519.

ON CURVATURE AND ORTHOGONAL SURFACES.

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THE principal object of the present Memoir is the establishment of the partial differential equation of the third order satisfied by the parameter of a family of surfaces belonging to a triple orthogonal system. It was first remarked by Bouquet that a given family of surfaces does not in general belong to an orthogonal system, but that (in order to its doing so) a condition must be satisfied; it was afterwards shown by Serret that the condition is that the parameter, considered as a function of the coordinates, must satisfy a partial differential equation of the third order: this equation was not obtained by him or the other French geometers engaged on the subject, although methods of obtaining it, essentially equivalent but differing in form, were given by Darboux and Levy; the last-named writer even found a particular form of the equation, viz. what the general equation becomes on writing therein $X=0$, $Y=0$ (X, Y, Z the first derived functions, or quantities proportional to the cosine-inclinations of the normal). Using Levy's method, I obtained the general equation, and communicated it to the French Academy, [518]. My result was, however, of a very complicated form, owing, as I afterwards discovered, to its being encumbered with the extraneous factor $X^2+Y^2+Z^2$; I succeeded, by some difficult reductions, in getting rid of this factor, and so obtaining the equation in the form given in the present memoir, viz.

$$\begin{aligned}&((A),\ (B),\ (C),\ (F),\ (G),\ (H)\between \delta a,\ \delta b,\ \delta c,\ 2\delta f,\ 2\delta g,\ 2\delta h)\\ &-2\,((A),\ (B),\ (C),\ (F),\ (G),\ (H)\between \bar{a},\ \ \bar{b},\ \ \bar{c},\ 2\bar{f}\ ,\ 2\bar{g}\ ,\ 2\bar{h}\ )=0:\end{aligned}$$

but the method was an inconvenient one, and I was led to reconsider the question. The present investigation, although the analytical transformations are very long, is in theory extremely simple: I consider a given surface, and at each point thereof take along the normal an infinitesimal length ρ (not a constant, but an arbitrary function of the coordinates), the extremities of these distances forming a new surface, say the vicinal surface; and the points on the same normal being considered as corresponding points, say this is the conormal correspondence of vicinal surfaces. In order that the two surfaces may belong to an orthogonal system, it is necessary and sufficient that at each point of the given surface the principal tangents (tangents to the curves of curvature) shall correspond to the principal tangents at the corresponding point of the vicinal surface; and the condition for this is that ρ shall satisfy a partial differential equation of the second order,

$$((A), (B), (C), (F), (G), (H)\between d_x, d_y, d_z)^2 \rho = 0,$$

where the coefficients depend on the first and second differential coefficients of U, if $U = 0$ is the equation of the given surface. Now, considering the given surface as belonging to a family, or writing its equation in the form $r - r(x, y, z) = 0$ (the last r a functional symbol), the condition in order that the vicinal surface shall belong to this family, or say that it shall coincide with the surface $r + \delta r - r(x, y, z) = 0$, is $\rho = \frac{\delta r}{V}$, where $V = \sqrt{X^2 + Y^2 + Z^2}$, if X, Y, Z are the first differential coefficients of $r(x, y, z)$, that is, of the parameter r considered as a function of the coordinates; we have thus the equation

$$((A), (B), (C), (F), (G), (H)\between d_x, d_y, d_z)^2 \frac{1}{V} = 0,$$

viz. the coefficients being functions of the first and second differential coefficients of r, and V being a function of the first differential coefficients of r, this is in fact a relation involving the first, second, and third differential coefficients of r, or it is the partial differential equation to be satisfied by the parameter r considered as a function of the coordinates. After all reductions, this equation assumes the form previously mentioned.

Article Nos. 1 to 21. *On the Curvature of Surfaces.*

1. Curvature is a metrical theory having reference to the circle at infinity; each point in space may be regarded as the vertex of a cone passing through this circle, say the circular cone; a line and plane through the vertex are at right angles to each other when they are polar line and polar plane in regard to the cone; and so two lines or two planes are at right angles when they are harmonics in regard to the cone, that is, when each line lies in the polar plane, or each plane passes through the polar line of the other. A plane through the vertex meets the cone in two lines, which are the "circular lines" in the plane and through the point; a line through the vertex has through it two tangent planes, which might be called the "circular planes" of the point and through the line; but the term is hardly required. Lines in the plane and through the point, at right angles to each other, are also harmonics (polar lines) in regard to the two circular lines.

2. Consider now a surface, and any point thereof; we have at this point a tangent plane and a normal. The tangent plane meets the surface in a curve having

at the point a node, and the tangents to the two branches of the curve (being of course lines in the tangent plane) are the "chief tangents" of the surface at the point.

3. The chief tangents are the intersections of the tangent plane by a quadric cone, which may be called the chief cone; but it is important to observe that this cone is not independent of the particular form under which the equation of the surface is presented. To explain this, suppose that the rational equation of the surface is $U=0$; taking ξ, η, ζ as current coordinates measured from the point as origin, the equation of the chief cone is $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2 U=0$, where x, y, z denote the coordinates of the point. But it is in the sequel necessary to present the equation of the surface in a different manner; say we have an equation between the coordinates (x, y, z) and a parameter r (r being therefore in general an irrational function of x, y, z), which, when $r=r_1$, reduces itself to $U=0$: we have then $r=r_1$ as the equation of the surface; and the corresponding equation of the chief cone is $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2 r=0$; this is not the same as the cone $(\xi\partial_x+\eta\partial_y+\zeta\partial_z)^2 U=0$, although of course it intersects the tangent plane in the same two lines, viz. the chief lines; and so in general there is a distinct chief cone corresponding to each form of the equation of the surface. But adopting a definite form of equation, we have a definite chief cone intersecting the tangent plane in the chief tangents.

4. Observe that the equations $U=0$, $r=r_1$, although each relating to one and the same surface, serve to represent this surface, and that in different ways, as belonging to a family of surfaces, viz. one of these is the family $U=$ const., and the other the family $r=$ const. In order to represent a given surface as belonging to a certain family, we need the irrational form of equation; thus r denoting the irrational function of x, y, z determined by the equation $\dfrac{x^2}{a+r}+\dfrac{y^2}{b+r}+\dfrac{z^2}{c+r}=1$, we have $r=0$ as the equation of the ellipsoid $\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}=1$, considered as belonging to a family of confocal quadrics.

5. Although at first sight presenting some difficulty, it is convenient to use the same letter r to denote the parameter considered as a function of the coordinates, and the special value of the parameter; thus in general the equation of a surface may be written $r(x, y, z)-r=0$ (in which form the first r may be regarded as a functional symbol), or simply $r-r=0$, viz. the first r here denotes the given function of (x, y, z), and the second r the particular value of the parameter.

6. By what precedes we have through the point and in the tangent plane two circular lines, the intersections of the tangent plane by the circular cone having the point for its vertex.

We have also through the point and in the tangent plane two other lines, termed the principal tangents, viz. the definition of these is that they are the double (or sibiconjugate) lines of the involution formed by the circular lines and the chief tangents, or, what is the same thing, they are the bisectors (and as such at right angles to each other) of the angles formed by the chief tangents.

7. The principal tangents may also be considered as the intersections of the tangent plane by a quadric cone, called the principal cone; this being a cone constructed by means of the circular cone and the chief cone, and thus depending on the particular chief cone, that is, on the form of the equation of the surface. The definition is that the principal cone is the locus of a line (through the point), such that the line itself, the perpendicular (or harmonic in regard to the circular cone) of the polar plane of the line in regard to the chief cone, and the normal of the surface are *in plano*.

8. Analytically, taking, as before, (x, y, z) for the coordinates of the point, and u, v, w as current coordinates measured from the point as origin, then the equation of the circular cone is $u^2+v^2+w^2=0$; and taking $Xu+Yv+Zw=0$ for the equation of the tangent plane, and $(a, b, c, f, g, h \between u, v, w)^2=0$ for that of the chief cone, then, if the line be $u : v : w = \xi : \eta : \zeta$, we have

$$(a, \ldots \between \xi, \eta, \zeta \between u, v, w)=0$$

for the equation of the polar plane, and thence

$$u : v : w = a\xi+h\eta+g\zeta : h\xi+b\eta+f\zeta : g\xi+f\eta+c\zeta$$

for those of the perpendicular, or harmonic in regard to the circular cone; also for the normal $u, v, w = X : Y : Z$; whence, if the three lines are *in plano*, we have

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ a\xi+h\eta+g\zeta, & h\xi+b\eta+f\zeta, & g\xi+f\eta+c\zeta \\ X, & Y, & Z \end{vmatrix}=0$$

as the equation of the principal cone. This is in the sequel written, for shortness, as

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix}=0.$$

9. Consider any point P', not in general on the surface, in the neighbourhood of the point on the surface, say P; then the point P' has in regard to the surface a polar plane, which plane, however, is dependent on the particular form of equation—viz. x', y', z' being the coordinates of P', and U' the same function of these that U is of x, y, z, then the form $U=0$ of the equation of the surface gives for P' the polar plane $(ud_{x'}+vd_{y'}+wd_{z'})\,U'=0$; and we may through P' draw hereto a perpendicular (or harmonic in regard to the circular cone), say this is the normal line of P'. Then for points P' in the neighbourhood of P, when these are such that their normal lines meet the normal at P, the locus of P' is the before-mentioned principal cone. The analytical investigation presents no difficulty.

10. Taking P' on the surface, the normal line of P' becomes the normal at a consecutive point P' of the surface (being now a line independent of the particular form of equation), and this normal meets the normal at P; that is, we have the

principal cone meeting the tangent plane in two lines, the principal tangents, such that at a consecutive point P' on either of these the normal meets the normal at P; viz. we have the principal tangents at the tangents of the two curves of curvature through the point P.

The plane through the normal and a principal tangent is termed a principal plane; we have thus at the point of the surface two principal planes, forming with the tangent plane an orthogonal triad of planes.

11. I proceed to further develop the theory, commencing with the following lemma:

Lemma. Given the line $Xu + Yv + Zw = 0$, and conic

$$(a, b, c, f, g, h \between u, v, w)^2 = 0,$$

then, to determine the coordinates (u_1, v_1, w_1), (u_2, v_2, w_2) of the points of intersection of the line and conic, we have

$$(a, .. \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2$$
$$= (\xi u_1 + \eta v_1 + \zeta w_1)(\xi u_2 + \eta v_2 + \zeta w_2),$$

or, what is the same thing, we have

$$(a, ... \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2 = 0$$

as the equation, in line coordinates, of the two points of intersection. The proof is obvious.

12. Making the equations refer to a plane and a cone, and writing throughout ξ, η, ζ as current point coordinates, the theorem is:

Given the plane $X\xi + Y\eta + Z\zeta = 0$, and cone

$$(a, b, c, f, g, h \between \xi, \eta, \zeta)^2 = 0;$$

then, to determine the lines of intersection of the plane and cone, we have

$$(a, .. \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2 = 0$$

as the equation of the pair of planes at right angles to the two lines respectively.

13. Denoting the coefficients by (a), (b), &c., that is, writing

$$(a, .. \between Y\zeta - Z\eta,\ Z\xi - X\zeta,\ X\eta - Y\xi)^2$$
$$= ((a), (b), (c), (f), (g), (h) \between \xi, \eta, \zeta)^2,$$

the values of these are

$$\begin{aligned}
(a) &= bZ^2 + cY^2 - 2fYZ,\\
(b) &= cX^2 + aZ^2 - 2gZX,\\
(c) &= aY^2 + bX^2 - 2hXY,\\
(f) &= -aYZ - fX^2 + gXY + hXZ,\\
(g) &= -bZX + fYX - gY^2 + hYZ,\\
(h) &= -cXY + fZX + gZY - hZ^2.
\end{aligned}$$

We have the following identities:

$$(a)\,X+(h)\ \ Y+(g\,)\,Z=0,$$
$$(h)\,X+(b)\ \ Y+(f\,)\,Z=0,$$
$$(g)\,X+(f\,)\ Y+(c)\ \ Z=0,$$
$$\big((b)\,(c)-(f\,)^2,\,..,\ (g)\,(h)-(a)\,(f\,),\,...\big)=-\,(X^2,\ Y^2,\ Z^2,\ YZ,\ ZX,\ XY)\,\phi,$$

that is, $(b)\,(c)-(f\,)^2=-\,X^2\phi$ &c., where

$$\phi=(bc-f^2,\,..\,gh-af,\,..\between X,\ Y,\ Z)^2.$$

Writing also

$$aX+hY+gZ,\ hX+bY+fZ,\ gX+fY+cZ=\delta X,\ \delta Y,\ \delta Z,$$

and $X^2+Y^2+Z^2=V^2$; also $a+b+c=\omega$, then

$$(a)\ =(b+c)\ V^2-\omega X^2\ +X\delta X\ -Y\delta Y-Z\delta Z,$$
$$(b)\ =(c+a)\ V^2-\omega Y^2\ -X\delta X+Y\delta Y-Z\delta Z,$$
$$(c)\ =(a+b)\ V^2-\omega Z^2\ \ -X\delta X\ -Y\delta Y+Z\delta Z,$$
$$(f\,)=-fV^2\qquad -\omega YZ\ +Y\delta Z\ \ +Z\delta Y,$$
$$(g)=-gV^2\ \ -\omega ZX\ +Z\delta X\ +X\delta Z,$$
$$(h)=-hV^2\ \ -\omega XY+X\delta Y+Y\delta X.$$

14. I give also the following lemma:

Lemma. The condition in order that the plane $X\xi+Y\eta+Z\zeta=0$ may meet the cones

$$(A\ ,\ B\ ,\ C\ ,\ F\ ,\ G\ ,\ H\between \xi,\ \eta,\ \zeta)^2=0,$$
$$(A',\ B',\ C',\ F',\ G',\ H'\between \xi,\ \eta,\ \zeta)^2=0$$

in two pairs of lines harmonically related to each other, is

$$(BC'+B'C-2FF',\,..,\ GH'+G'H-AF'-A'F,\,..\between X,\ \ Y,\ Z)^2=0.$$

Writing here

$$(A,\,..\between Y\zeta-Z\eta,\ Z\xi-X\zeta,\ X\eta-Y\xi)^2$$
$$=\big((A),\ (B),\ (C),\ (F),\ (G),\ (H)\between \xi,\ \eta,\ \zeta)^2,$$

that is, $(A)=BZ^2+CY^2-2FYZ$, &c., the condition may be written

$$(A)\,A'+(B)\,B'+(C)\,C'+2\,(F)\,F'+2\,(G)\,G'+2(H)\,H'=0,$$

or say

$$((A),\,..\between A',\,..)=0\,;$$

and we may, it is clear, interchange the accented and unaccented letters respectively.

15. I take $r - r = 0$ for the equation of a surface, X, Y, Z for the first derived functions of r, (a, b, c, f, g, h) for the second derived functions. The equation of the tangent plane at the point (x, y, z), taking ξ, η, ζ as current coordinates measured from this point, is

$$X\xi + Y\eta + Z\zeta = 0;$$

the equation of the chief cone in regard to this form of the equation of the surface is

$$(a, b, c, f, g, h \between \xi, \eta, \zeta)^2 = 0,$$

and the equation of the circular cone is $\xi^2 + \eta^2 + \zeta^2 = 0$, or, what is the same thing,

$$(1, 1, 1, 0, 0, 0 \between \xi, \eta, \zeta)^2 = 0.$$

Imagine a quadric cone

$$(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0,$$

such that it meets the tangent plane in the sibiconjugate lines of the involution formed by the intersections of the tangent plane by the chief cone and the circular cone respectively; that is, in a pair of lines harmonically related to the intersections with the chief cone, and also to the intersections with the circular cone; the conditions are

$$((A), \ldots \between a, \ldots) = 0,$$

and

$$(A) + (B) + (C) = 0,$$

viz. if only these two conditions are satisfied the cone will intersect the tangent plane in the two principal tangents.

16. The principal cone, writing, for shortness,

$$a\xi + h\eta + g\zeta,\ h\xi + b\eta + f\zeta,\ g\xi + f\eta + c\zeta = \delta\xi,\ \delta\eta,\ \delta\zeta,$$

was before taken to be the cone

$$\begin{vmatrix} \xi, & \eta, & \zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} = 0.$$

Representing this equation by

$$\tfrac{1}{2}(A, B, C, F, G, H \between \xi, \eta, \zeta)^2 = 0,$$

the expressions of the coefficients are

$$\begin{aligned} A &= 2hZ - 2gY, \\ B &= 2fX - 2hZ, \\ C &= 2gY - 2fX, \\ F &= \ hY - \ gZ - (b - c)\,X, \\ G &= \ fZ - \ hX - (c - a)\,Y, \\ H &= \ gX - \ fY - (a - b)\,Z. \end{aligned}$$

These values give

$$AX + HY + GZ = Z\delta Y - Y\delta Z,$$
$$HX + BY + FZ = X\delta Z - Z\delta X,$$
$$GX + FY + CZ = Y\delta X - X\delta Y;$$

whence also

$$(A, \ldots \between X, Y, Z)^2 = 0,$$

as is, in fact, at once obvious from the determinant-form; and also

$$A + B + C = 0.$$

17. Writing for shortness

$$(\bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{g}, \bar{h}) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$

we find

$$Aa + Hh + Gg = \omega\,(hZ - gY) + \bar{h}Z - \bar{g}Y,$$
$$Hh + Bb + Ff = \omega\,(fX - hZ) + \bar{f}X - \bar{h}Z,$$
$$Gg + Ff + Cc = \omega\,(gY - fX) + \bar{g}Y - \bar{f}X;$$

whence

$$(A, \ldots \between a, \ldots) = 0.$$

18. By what precedes, we have

$$((A), \ldots \between \xi, \eta, \zeta)^2 = 0$$

for the equation of the two principal planes, where the coefficients (A), (B), &c. are functions of A, B, &c. and of X, Y, Z, as mentioned above. These coefficients satisfy of course the several relations similar to those satisfied by (a), (b), &c., and other relations dependent on the expressions of A, B, &c. in terms of a, b, &c. and X, Y, Z.

19. Proceeding to consider the coefficients (A), (B), &c., we have then

$$(A) + (B) + (C) = (A + B + C)\,V^2 - (A, \ldots \between X, Y, Z)^2,$$

that is

$$(A) + (B) + (C) = 0.$$

Observing the relation $A + B + C = 0$, the equations analogous to

$(a) = (b + c)\,V^2 - (a + b + c)\,X^2 + \&c.$, are $(A) = -AV^2 + X\delta'X - Y\delta'Y - Z\delta'Z$, &c.

if for a moment we write $\delta'X$, $\delta'Y$, $\delta'Z$ to denote the functions

$$AX + HY + GZ,\ HX + BY + FZ,\ GX + FY + CZ.$$

But, from the above values, $X\delta'X + Y\delta'Y + Z\delta'Z = 0$, or the equation is $(A) = -AV^2 + 2X\delta'X$, that is $= -AV^2 + 2X\,(Z\delta Y - Y\delta Z)$. The equation for (F) is $(F) = -FV^2 + Y\delta'Z + Z\delta'Y$, where $Y\delta'Z + Z\delta'Y$ is $= Y\,(Y\delta X - X\delta Y) + Z\,(X\delta Z - Z\delta X)$, viz. this is

$$= (Y^2 - Z^2)\,\delta X - XY\delta Y + XZ\delta Z.$$

We have thus the system of equations

$$\begin{aligned}
(A) &= -AV^2 \quad\quad\quad\quad \cdot \quad\quad\quad\quad + 2XZ\delta Y \quad\quad - 2XY\delta Z,\\
(B) &= -BV^2 - 2YZ\delta X \quad\quad\quad \cdot \quad\quad\quad\quad + 2XY\delta Z,\\
(C) &= -CV^2 + 2YZ\delta X \quad\quad - 2XZ\delta Y \quad\quad\quad \cdot\\
(F) &= -FV^2 + (Y^2 - Z^2)\,\delta X - XY\delta Y \quad\quad + XZ\delta Z,\\
(G) &= -GV^2 + XY\delta X \quad\quad + (Z^2 - X^2)\,\delta Y - YZ\delta Z,\\
(H) &= -HV^2 - XZ\delta X \quad\quad + YZ\delta Y \quad\quad + (X^2 - Y^2)\,\delta Z.
\end{aligned}$$

20. We hence find

$$\begin{aligned}
(A)\,a + (H)\,h + (G)\,g &= -(Aa + Hh + Gg)\,V^2 + (Z\delta Y - Y\delta Z)\,\delta X + XP,\\
(H)\,h + (B)\,b + (F)\,f &= -(Hh + Bb + Ff)\,V^2 + (X\delta Z - Z\delta X)\,\delta Y + YQ,\\
(G)\,g + (F)\,f + (C)\,c &= -(Gg + Ff + Cc)\,V^2 + (Y\delta X - X\delta Y)\,\delta Z + ZR,
\end{aligned}$$

if for shortness

$$\begin{aligned}
P &= (gY - hZ)\,\delta X + (aZ - gX)\,\delta Y + (hX - aY)\,\delta Z,\\
Q &= (fY - bZ)\,\delta X + (hZ - fX)\,\delta Y + (bX - hY)\,\delta Z,\\
R &= (cY - fZ)\,\delta X + (gZ - cX)\,\delta Y + (fX - gY)\,\delta Z.
\end{aligned}$$

Forming the sum $PX + QY + RZ$, the coefficient of δX is found to be

$$= -Z\,(hX + bY + fZ) + Y\,(gX + fY + cZ), = -Z\delta Y + Y\delta Z;$$

hence the whole is

$$= \delta X\,(Y\delta Z - Z\delta Y) + \delta Y\,(Z\delta X - X\delta Z) + \delta Z\,(X\delta Y - Y\delta X), \text{ which is } = 0, \text{ that is,}$$

$$PX + QY + RZ = 0.$$

21. Hence, adding, we find

$$((A), \ldots \between a, \ldots) = 0;$$

viz. in this and the before-mentioned equation

$$(A) + (B) + (C) = 0$$

we have the *à posteriori* verification that the cone $(A, \ldots \between \xi, \eta, \zeta)^2 = 0$ cuts the tangent plane in the double lines of the involution.

In what precedes I have given only those relations between the several sets of quantities a, $\overline{a}$, (a), A, (A), &c. which have been required for establishing the results last obtained; but there are various other relations required in the sequel, and which will be obtained as they are wanted.

The Conormal Correspondence of Vicinal Surfaces.
Art. Nos. 22 to 35.

22. We consider a surface $U=0$ (or $r=r$), and at each point P thereof measure along the normal an infinitesimal length ρ, dependent on the position of the point P (that is, ρ is a function of x, y, z). We have thus a point P', the coordinates of which are

$$x',\ y',\ z' = x+\rho\alpha,\ y+\rho\beta,\ z+\rho\gamma,$$

where α, β, γ are the cosine-inclinations of the normal, that is,

$$\alpha,\ \beta,\ \gamma = \frac{X}{V},\ \frac{Y}{V},\ \frac{Z}{V},\ \text{if } V=\sqrt{X^2+Y^2+Z^2};$$

the locus of P' is of course a surface, say the vicinal surface, and we require to find the direction of the normal at P', or, what is the same thing, the differential equation $X'dx'+Y'dy'+Z'dz'$ of the surface. We have

$$\begin{aligned}
dx' &= (1+d_x\rho\alpha)\,dx + d_y\rho\alpha\,.\,dy + d_z\rho\alpha\,.\,dz,\\
dy' &= d_x\rho\beta\,.\,dx + (1+d_y\rho\beta)\,dy + d_z\rho\beta\,.\,dz,\\
dz' &= d_x\rho\gamma\,.\,dx + d_y\rho\gamma\,.\,dy + (1+d_z\rho\gamma)\,dz,\\
0 &= X\,dx + Y\,dy + Z\,dz;
\end{aligned}$$

whence, eliminating dx, dy, dz, we have between dx', dy', dz' a linear equation, the coefficients of which may be taken to be X', Y', Z'. Taking these only as far as the first power of ρ, we have

$$X' = X(1+d_y\rho\beta+d_z\rho\gamma) - Yd_x\rho\beta - Zd_x\rho\gamma,$$

or, what is the same thing,

$$X' = X(1+d_x\rho\alpha+d_y\rho\beta+d_z\rho\gamma) - Xd_x\rho\alpha - Yd_x\rho\beta - Zd_x\rho\gamma,$$

with the like expressions for Y' and Z'. I proceed to reduce these. The formula for X' is

$$\begin{aligned}
X' = X\{1+\rho(d_x\alpha+d_y\beta+d_z\gamma)+\alpha d_x\rho+\beta d_y\rho+\gamma d_z\rho\}&\\
-\rho(Xd_x\alpha+Yd_x\beta+Zd_x\gamma)-(\alpha X+\beta Y+\gamma Z)\,d_x\rho.&
\end{aligned}$$

23. I write, for shortness, $\delta = Xd_x+Yd_y+Zd_z$, whence δX, δY, $\delta Z = aX+hY+gZ$, $hX+bY+fZ$, $gX+fY+cZ$, agreeing with the former significations of δX, δY, δZ; also Vd_xV, Vd_yV, $Vd_zV=\delta X$, δY, δZ, and $V\delta V = X\delta X+Y\delta Y+Z\delta Z$. It is now easy to form the values of

$$\begin{aligned}
&d_x\alpha,\ d_x\beta,\ d_x\gamma, && \text{viz. these are } \frac{a}{V}-\frac{X\delta X}{V^3},\ \frac{h}{V}-\frac{Y\delta X}{V^3},\ \frac{g}{V}-\frac{Z\delta X}{V^3},\\
&d_y\alpha,\ d_y\beta,\ d_y\gamma, && \frac{h}{V}-\frac{X\delta Y}{V^3},\ \frac{b}{V}-\frac{Y\delta Y}{V^3},\ \frac{f}{V}-\frac{Z\delta Y}{V^3},\\
&d_z\alpha,\ d_z\beta,\ d_z\gamma, && \frac{g}{V}-\frac{X\delta Z}{V^3},\ \frac{f}{V}-\frac{Y\delta Z}{V^3},\ \frac{c}{V}-\frac{Z\delta Z}{V^3};
\end{aligned}$$

and hence

$$d_x\alpha + d_y\beta + d_z\gamma = \frac{a+b+c}{V} - \frac{\delta V}{V^2},$$

$$Xd_x\alpha + Yd_x\beta + Zd_x\gamma = \frac{\delta X}{V} - \frac{V^2}{V^3}\delta X, = 0,$$

$$\alpha d_x\rho + \beta d_y\rho + \gamma d_z\rho = \frac{1}{V}\delta\rho,$$

$$\alpha X + \beta Y + \gamma Z \qquad = V;$$

and we have

$$X' = X\left\{1 + \rho\left(\frac{a+b+c}{V} - \frac{\delta V}{V^2}\right) + \frac{1}{V}\delta\rho\right\} - Vd_x\rho,$$

with the like values of Y' and Z'. But we are only concerned with the ratios $X' : Y' : Z'$; whence, dividing the foregoing values by the coefficient in { }, and taking the second terms only to the first order in ρ, we have simply

$$X',\ Y',\ Z' = X - Vd_x\rho,\ \ Y - Vd_y\rho,\ \ Z - Vd_z\rho.$$

24. We may investigate the condition in order that the surface x', y', z' may be the consecutive surface $r + dr = r(x, y, z)$. This will be the case of

$$r + dr = r\left(x + \rho\frac{X}{V},\ \ y + \rho\frac{Y}{V},\ \ z + \rho\frac{Z}{V}\right),$$

that is, $r + dr = r + \rho V$, or $\rho = \frac{dr}{V}$. This value of ρ gives $d_x\rho = -\frac{dr}{V^2}d_xV = -\frac{\rho}{V^2}\delta X$, and similarly $d_y\rho = -\frac{\rho}{V^2}\delta Y$, $d_z\rho = -\frac{\rho}{V^2}\delta Z$; whence

$$X',\ Y',\ Z' = X + \frac{\rho}{V}\delta X,\ Y + \frac{\rho}{V}\delta Y,\ Z + \frac{\rho}{V}\delta Z,$$

which is as it should be, viz. these are what X, Y, Z become on substituting therein for x, y, z the values $x + \rho\alpha$, $y + \rho\beta$, $z + \rho\gamma$.

25. I return to the case where ρ is arbitrary, and I investigate the values of a, b, ... for the point P' on the vicinal surface; say these are a', b', &c., then we have $a' = d_{x'}X'$ &c. The relation between the differentials may be written

$$\begin{aligned}
dx &= (1 - d_x\rho\alpha)\,dx' - d_y\rho\alpha\ dy' - d_z\rho\alpha\ dz',\\
dy &= -d_x\rho\beta\ dx' + (1 - d_y\rho\beta)\,dy' - d_z\rho\beta\ dz',\\
dz &= -d_x\rho\gamma\ dx' - d_y\rho\gamma\ dy' + (1 - d_z\rho\gamma)\,dz',
\end{aligned}$$

and we thence have $d_{x'} = (1 - d_x\rho\alpha)\, d_x - d_x\rho\beta d_y - d_x\rho\gamma d_z$ &c.; hence

$$\begin{aligned} a' &= \{(1 - d_x\rho\alpha)\, d_x - d_x\rho\beta d_y - d_x\rho\gamma d_z\}\,(X - Vd_x\rho) \\ &= (1 - d_x\rho\alpha)\, a - d_x\rho\beta\,.\,h - d_x\rho\gamma\,.\,g - d_x\,(Vd_x\rho) \\ &= a - \rho\,(ad_x\alpha + hd_x\beta + gd_x\gamma) \\ &\quad - (a\alpha + h\beta + g\gamma)\, d_x\rho \\ &\quad - \frac{1}{V}\,\delta X d_x\rho - Vd_x^2\rho\,; \end{aligned}$$

and similarly, $f' = d_{y'}Z'$ (or $d_{z'}Y'$), that is

$$\begin{aligned} f' &= f - \rho\,(gd_y\alpha + fd_y\beta + cd_y\gamma) \\ &\quad - (g\alpha + f\beta + c\gamma)\, d_y\rho \\ &\quad - \frac{1}{V}\,\delta Y d_z\rho - Vd_y d_z\rho. \end{aligned}$$

26. Completing the reduction, we find

$$\begin{aligned} a' &= a - \rho\left(\frac{a\omega - \bar{b} - \bar{c}}{V} - \frac{(\delta X)^2}{V^3}\right) - \frac{2}{V}\,\delta X d_x\rho - Vd_x^2\rho, \\ b' &= b - \rho\left(\frac{b\omega - \bar{c} - \bar{a}}{V} - \frac{(\delta Y)^2}{V^3}\right) - \frac{2}{V}\,\delta Y d_y\rho - Vd_y^2\rho, \\ c' &= c - \rho\left(\frac{c\omega - \bar{a} - \bar{b}}{V} - \frac{(\delta Z)^2}{V^3}\right) - \frac{2}{V}\,\delta Z d_z\rho - Vd_z^2\rho, \\ f' &= f - \rho\left(\frac{\omega f + \bar{f}}{V} - \frac{\delta Y\delta Z}{V^3}\right) - \frac{1}{V}\,(\delta Y d_z\rho + \delta Z d_x\rho\,) - Vd_y d_z\rho, \\ g' &= g - \rho\left(\frac{\omega g + \bar{g}}{V} - \frac{\delta Z\delta X}{V^3}\right) - \frac{1}{V}\,(\delta Z d_x\rho + \delta X d_y\rho) - Vd_z d_x\rho, \\ h' &= h - \rho\left(\frac{\omega h + \bar{h}}{V} - \frac{\delta X\delta Y}{V^3}\right) - \frac{1}{V}\,(\delta X d_y\rho + \delta Y d_z\rho) - Vd_x d_y\rho\,; \end{aligned}$$

say these expressions are $a' = a + \Delta a$, &c.

27. Taking ξ, η, ζ for the coordinates, referred to P as origin, of a point on the given surface near to P, and ξ', η', ζ' for the coordinates, referred to P' as origin, of the corresponding point on the vicinal surface, the relation between ξ', η', ζ' and ξ, η, ζ is the same as that between dx', dy', dz' and dx, dy, dz; viz. we have

$$\begin{aligned} \xi &= (1 - d_x\rho\alpha)\,\xi' - d_y\rho\alpha\,.\,\eta' - d_z\rho\alpha\,.\,\zeta', \\ \eta &= - d_x\rho\beta\,.\,\xi' + (1 - d_y\rho\beta)\,\eta' - d_z\rho\beta\,.\,\zeta', \\ \zeta &= - d_x\rho\gamma\,.\,\xi' - d_y\rho\gamma\,.\,\eta' + (1 - d_z\rho\gamma)\,\zeta'\,: \end{aligned}$$

or conversely

$$\begin{aligned}\xi' &= (1+d_x\rho\alpha)\,\xi + d_y\rho\alpha\,.\,\eta + d_z\rho\alpha\,.\,\zeta,\\ \eta' &= d_x\rho\beta\,.\,\xi + (1+d_y\rho\beta)\,\eta + d_z\rho\beta\,.\,\zeta,\\ \zeta' &= d_x\rho\gamma\,.\,\xi + d_y\rho\gamma\,.\,\eta + (1+d_z\rho\gamma)\,\zeta,\end{aligned}$$

say $\xi', \eta', \zeta' = \xi+\Delta\xi,\ \eta+\Delta\eta,\ \zeta+\Delta\zeta$; hence

$$\begin{aligned}X'\xi'+Y'\eta'+Z'\zeta' = (X-Vd_x\rho)(\xi+\Delta\xi) + \&\text{c.}\\ = X\xi+Y\eta+Z\zeta\\ +X\Delta\xi+Y\Delta\eta+Z\Delta\zeta\\ -V(\xi d_x\rho+\eta d_y\rho+\zeta d_z\rho),\end{aligned}$$

where second line is

$$\begin{aligned}&(X\alpha+Y\beta+Z\gamma)(\xi d_x\rho+\eta d_y\rho+\zeta d_z\rho)\\ &+\rho\,\{(Xd_x\alpha+Yd_x\beta+Zd_x\gamma)\,\xi+(Xd_y\alpha+Yd_y\beta+Zd_y\gamma)\,\eta+(Xd_z\alpha+Yd_z\beta+Zd_z\gamma)\,\zeta\}.\end{aligned}$$

But

$$\begin{aligned}Xd_x\alpha+Yd_x\beta+Zd_x\gamma &= \frac{\delta X}{V}-\frac{V^2}{V^3}\delta X = 0,\\ Xd_y\alpha+Yd_y\beta+Zd_y\gamma &= 0,\\ Xd_z\alpha+Yd_z\beta+Zd_z\gamma &= 0,\end{aligned}$$

or second line is $= V(\xi d_x\rho+\eta d_y\rho+\zeta d_z\rho)$; and we have therefore

$$X'\xi'+Y'\eta'+Z'\zeta' = X\xi+Y\eta+Z\zeta.$$

We require

$$(A', B', C', F', G', H' \between \xi', \eta', \zeta')^2;$$

viz., to the first order in ρ, this is

$$\begin{aligned}&=(A', \ldots \between \xi, \eta, \zeta)^2\\ &\quad+2\,(A, \ldots \between \Delta\xi, \Delta\eta, \Delta\zeta \between \xi, \eta, \zeta).\end{aligned}$$

28. Here second line is

$$2\,\{(A\xi+H\eta+G\zeta)\,\Delta\xi+(H\xi+B\eta+F\zeta)\,\Delta\eta+(G\xi+F\eta+C\zeta)\,\Delta\zeta\}:$$

but

$$A\xi+H\eta+G\zeta = Z\delta\eta-Y\delta\zeta+\begin{vmatrix}a, & h, & g\\ X, & Y, & Z\\ \xi, & \eta, & \zeta\end{vmatrix},$$

$$H\xi+B\eta+F\zeta = X\delta\zeta-Z\delta\xi+\begin{vmatrix}h, & b, & f\\ X, & Y, & Z\\ \xi, & \eta, & \zeta\end{vmatrix},$$

$$G\xi+F\eta+C\zeta = Y\delta\xi+X\delta\eta+\begin{vmatrix}g, & f, & c\\ X, & Y, & Z\\ \xi, & \eta, & \zeta\end{vmatrix},$$

whence term in { } is

$$\begin{vmatrix} \Delta\xi, & \Delta\eta, & \Delta\zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} + \begin{vmatrix} a\Delta\xi + h\Delta\eta + g\Delta\zeta, & h\Delta\xi + b\Delta\eta + f\Delta\zeta, & g\Delta\xi + f\Delta\eta + c\Delta\zeta \\ X & , \quad Y & , \quad Z \\ \xi & , \quad \eta & , \quad \zeta \end{vmatrix},$$

which might be written

$$\begin{vmatrix} \Delta\xi, & \Delta\eta, & \Delta\zeta \\ \delta\xi, & \delta\eta, & \delta\zeta \\ X, & Y, & Z \end{vmatrix} - \begin{vmatrix} \delta\Delta\xi, & \delta\Delta\eta, & \delta\Delta\zeta \\ \xi, & \eta, & \zeta \\ X, & Y, & Z \end{vmatrix},$$

but it is perhaps more convenient to retain the second term in its original form.

29. As regards the first line, we have

$$\begin{aligned} A' &= 2h'Z' - 2g'Y' \\ &= 2(h+\Delta h)(Z - Vd_z\rho) - 2(g+\Delta g)(Y - Vd_y\rho) \\ &= A + 2(Z\Delta h - Y\Delta g) - 2V(hd_z\rho - gd_y\rho), \end{aligned}$$

with similar expressions for the other coefficients. Attending only to the terms of the first order, we thus obtain

$$\begin{aligned} A' &= A + 2(Z\Delta h - Y\Delta g) - 2V(hd_z - gd_y)\rho, \\ B' &= B + 2(X\Delta f - Z\Delta h) - 2V(fd_x - hd_z)\rho, \\ C' &= C + 2(Y\Delta g - X\Delta f) - 2V(gd_y - fd_x)\rho, \\ F' &= F + Y\Delta h - Z\Delta g - X(\Delta b - \Delta c) - V\big(hd_y - gd_z - (b-c)\,d_x\big)\rho, \\ G' &= G + Z\Delta f - X\Delta h - Y(\Delta c - \Delta a) - V\big(fd_z - hd_x - (c-a)\,d_y\big)\rho, \\ H' &= H + X\Delta g - Y\Delta f - Z(\Delta a - \Delta b) - V\big(gd_x - fd_y - (a-b)\,d_z\big)\rho, \end{aligned}$$

say these are $A' = A + \theta A$, &c., where θ is a functional symbol; we thus have

$$(A', \ldots \between \xi', \eta', \zeta')^2 = (A, \ldots \between \xi, \eta, \zeta)^2 + (\theta A, \ldots \between \xi, \eta, \zeta)^2 + 2(A, \ldots \between \xi, \eta, \zeta \between \Delta\xi, \Delta\eta, \Delta\zeta),$$

which, for shortness, I represent by

$$= (A, \ldots \between \xi, \eta, \zeta)^2 + (A'', \ldots \between \xi, \eta, \zeta)^2;$$

and I proceed to complete the calculation of the coefficients A'', B'', &c.

30. We have

$$\begin{aligned} A'' = \theta A &+ \text{coeff. } \xi^2 \text{ in} \\ &2\left[(A\xi + H\eta + G\zeta)\Delta\xi + (H\xi + B\eta + F\zeta)\Delta\eta + (G\xi + F\eta + C\zeta)\Delta\zeta\right] \\ &= \theta A + 2(Ad_x\rho\alpha + Hd_x\rho\beta + Gd_x\rho\gamma), \end{aligned}$$

that is,

$$\begin{aligned} A'' = \theta A &+ 2\,\frac{1}{V}(AX + HY + GZ)\,d_x\rho \\ &+ 2\rho\,(Ad_x\alpha + Hd_x\beta + Gd_x\gamma), \end{aligned}$$

where coeff. 2ρ is

$$= \frac{Aa+Hh+Gg}{V} - \frac{(AX+HY+GZ)\,\delta X}{V^3}$$

$$= \frac{1}{V}\left\{\omega\,(hZ-gY)+\bar{h}Z-\bar{g}Y\right\} - \frac{\delta X}{V^3}(Z\delta Y - Y\delta Z).$$

31. And similarly,

$$\begin{aligned} F'' &= \theta F + (H\alpha + B\beta + F\gamma)\,d_z\rho + (G\alpha + F\beta + C\gamma)\,d_y\rho \\ &\quad + \rho\,\{(Hd_z\alpha + Bd_z\beta + Fd_z\gamma) + (Gd_y\alpha + Fd_y\beta + Cd_y\gamma)\} \\ &= \theta F + \frac{1}{V}\{(HX+BY+FZ)\,d_z\rho + (GX+FY+CZ)\,d_y\rho\} \\ &\quad + \rho\left\{\frac{Hg+Bf+Fc}{V} - \frac{(HX+BY+FZ)\,\delta Z}{V^3}\right. \\ &\quad \left. + \frac{Gh+Fb+Cf}{V} - \frac{(GX+FY+CZ)\,\delta Y}{V^3}\right\}, \end{aligned}$$

$$\begin{aligned} Gh+Fb+Cf &= \omega\,(hY-bX)+\bar{h}Y-\bar{b}X+\bar{\omega}X+\bar{a}X+\bar{h}Y+\bar{g}Z, \\ Hg+Bf+Fc &= -\omega\,(gZ-cX)-\bar{g}Y+\bar{c}X-\bar{\omega}X-\bar{a}X-\bar{h}Y-\bar{g}Z. \end{aligned}$$

Sum is $\omega\,\{hY-gZ-(b-c)X\}+\bar{h}Y-\bar{g}Z-(\bar{b}-\bar{c})\,X$, which is $=\omega F+\bar{h}Y-\bar{g}Z-(\bar{b}-\bar{c})\,X$: hence

$$\begin{aligned} F'' &= \theta F + (X\delta Z - Z\delta X)\left(\frac{1}{V}d_z\rho - \frac{\rho\delta Z}{V^3}\right) + (Y\delta X - X\delta Y)\left(\frac{1}{V}d_y\rho - \frac{\rho\delta Y}{V^3}\right) \\ &\quad + \frac{\rho}{V}\{\omega F + \bar{h}Y - \bar{g}Z - (\bar{b}-\bar{c})\,X\}. \end{aligned}$$

32. We may write

$$A'' = \theta A + 2\left(\frac{1}{V}d_x\rho - \frac{\rho\delta X}{V^3}\right)(Z\delta Y - Y\delta Z) + \frac{\rho}{V}\{\omega A + \bar{A}\},$$

$$B'' = \theta B + 2\left(\frac{1}{V}d_y\rho - \frac{\rho\delta Y}{V^3}\right)(X\delta Z - Z\delta X) + \frac{\rho}{V}\{\omega B + \bar{B}\},$$

$$C'' = \theta C + 2\left(\frac{1}{V}d_z\rho - \frac{\rho\delta Z}{V^3}\right)(Y\delta X - X\delta Y) + \frac{\rho}{V}\{\omega C + \bar{C}\},$$

$$F'' = \theta F + \left(\frac{1}{V}d_z\rho - \frac{\rho\delta Z}{V^3}\right)(X\delta Z - Z\delta X) + \left(\frac{1}{V}d_y\rho - \frac{\rho\delta Y}{V^3}\right)(Y\delta X - X\delta Y) + \frac{\rho}{V}\{\omega F + \bar{F}\},$$

$$G'' = \theta G + \left(\frac{1}{V}d_x\rho - \frac{\rho\delta X}{V^3}\right)(Y\delta X - X\delta Y) + \left(\frac{1}{V}d_z\rho - \frac{\rho\delta Z}{V^3}\right)(Z\delta Y - Y\delta Z) + \frac{\rho}{V}\{\omega G + \bar{G}\},$$

$$H'' = \theta H + \left(\frac{1}{V}d_y\rho - \frac{\rho\delta Y}{V^3}\right)(Z\delta Y - Y\delta Z) + \left(\frac{1}{V}d_x\rho - \frac{\rho\delta X}{V^3}\right)(X\delta Z - Z\delta X) + \frac{\rho}{V}\{\omega H + \bar{H}\},$$

in which equations $\bar{A}$, $\bar{B}$, &c. are the like functions of $\bar{a}$, $\bar{b}$, &c. that A, B, &c. are of a, b, &c.; viz. $\bar{A} = 2\bar{h}Z - 2\bar{g}Y$, &c.

The value of θA is

$$\theta A = 2Z\left(\left\{-\frac{\rho}{V}(h\omega+\bar{h})+\frac{\rho}{V^3}\delta X\delta Y\right\}-\frac{1}{V}(\delta Y d_x\rho+\delta X d_y\rho)-Vd_xd_y\rho\right)$$

$$-2Y\left(\left\{-\frac{\rho}{V}(g\omega+\bar{g})+\frac{\rho}{V^3}\delta Z\delta X\right\}-\frac{1}{V}(\delta Z d_x\rho+\delta X d_z\rho)-Vd_xd_z\rho\right)$$

$$-2V(hd_z-gd_y)\rho,$$

which is

$$=-\frac{\rho}{V}(\omega A+\bar{A}) \quad +\frac{2\rho}{V^3}\delta X(Z\delta Y-Y\delta Z)$$

$$-\frac{2\delta X}{V}(Zd_y-Yd_z)\rho \quad -2V(hd_z-gd_y)\rho$$

$$-\frac{2}{V}(Z\delta Y-Y\delta Z)d_x\rho-2V(Zd_y-Yd_z)d_x\rho.$$

Hence the value of A'' is equal to the last-mentioned expression, together with the following terms:—

$$+\frac{\rho}{V}(\omega A+\bar{A})-\frac{2\rho}{V^3}\delta X(Z\delta Y-Y\delta Z)+\frac{2}{V}(Z\delta Y-Y\delta Z)d_x\rho,$$

which destroy certain of the foregoing ones; viz. we have

$$A''=\left(2Vg-\frac{2Z\delta X}{V}\right)d_y\rho-2\left(Vh-\frac{2Y\delta X}{V}\right)d_z\rho-2V(Zd_y-Yd_z)d_x\rho.$$

33. Similarly, the value of θF is

$$\theta F=Y\left(-\frac{\rho}{V}(h\omega+\bar{h})+\frac{\rho}{V^3}\delta X\delta Y-\frac{1}{V}(\delta Y d_x\rho+\delta X d_y\rho)-Vd_xd_y\rho\right)$$

$$-Z\left(-\frac{\rho}{V}(g\omega+\bar{g})+\frac{\rho}{V^3}\delta Z\delta X-\frac{1}{V}(\delta Z d_x\rho+\delta X d_z\rho)-Vd_xd_z\rho\right)$$

$$-X\left(-\frac{\rho}{V}\{(b-c)\omega+\bar{b}-\bar{c}\}+\frac{\rho\delta Y^2}{V^3}-\frac{\rho\delta Z^2}{V^3}-\frac{2}{V}\delta Y d_y\rho+\frac{2}{V}\delta Z d_z\rho-Vd_y{}^2\rho+Vd_z{}^2\rho\right)$$

$$-V\left(hd_y-gd_z-(b-c)d_x\right)\rho,$$

which is

$$=\frac{\rho}{V}(-F\omega-\bar{F})+\frac{\rho\delta Z}{V^3}(X\delta Z-Z\delta X)+\frac{\rho\delta Y}{V^3}(Y\delta X-X\delta Y)$$

$$+\left\{-\frac{1}{V}(Y\delta Y-Z\delta Z)+V(b-c)\right\}d_x\rho$$

$$+\left\{\frac{Y}{V}\delta X+\frac{2X}{V}\delta Y-Vh\right\}d_y\rho$$

$$+\left\{\frac{Z}{V}\delta X+\frac{2X}{V}\delta Y+Vg\right\}d_z\rho$$

$$+(-VYd_xd_y+VZd_xd_z+VXd_y{}^2-VXd_z{}^2)\rho.$$

Hence F'' is equal to the foregoing expression, together with the following terms:—

$$+\frac{\rho}{V}(F\omega+\bar{F})-\frac{\rho\delta Z}{V^3}(X\delta Z-Z\delta X)-\frac{\rho\delta Y}{V^3}(Y\delta X-X\delta Y)$$

$$+\frac{1}{V}(Y\delta X-X\delta Y)\,d_y\rho+\frac{1}{V}(X\delta Z-Z\delta X)\,d_z\rho,$$

which destroy certain of the foregoing terms; viz. we thus have

$$F''=\left\{-\frac{1}{V}(Y\delta Y-Z\delta Z)+V(b-c)\right\}d_x\rho+\left\{\frac{X}{V}\delta Y-Vh\right\}d_y\rho+\left\{-\frac{X}{V}\delta Z-Vg\right\}d_z\rho$$
$$+V(-Yd_xd_y+Zd_zd_x+Xd_y{}^2-Xd_z{}^2)\,\rho.$$

34. We thus have

$$A''=\qquad 2\left(Vg-\frac{Z\delta X}{V}\right)d_y\rho-2\left(Vh-\frac{Y\delta X}{V}\right)d_z\rho+2V(-Zd_yd_x+Yd_zd_x)\,\rho,$$

$$B''=-2\left(Vf-\frac{Z\delta Y}{V}\right)d_x\rho\qquad+2\left(Vh-\frac{X\delta Y}{V}\right)d_z\rho+2V(-Xd_zd_y+Zd_xd_y)\,\rho,$$

$$C''=+2\left(Vf-\frac{Y\delta Z}{V}\right)d_x\rho-2\left(Vg-\frac{X\delta Z}{V}\right)d_y\rho\qquad+2V(-Yd_xd_z+Xd_yd_z)\,\rho,$$

$$F''=\left\{V(b-c)-\frac{1}{V}(Y\delta Y-Z\delta Z)\right\}d_x\rho-\left(Vh-\frac{X\delta Y}{V}\right)d_y\rho+\left(Vg-\frac{X\delta Z}{V}\right)d_z\rho$$
$$+V(-Yd_xd_y+Zd_xd_z+Xd_y{}^2-Xd_z{}^2)\,\rho,$$

$$G''=\left(Vh-\frac{Y\delta X}{V}\right)d_x\rho+\left\{V(c-a)-\frac{1}{V}(Z\delta Z-X\delta X)\right\}d_y\rho-\left(Vf-\frac{Y\delta Z}{V}\right)d_z\rho$$
$$+V(-Zd_yd_z+Xd_yd_x+Yd_z{}^2-Yd_x{}^2)\,\rho,$$

$$H''=-\left(Vg-\frac{Z\delta X}{V}\right)d_x\rho-\left(Vf-\frac{Z\delta Y}{V}\right)d_y\rho+\left\{V(a-b)-\frac{1}{V}(X\delta X-Y\delta Y)\right\}d_z\rho$$
$$+V(-Xd_zd_x+Yd_zd_y+Zd_x{}^2-Zd_y{}^2)\,\rho.$$

35. It will be recollected that we have $X'\xi'+Y'\eta'+Z'\zeta'=X\xi+Y\eta+Z\zeta$; by what precedes it appears that for the given surface the principal tangents are determined by the equations

$$(A,\ldots\between\xi,\ \eta,\ \zeta)^2=0,$$
$$X\xi+Y\eta+Z\zeta=0,$$

and that the lines which (in the tangent plane of the given surface) correspond to the principal tangents of the corresponding point of the vicinal surface are determined by the equations

$$(A,\ldots\between\xi,\ \eta,\ \zeta)^2+(A'',\ldots\between\xi,\ \eta,\ \zeta)^2=0,$$
$$X\xi+Y\eta+Z\zeta=0.$$

Condition that the two surfaces may belong to an Orthogonal System.
Art. Nos. 36 to 41.

36. The condition in order that the two surfaces may belong to an orthogonal system is that the principal tangents shall correspond, or, what is the same thing, the lines which (in the tangent plane of the given surface) correspond to the principal tangents of the vicinal surface must be the principal tangents of the given surface. When this is the case, the plane and cone $X\xi + Y\eta + Z\zeta = 0$, $(A'', ..\between \xi, \eta, \zeta)^2 = 0$ intersect in the principal tangents, and this is therefore the required condition.

The plane $X\xi + Y\eta + Z\zeta = 0$ meets the cone $(A'', ..\between \xi, \eta, \zeta)^2 = 0$ in the principal tangents, that is, in a pair of lines harmonically related to the circular lines and also to the chief tangents. Forming then the coefficients (A''), (B''), (C''), (F''), (G''), (H'') from A'', &c. in the same way as (A) &c. are formed from A, &c., that is, writing $(A'') = B''Z^2 + C''Y^2 - 2F''YZ$, &c., the conditions are

$$(A'') + (B'') + (C'') = 0,$$

$$\big((A''), ..\between a, ...\big) = 0,$$

or, what is the same thing,

$$(A'', ...\between (a), ...) = 0.$$

The former of these, as about to be shown, is satisfied identically; we have therefore the second of them, say $(A'', ..\between (a), ..) = 0$ as the required condition.

37. We have

$$(A'') + (B'') + (C'') = (A'' + B'' + C'')\, V^2 - (A'', ..\between X, Y, Z)^2,$$

$$A'' + B'' + C'' = \frac{2}{V}\{(Z\delta Y - Y\delta Z)\, d_x\rho + (X\delta Z - Z\delta X)\, d_y\rho + (Y\delta X - X\delta Y)\, d_z\rho\}.$$

Forming next the expressions of $A''X + H''Y + G''Z$ &c., and, for convenience, writing down separately the terms which involve the second differential coefficients of ρ, we have

$A''X + H''Y + G''Z =$

$$d_x\rho \,.\, V(hZ - gY) + d_y\rho\,[V\delta Z - Z\delta V + V(gX - aZ)] + d_z\rho\,[-(V\delta Y - Y\delta V) - V(hX - aY)],$$

$H''X + B''Y + F''Z =$

$$d_x\rho\,[-(V\delta Z - Z\delta V) - V(fY - bZ)] + d_y\rho \,.\, V(fX - hZ) + d_z\rho\,[(V\delta X - X\delta V) + V(hY - bX)],$$

$G''X + F''Y + C''Z =$

$$d_x\rho\,[V\delta Y - Y\delta V + V(fZ - cY)] + d_y\rho\,[-(V\delta X - X\delta V) - V(gZ - cX)] + d_z\rho \,.\, V(gY - fX),$$

where δV stands for $\frac{1}{V}(X\delta X + Y\delta Y + Z\delta Z)$, and where the three expressions contain also the following terms respectively:

$$\{\quad . \quad - YZd_y^2 + YZd_z^2 + (Y^2 - Z^2)\, d_y d_z + \quad XYd_z d_x - \quad XZd_x d_y\}\, \rho,$$

$$\{ZXd_x^2 \quad . \quad - ZXd_z^2 - \quad XYd_y d_z + (Z^2 - X^2)\, d_z d_x + \quad YZd_x d_y\}\, \rho,$$

$$\{-XYd_x^2 + XYd_y^2 \quad . \quad + \quad XZd_y d_z - \quad YZd_z d_x + (X^2 - Y^2)\, d_x d_y\}\, \rho.$$

Multiplying by X, Y, Z, and adding, the terms which contain the second differential coefficients disappear, and we obtain

$$(A'', \ldots \between X, Y, Z)^2 = 2V[(Z\delta Y - Y\delta Z)\, d_x\rho + (X\delta Z - Z\delta X)\, d_y\rho + (Y\delta X - X\delta Y)\, d_z\rho];$$

so that, attending to the above value of $A'' + B'' + C''$, we have the required equation

$$(A'') + (B'') + (C'') = 0.$$

38. Proceeding now to form the value of $(A'', \ldots \between (a), \ldots)$, that is,

$$A''(a) + B''(b) + C''(c) + 2F''(f) + 2G''(g) + 2H''(h),$$

it will be shown that the terms involving the first differential coefficients of ρ vanish of themselves; as regards those containing the second differential coefficients, forming the auxiliary equations

$$\begin{aligned}
(A) &= 2(h)Z - 2(g)Y, \\
(B) &= 2(f)X - 2(h)Z, \\
(C) &= 2(g)Y - 2(f)X, \\
(F) &= (h)Y - (g)Z - ((b) - (c))X, \\
(G) &= (f)Z - (h)X - ((c) - (a))Y, \\
(H) &= (g)X - (f)Y - ((a) - (b))Z,
\end{aligned}$$

we find without difficulty that the terms in question (being, in fact, the complete value of the expression) are

$$= V((A), \ldots \between d_x, d_y, d_z)^2 \rho.$$

39. As regards the terms involving the first differential coefficients, observe that the whole coefficient of $d_x\rho$ is

$$\begin{aligned}
&-2(b)\left(Vf - \frac{Z\delta Y}{V}\right) \\
&+2(c)\left(Vg - \frac{Y\delta Z}{V}\right) \\
&+2(f)\left(V(b-c) - \frac{1}{V}(Y\delta Y - Z\delta Z)\right) \\
&+2(g)\left(Vh - \frac{Y\delta X}{V}\right) \\
&-2(h)\left(Vg - \frac{Z\delta X}{V}\right),
\end{aligned}$$

which is

$$\begin{aligned}
&= 2V\{(g)h + (f)b + (c)g - ((h)g + (b)f + (f)c)\} \\
&+ \frac{2}{V}\{Z((h)\delta X + (b)\delta Y + (f)\delta Z) - Y((g)\delta X + (f)\delta Y + (c)\delta Z)\}.
\end{aligned}$$

40. The reduction depends on the following auxiliary formulæ:

$$\begin{array}{l|l|l} a(a)+h(h)+g(g)=V\bar\delta V-X\bar\delta X, & a(h)+h(b)+g(f)=\quad -X\bar\delta Y, & a(g)+h(f)+g(c)=\quad -X\bar\delta Z, \\ h\ ,,\ +b\ ,,\ +f\ ,,\ =\quad -Y\bar\delta X, & h\ ,,\ +b\ ,,\ +f\ ,,\ =V\bar\delta V-Y\bar\delta Y, & h\ ,,\ +b\ ,,\ +f\ ,,\ =\quad -Y\bar\delta Z, \\ g\ ,,\ +f\ ,,\ +c\ ,,\ =\quad -Z\bar\delta X, & g\ ,,\ +f\ ,,\ +c\ ,,\ =\quad -Z\bar\delta Y, & g\ ,,\ +f\ ,,\ +c\ ,,\ =V\bar\delta V-Z\bar\delta Z, \end{array}$$

where, for shortness, I have written $\bar\delta X$, $\bar\delta Y$, $\bar\delta Z$ to stand for $\bar a X+\bar h Y+\bar g Z$, $\bar h X+\bar h Y+\bar f Z$, $\bar g X+\bar f Y+\bar c Z$ respectively, and $V\bar\delta V$ for $X\bar\delta X+Y\bar\delta Y+Z\bar\delta Z$, $(=\bar a,\ldots\between X,\ Y,\ Z)^2$.

From these we immediately have

$$(a)\,\delta X+(h)\ \delta Y+(g)\ \delta Z=V(X\bar\delta V-V\bar\delta X),$$
$$(h)\,\delta X+(b)\ \delta Y+(f)\,\delta Z=V(Y\bar\delta V-V\bar\delta Y),$$
$$(g)\,\delta X+(f)\,\delta Y+(c)\ \delta Z=V(Z\bar\delta V-V\bar\delta Z).$$

Hence, in the coefficient of $d_x\rho$, the first line is

$$=2V(-Y\bar\delta Z+Z\bar\delta Y),$$

and the second line is

$$=\frac{2}{V}\{VZ(Y\bar\delta V-V\bar\delta Y)-VY(Z\bar\delta V-V\bar\delta Z)\},\ =2V(Y\bar\delta Z-Z\bar\delta Y);$$

so that the sum, or whole coefficient of $d_x\rho$, is $=0$. Similarly, the coefficients of $d_y\rho$ and $d_z\rho$ are each $=0$.

41. We have thus arrived at the equation

$$((A),\ldots\between d_x,\ d_y,\ d_z)^2\,\rho=0$$

as the condition to be satisfied by the normal distance ρ in order that the given surface and the vicinal surface may belong to an orthogonal system, viz. this is a partial differential equation of the second order, its coefficients being given functions of X, Y, Z, a, b, c, f, g, h, the first and second differential coefficients of r (where $r=r(x,\ y,\ z)$ is the equation of the given surface).

The equation, it is clear, may also be written in the two forms

$$(A,\ldots\between Zd_y-Yd_z,\ Xd_z-Zd_x,\ Yd_x-Xd_y)^2\,\rho=0,$$

and

$$\left|\begin{array}{ccc} P\ , & Q\ , & R \\ aP+hQ+gR, & hP+bQ+fR, & gP+fQ+cR \\ X\ , & Y\ , & Z \end{array}\right|\rho=0,$$

if, for shortness, P, Q, R are written to denote Zd_y-Yd_z, Xd_z-Zd_x, Yd_x-Xd_y respectively, it being understood that in each of these forms the d_x, d_y, d_z operate on the ρ only.

Condition that a family of surfaces may belong to an Orthogonal System.

Art. Nos. 42 to 49.

42. We pass at once to the condition in order that the family of surfaces

$$r - r(x, y, z) = 0$$

may belong to an orthogonal system, viz. when the vicinal surface belongs to the family, we have ρ proportional to $\frac{1}{V}\left(=\frac{1}{\sqrt{X^2+Y^2+Z^2}}\right)$, and the condition is

$$((A), \ldots \between d_x,\ d_y,\ d_z)^2 \frac{1}{V} = 0,$$

where r is a function of (x, y, z), the first and second differential coefficients of which are $X, Y, Z, a, b, c, f, g, h$; and the equation is thus a partial differential equation of the third order satisfied by r. The form is by no means an inconvenient one, but it admits of further reduction.

43. We have $d_x \frac{1}{V}$, $d_y \frac{1}{V}$, $d_z \frac{1}{V}$ equal to $-\frac{1}{V^3}\delta X$, $-\frac{1}{V^3}\delta Y$, $-\frac{1}{V^3}\delta Z$ respectively, and thence

$$d_x{}^2 \frac{1}{V} = -\frac{1}{V^3}(a^2+h^2+g^2+\delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_z \frac{1}{V} = -\frac{1}{V^3}(gh+bf+cf+\delta f) + \frac{3}{V^5}\delta Y \delta Z,$$

or, as these may be written,

$$d_x{}^2 \frac{1}{V} = -\frac{1}{V^3}(a\omega - \bar{\omega} + \bar{a} + \delta a) + \frac{3}{V^5}(\delta X)^2,$$

$$d_y d_z \frac{1}{V} = -\frac{1}{V^3}(f\omega \qquad + \bar{f} + \delta f) + \frac{3}{V^5}\delta Y \delta Z,$$

with the like values for $d_y{}^2 \frac{1}{V}$, &c. Substituting, the equation contains a term multiplied by ω, viz. this is

$$-\frac{1}{V^3}\omega\left((A), \ldots \between a, \ldots\right),$$

which vanishes; and a term multiplied by $\bar{\omega}$, viz. this is

$$\frac{1}{V^3}\bar{\omega}\left((A)+(B)+(C)\right),$$

which also vanishes. Writing down the remaining terms, and multiplying the whole by $-V^3$, the equation becomes

$$\left((A), \ldots \between \bar{a}, \ldots\right) + \left((A), \ldots \between \delta a, \ldots\right) - \frac{3}{V^2}\left((A), \ldots \between \delta X,\ \delta Y,\ \delta Z\right)^2 = 0.$$

44. The last term admits of reduction; from the equations

$$(A) = -AV^2 + 2XZ\delta Y - 2XY\delta Z, \text{ \&c., we find}$$

$$(A)\,\delta X + (H)\,\delta Y + (G)\,\delta Z = -V^2(A\delta X + H\delta Y + G\delta Z) + V\delta V(Z\delta Y - Y\delta Z),$$

$$(H)\,\delta X + (B)\,\delta Y + (F)\,\delta Z = -V^2(H\delta X + B\delta Y + F\delta Z) + V\delta V(X\delta Z - Z\delta X),$$

$$(G)\,\delta X + (F)\,\delta Y + (C)\,\delta Z = -V^2(G\delta X + F\delta Y + C\delta Z) + V\delta V(Y\delta X - X\delta Y),$$

and hence

$$((A), ..\between \delta X, \delta Y, \delta Z)^2 = -V^2(A, ...\between \delta X, \delta Y, \delta Z)^2;$$

wherefore the equation becomes

$$((A), ..\between \bar{a}, ..) + ((A), ..\between \delta a ..) + 3(A, ..\between \delta X, \delta Y, \delta Z)^{2\prime} = 0.$$

45. It will be shown that we have identically

$$((A), ...\between \bar{a}, ...) = -(A, ..\between \delta X, \delta Y, \delta Z)^2 = 2\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix}.$$

The partial differential equation thus assumes the form

$$((A), ..\between \delta a, ...) + \Omega = 0,$$

where Ω may be expressed indifferently in the three forms,

$$= +2(A, ..\between \bar{a}, ..),$$

$$= +2(A, ..\between \delta X, \delta Y, \delta Z)^2,$$

$$= -4\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta} X, & \bar{\delta} Y, & \bar{\delta} Z \end{vmatrix}.$$

46. Taking the first of these, the partial differential equation is

$$((A), ...\between \delta a, ..) - 2((A), ..\between \bar{a}, ...) = 0;$$

or, written at full length, it is

$$(A)\,\delta a + (B)\,\delta b + (C)\,\delta c + 2(F)\,\delta f + 2(G)\,\delta g + 2(H)\,\delta h$$

$$-2\{(A)\;\bar{a} + (B)\;\bar{b} + (C)\;\bar{c} + 2(F)\;\bar{f} + 2(G)\;\bar{g} + 2(H)\,\bar{h}\} = 0,$$

where the coefficients are given functions of $X, Y, Z, a, b, c, f, g, h$, the first and second differential coefficients of r; and δ is written to denote $Xd_x + Yd_y + Zd_z$.

47. It remains to prove the above-mentioned identities.

To reduce the term $(A,..\between\delta X, \delta Y, \delta Z)^2$, we have

$$\begin{aligned}
&A\delta X + H\delta Y + G\delta Z \\
&\quad = A(aX+hY+gZ) + H(hX+bY+fZ) + G(gX+fY+cZ) \\
&\quad = X\{\ \ \omega(hZ-gY) + \bar{h}Z - \bar{g}Y\} \\
&\quad\quad + Y\{-\omega(fY-bZ) - (\bar{f}Y-\bar{b}Z) - \omega Z - \bar{\delta}Z\} \\
&\quad\quad + Z\{\ \ \omega(fZ-cY) + fZ - \bar{c}Y + \bar{\omega}Y + \bar{\delta}Z\} \\
&\quad = \ \omega(Z\delta Y - Y\delta Z) + (Z\bar{\delta}Y - Y\bar{\delta}Z) + (Z\bar{\delta}Y - Y\bar{\delta}Z),
\end{aligned}$$

that is,

$$A\delta X + H\delta Y + G\delta Z = \omega(Z\delta Y - Y\delta Z) + 2(Z\bar{\delta}Y - Y\bar{\delta}Z);$$

and similarly

$$H\delta X + B\delta Y + F\delta Z = \omega(X\delta Z - Z\delta X) + 2(X\bar{\delta}Z - Z\bar{\delta}X),$$

$$G\delta X + F\delta Y + C\delta Z = \omega(Y\delta X - X\delta Y) + 2(Y\bar{\delta}X - X\bar{\delta}Y),$$

whence

$$(A,..\between\delta X, \delta Y, \delta Z)^2 = -2\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta}X, & \bar{\delta}Y, & \bar{\delta}Z \end{vmatrix}.$$

48. Now, from the equations $AX + HY + GZ = Z\delta Y - Y\delta Z$, &c. we have for the value of twice the foregoing determinant

$$\begin{aligned}
2\text{ det.} = 2\{&(\bar{a}X + \bar{h}Y + \bar{g}Z)(AX + HY + GZ) \\
+ &(\bar{h}X + bY + \bar{f}Z)(HX + BY + FZ) \\
+ &(\bar{g}X + \bar{f}Y + \bar{c}Z)(GX + FY + CZ)\};
\end{aligned}$$

and subtracting herefrom the function $((A), ..\between\bar{a}, ..)$, which is

$$\begin{aligned}
= \quad & (BZ^2 + CY^2 - 2FYZ)\bar{a} \\
& + (CX^2 + AZ^2 - 2GZX)\bar{b} \\
& + (AY^2 + BX^2 - 2HYZ)\bar{c} \\
& + 2(-AYZ - FX^2 + GXY + HXZ)\bar{f} \\
& + 2(-BZX + FXY - GY^2 + HYZ)\bar{g} \\
& + 2(-CXY + FXZ + GYZ - HZ^2\ )\bar{h},
\end{aligned}$$

the difference is found to be

$$\begin{aligned}
= \quad & \bar{a}\{(A,..\between X,\ Y,\ Z)^2 + AV^2\} \\
+ \ & \bar{b}\{(A,..\between X,\ Y,\ Z)^2 + BV^2\} \\
+ \ & \bar{c}\{(A,..\between X,\ Y,\ Z)^2 + CV^2\} \\
+ \ & 2\bar{f}\{(A+B+C)YZ + FV^2\} \\
+ \ & 2\bar{g}\{(A+B+C)ZX + GV^2\} \\
+ \ & 2\bar{h}\{(A+B+C)XY + HV^2\},
\end{aligned}$$

which, on account of $(A, ..\between X, Y, Z)^2 = 0$, and $A + B + C = 0$, reduces itself to

$$(A, ..\between \bar{a}, ...) . V^2.$$

49. We have

$$\begin{aligned} A\bar{a} + H\bar{h} + G\bar{g} = \ & \bar{a}\,(2hZ - 2gY) \\ & + \bar{h}\,(\ gX - \ fY - (a-b)\,Z) \\ & + \bar{g}\,(\ fZ - \ hX - (c-a)\,Y) \\ = \ & X\,(g\bar{h} - h\bar{g}) \\ & + Y\,(a\bar{g} - g\bar{a} - (g\bar{a} + f\bar{h} + c\bar{g})) \\ & + Z\,(h\bar{a} - a\bar{h} + (h\bar{a} + b\bar{h} + f\bar{g})) ; \end{aligned}$$

or, observing that in the coefficients of Y and Z the second terms each vanish, this is

$$A\bar{a} + H\bar{h} + G\bar{g} = X\,(\bar{h}g - \bar{g}h) + Y\,(\bar{g}a - \bar{a}g) + Z\,(\bar{a}h - \bar{h}a);$$

and similarly

$$\begin{aligned} H\bar{h} + B\bar{b} \ + F\bar{f} &= X\,(\bar{b}f - \bar{f}b) + Y\,(\bar{f}h - \bar{h}f) + Z\,(\bar{h}b - \bar{b}h), \\ G\bar{g} \ + H\bar{f} + C\bar{c} &= X\,(\bar{f}c - \bar{c}f) + Y\,(\bar{c}g - \bar{g}c) + Z\,(\bar{g}f - \bar{f}g). \end{aligned}$$

Adding these equations, the coefficient of X is the difference of two expressions each of which vanishes; and the like as regards the coefficients of Y and Z; that is, we have

$$(A, ..\between \bar{a}, ..) = 0;$$

and consequently

$$2\begin{vmatrix} \delta X, & \delta Y, & \delta Z \\ X, & Y, & Z \\ \bar{\delta X}, & \bar{\delta Y}, & \bar{\delta Z} \end{vmatrix} = ((A), ..\between \bar{a}, ...) = -(A, ...\between \delta X, \delta Y, \delta Z)^2,$$

the required relation.