge; and if the point be taken anywhere on the nodal curve of the centro-surface, then there are two pairs of corresponding surfaces; that is, through each point of the nodal curve there are two sociate-edges: this explains the method above made use of for finding the equations of the nodal curve, by giving to the equations $S=0$, $T=0$, considered as equations in $m$, two equal roots.

83. The *à posteriori* verification that the surfaces $V=0$, $V'=0$, $V''=0$ intersect in a point of the centro-surface, is not without interest; the parameters $\xi_1$, $\eta_1$ of the point of intersection are found to be $\xi_1 = \xi$, $\eta_1 = m - a^2 - b^2 - c^2 - 3\xi$; whence in the equation $V=0$, writing $-\beta\gamma a^2x^2 = (a^2+\xi_1)^3(a^2+\eta_1)$ and $m = a^2+b^2+c^2+3\xi_1+\eta_1$, the resulting equation considered as an equation in $\xi$ should have three roots $\xi = \xi_1$: the fourth root is at once seen to be $\xi = \eta_1$, and we ought therefore to have identically

$$\frac{-\alpha(a^2+\xi_1)^3(a^2+\eta_1)}{a^2+\xi} - \&\text{c.} + \alpha\beta\gamma(\xi - 3\xi_1 - \eta_1 - a^2 - b^2 - c^2)$$
$$= \frac{\alpha\beta\gamma(\xi-\xi_1)^3(\xi-\eta_1)}{(a^2+\xi)(b^2+\xi)(c^2+\xi)};$$

and by decomposing the right-hand side into its component fractions this is at once seen to be true.

*Third generation of the Centro-surface.* Art. Nos. 84 and 85.

84. Instead of the foregoing equation $V=0$, consider the equation

$$W = \left(\frac{a^2x^2}{a^2+\xi} + \frac{b^2y^2}{b^2+\xi} + \frac{c^2z^2}{c^2+\xi} + \xi\right) - \left(\frac{a^2x^2}{a^2+\eta} + \frac{b^2y^2}{b^2+\eta} + \frac{c^2z^2}{c^2+\eta} + \eta\right) = 0.$$

The equations $d_\xi W=0$, $d_\xi{}^2 W=0$ contain only $\xi$, and are in fact identically the same as the equations $d_\xi V=0$, $d_\xi{}^2 V=0$; the elimination of $\xi$ from the equations $d_\xi V=0$, $d_\xi{}^2 W=0$ would therefore lead to the equation of the centro-surface: and the centro-surface is connected with the surfaces $W=0$, $d_\xi W=0$, $d_\xi{}^2 W=0$ and the parameters $\xi$, $\eta$ in the same way as it is with the surfaces $V=0$, $d_\xi V=0$, $d_\xi{}^2 V=0$ and the parameters $\xi$, $m$. That is, if from the equations $W=0$, $d_\xi W=0$ we eliminate $\xi$ we have a surface $\Omega=0$, depending upon $\eta$ and having a cuspidal curve; and the locus of the cuspidal curve (as $\eta$ varies) is the centro-surface. But the equation $W=0$ divides by $\xi-\eta$, and throwing out this factor it becomes

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}-1=0,$$

so that the surface $\Omega=0$ is obtained by eliminating $\xi$ from this equation and the derived equation in regard to $\xi$; or, what is the same thing, by equating to zero the discriminant in regard to $\xi$ of the cubic function

$$(a^2+\xi)(b^2+\xi)(c^2+\xi)-\Sigma\frac{a^2x^2}{a^2+\eta}(b^2+\xi)(c^2+\xi).$$

This surface is in fact the torse generated by the normals at the several points of the curve of curvature belonging to the parameter $\eta$; the cuspidal curve is the edge of regression of this torse, that is, it is the sequential centro-curve of the curve of curvature; and we thus fall back upon the original investigation for the centro-surface.

85. In verification I remark that if $X$, $Y$, $Z$ be the coordinates of a point on the curve of curvature in question, and $(x, y, z)$ current coordinates, then the tangent plane of the torse, or plane through the normal and the tangent of the curve of curvature, has for its equation

$$\frac{Xx}{a^2+\eta}+\frac{Yy}{b^2+\eta}+\frac{Zz}{c^2+\eta}-1=0,$$

and if in this equation we consider the point $(X, Y, Z)$ to be the point belonging to the parameters $(\eta, \xi)$, viz. if we have $-\beta\gamma X^2=a^2(a^2+\xi)(a^2+\eta)$, &c., then this plane will be always touched by the before-mentioned ellipsoid,

$$\frac{a^2x^2}{(a^2+\xi)(a^2+\eta)}+\frac{b^2y^2}{(b^2+\xi)(b^2+\eta)}+\frac{c^2z^2}{(c^2+\xi)(c^2+\eta)}=1.$$

The condition for the contact in fact is

$$\Sigma\frac{X^2}{(a^2+\eta)^2}\,\frac{(a^2+\xi)(a^2+\eta)}{a^2}=1,$$

viz. substituting for $(X, Y, Z)$ their values, this is

$$-\frac{1}{\alpha\beta\gamma}\,\Sigma\alpha(a^2+\xi)^2=1,$$

which is true. And this being so, the ellipsoid and the plane have each the same envelope, viz. this is the torse in question.

*Reciprocal Surface.* Art. No. 86.

86. The centro-surface is the envelope of

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}-1=0;$$

hence the reciprocal surface in regard to the sphere $x^2+y^2+z^2-k^2=0$ is the envelope of

$$\frac{(a^2+\xi)^2}{a^2}X^2+\frac{(b^2+\xi)^2}{b^2}Y^2+\frac{(c^2+\xi)^2}{c^2}Z^2-k^2=0,$$

that is,

$$a^2X^2+b^2Y^2+c^2Z^2-k^2+2\xi(X^2+Y^2+Z^2)+\xi^2\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)=0,$$

viz. the envelope is

$$(a^2X^2+b^2Y^2+c^2Z^2-k^4)\left(\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}\right)-(X^2+Y^2+Z^2)^2=0,$$

or, expanding and multiplying by $a^2b^2c^2$, this is

$$a^2(b^2-c^2)^2\,Y^2Z^2+b^2(c^2-a^2)^2\,Z^2X^2+c^2(a^2-b^2)^2\,X^2Y^2$$
$$-k^4(b^2c^2\,X^2+c^2a^2\,Y^2+a^2b^2\,Z^2)=0,$$

or, what is the same thing,

$$a^2\alpha^2\,Y^2Z^2+b^2\beta^2\,Z^2X^2+c^2\gamma^2\,X^2Y^2-k^4(b^2c^2\,X^2+c^2a^2\,Y^2+a^2b^2\,Z^2)=0,$$

which may be written

$$\mathrm{a}^2Y^2Z^2+\mathrm{b}^2Z^2X^2+\mathrm{c}^2X^2Y^2+\mathrm{f}^2X^2+\mathrm{g}^2Y^2+\mathrm{h}^2Z^2=0,$$

where

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h})=(a\alpha,\ b\beta,\ c\gamma,\ 2k^2bc,\ 2k^2ca,\ 2k^2ab),$$

and consequently,

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=2k^2abc\,(\alpha+\beta+\gamma)=0.$$

It would doubtless be interesting to discuss this surface as it here presents itself, and with reference to its geometrical signification as the locus of the pole, in regard to the sphere, of the plane through two intersecting consecutive normals of the ellipsoid: but I abstain from any consideration of the question.

*Delineation of the centro-surface for given numerical values of the semiaxes.* Art. Nos. 87 and 88.

87. I constructed on a large scale a drawing of the centro-surface for the values

$$a^2=50,\quad b^2=25,\quad c^2=15.$$

(These were chosen so that $a$, $b$, $c$ should have approximately the integer values 7, 5, 4, and that $a^2+c^2$ should be well greater than $2b^2$; they give a good form of surface, though perhaps a better selection might have been made; there is a slight objection to the existence of the relation $a^2 = 2b^2$, as in the $xy$-section it brings a cusp of the evolute on the ellipse.) We have therefore

$$\alpha = 10, \quad \beta = -35, \quad \gamma = 25;$$

the ellipses in the principal planes of the centro-surface are

$$\frac{y^2}{(5)^2} + \frac{z^2}{(8\cdot937)^2} = 1,$$

$$\frac{z^2}{(2\cdot582)^2} + \frac{x^2}{(3\cdot535)^2} = 1,$$

$$\frac{x^2}{(4\cdot950)^2} + \frac{y^2}{(2)^2} = 1,$$

and these determine on each axis the two points which are the cusps of the evolutes. We have moreover for the umbilicar centre $x = 2\cdot988$, $y = 0$, $z = 1\cdot380$, and for the outcrop $x = 1\cdot127$, $y = 1\cdot947$, $z = 0$.

88. For the delineation of the nodal curve (crunodal portion) we have first to find the values of $\xi$, $\xi_1$; these are given in terms of x, y *ante* No. 33 [p. 334], where y is a given function of x, and x extends between the values $\{-b^2$ and $-\frac{1}{3}(a^2+b^2+c^2)\} - 25$ and $-26\frac{2}{3}$. It was thought sufficient to divide the interval into 6 equal parts, that is, the values of $x$ were taken to be $-25$, $-25\cdot\dot{3}$, ... $-26\cdot\dot{6}$. The values of $\xi$, $\xi_1$ being found, those of $\eta$, $\eta_1$ were obtained from them by means of the original equations $(a^2+\xi)^3(a^2+\eta) = (a^2+\xi_1)(a^2+\eta)$ &c. viz. we have thus for the determination of $\eta$, $\eta_1$ three simple equations, affording a verification of each other.

For the performance of these calculations (viz. of the values of $y$, $\xi$, $\xi_1$, $\eta$, $\eta_1$) I am indebted to the kindness of Mr J. W. L. Glaisher, of Trinity College. The results being obtained it is then easy to calculate as well the coordinates $(x, y, z)$ of the point on the nodal curve as also the coordinates $(X, Y, Z)$ and $(X_1, Y_1, Z_1)$ of the corresponding two points on the ellipsoid (these last are of course not required for the delineation of the nodal curve, but it was interesting to obtain them). The whole series of the results is given in the annexed Table, and from them the drawing was constructed.

I find also in the neighbourhood of the umbilicar centre (if $\xi = -25 + q$),

$$\delta x = \phantom{\pm}\cdot02868\, q^2,$$
$$\delta y = \pm\cdot02484\, q^2,$$
$$\delta z = \phantom{\pm}\cdot02191\, q^2,$$

and in the neighbourhood of the outcrop (if $\xi_1 = -38\cdot333 + \frac{70}{9}\varpi$),

$$\delta x = \phantom{-}1\cdot127\, \varpi,$$
$$\delta y = -1\cdot704\, \varpi,$$
$$\delta z = \pm 4\cdot582\, \varpi^{\frac{3}{2}}.$$

| x | y | $\xi$ | $\xi_1$ | $\eta$ | $\eta_1$ | $x$ | $y$ | $z$ | $X$ | $Y$ | $Z$ | $X_1$ | $Y_1$ | $Z_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-25$ | 0· | $-25\cdot$ | 25· | $-25\cdot$ | $-25\cdot$ | 2·988 | 0· | 1·380 | 5·326 | 0· | 2·070 | 5·326 | 0· | 2·070 |
| $-25\cdot\dot{3}$ | 4·2669 | $-21\cdot0664$ | $-29\cdot6002$ | $-38\cdot3193$ | $-16\cdot6728$ | 2·543 | 0·360 | 0·996 | 4·394 | 2·289 | 2·491 | 6·233 | 1·957 | 1·023 |
| $-25\cdot\dot{6}$ | 6·3191 | $-19\cdot3475$ | $-31\cdot9858$ | $-42\cdot9911$ | $-15\cdot4693$ | 2·148 | 0·721 | 0·662 | 3·504 | 3·189 | 2·283 | 5·956 | 2·580 | 0·584 |
| $-26\cdot$ | 8·1240 | $-17\cdot8760$ | $-34\cdot1240$ | $-45\cdot7879$ | $-15\cdot1047$ | 1·786 | 1·175 | 0·373 | 2·780 | 3·847 | 1·948 | 5·626 | 3·005 | 0·293 |
| $-26\cdot\dot{3}$ | 9·8760 | $-16\cdot4573$ | $-36\cdot2094$ | $-47\cdot5684$ | $-15\cdot0106$ | 1·448 | 1·500 | 0·139 | 2·159 | 4·391 | 1·426 | 5·251 | 3·346 | 0·098 |
| $-26\cdot\dot{6}$ | 11·6667 | $-15\cdot0000$ | $-38\cdot3333$ | $-48\cdot7037$ | $-15\cdot0000$ | 1·127 | 1·947 | 0· | 1·610 | 4·869 | 0· | 4·829 | 3·651 | 0· |

The calculations were performed before I had obtained the formulæ in $\sigma$, which would have given the results more easily.