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ON A CORRESPONDENCE OF POINTS IN RELATION TO TWO TETRAHEDRA.

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THE following question has been considered by R. Sturm in an interesting paper, "Das Problem der Projectivität und seine Anwendung auf die Flächen zweiten Grades," *Math. Ann.*, t. I. (1870), pp. 533—574: Given *in plano* two groups of the same number (5, 6, or 7) of points, to find points P, P' homographically related to these two groups respectively; viz. the lines from P to the points of the first group and those from P' to the points of the second group are to be homographic pencils. In the present paper I require only a particular form of these results; viz. in each group two of the points are the circular points at infinity; or, disregarding these, we have two groups of 3, 4, or 5 points such that the points of the first group at P, and those of the second group at P', subtend equal angles. I give for this particular case an independent analytical investigation; but I will first state the results included in the more general ones obtained by Sturm.

If the points A, B, C at P and the points A', B', C' at P' subtend equal angles, then to any given position of the one point corresponds a single position of the other point; viz. the two points have a (1, 1) correspondence; the nature of this being, that to any line in the one figure corresponds in the other figure a quintic curve, having 6 dps.; viz. the three points, the two circular points at infinity I, J, and one other fixed point of that figure {say for the first figure this fixed point is (ABC)}.

If the points A, B, C, D at P and the points A', B', C', D' at P' subtend equal angles, then the locus of each point is a cubic curve; viz. the locus of P passes through A, B, C, D, I, J and the four fixed points (ABC), (ABD), (ACD), (BCD); and the like for the locus of P'.

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Finally (although this is a theorem which I do not require), if the points A, B, C, D, E at P and the points A', B', C', D', E' at P' subtend equal angles, then there are three positions of each point.

The problem I propose to consider is: Given the tetrahedra $ABCD$ and $A'B'C'D'$, it is required in the planes ABC and $A'B'C'$ respectively to find the points P, P' such that A, B, C, D at P, and A', B', C', D' at P', subtend equal angles. I was led to this by the more general problem, which I do not at present discuss: Given the two tetrahedra, it is required to find the loci of the points P, P' such that A, B, C, D at P, and A', B', C', D' at P', subtend equal angles.

Here, drawing from D, D' the perpendiculars $DK, D'K'$ on the planes ABC and $A'B'C'$ respectively, we have A, B, C, K at P, and A', B', C', K' at P', subtending equal angles, and such that the distances PK and $P'K'$ are proportional to the heights of the tetrahedra (for the triangles PDK and $P'D'K'$ are obviously similar). The required points P, P' are each the intersection of two loci, viz.:

1. P is such that A, B, C, K at P, and A', B', C', K' at P', subtend equal angles; locus is a cubic through A, B, C, K, I, J, (ABC), (ABK), (ACK), (BCK).

2. P is such that A, B, K at P, and A', B', K' at P', subtend equal angles, and that PK and $P'K'$ are in a given ratio; locus is a certain octic curve Ω;

and the required positions of P are obtained as the intersections of the two loci.

I proceed to the analytical investigation.

Preliminary Formulæ.

1. Consider a triangle ABC, and let the position of a point P be determined by means of its coordinates x, y, z, which are equal to the perpendicular distances of P from the sides, each divided by the perpendicular distance of the opposite vertex (as usual, x, y, z are positive for a point within the triangle); or what is the same thing, $x, y, z = PBC, PCA, PAB$, divided each by ABC, whence identically $x + y + z = 1$.

Suppose for a moment the rectangular coordinates of A, B, C are (α_1, β_1), (α_2, β_2), (α_3, β_3) respectively; and that those of P are X, Y. Also let the sides BC, CA, AB be $= a, b, c$ respectively.

We have

$$\begin{aligned} X &= \alpha_1 x + \alpha_2 y + \alpha_3 z, \\ Y &= \beta_1 x + \beta_2 y + \beta_3 z, \\ 1 &= x + y + z; \end{aligned}$$

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and if we consider a second point P', the coordinates of which are x', y', z' and X', Y', we have the like relations between these quantities. Calling δ the distance of the two points P, P', we may write

$$\begin{aligned}\delta^2 = \;& \{\alpha_1(x-x') + \alpha_2(y-y') + \alpha_3(z-z')\}^2 \\ & + \{\beta_1(x-x') + \beta_2(y-y') + \beta_3(z-z')\}^2 \\ & - \{(\alpha_1^2+\beta_1^2)(x-x') + (\alpha_2^2+\beta_2^2)(y-y') + (\alpha_3^2+\beta_3^2)(z-z')\} \\ & \times \{(x-x') + \quad (y-y') + \quad (z-z')\},\end{aligned}$$

the last term being in fact $=0$; viz. this is

$$\begin{aligned}\delta^2 = & -\{(\alpha_2-\alpha_3)^2 + (\beta_2-\beta_3)^2\}(y-y')(z-z') \\ & -\{(\alpha_3-\alpha_1)^2 + (\beta_3-\beta_1)^2\}(z-z')(x-x') \\ & -\{(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\}(x-x')(y-y');\end{aligned}$$

or what is the same thing, the expression for the distance δ^2 of the two points P, P' is

$$\delta^2 = -a^2(y-y')(z-z') - b^2(z-z')(x-x') - c^2(x-x')(y-y'),$$

which expression may be modified by means of the identical equations

$$1 = x+y+z, \quad 1 = x'+y'+z';$$

viz. writing

$$yz'-y'z,\; zx'-z'x,\; xy'-x'y = \xi,\; \eta,\; \zeta,$$

we have

$$\begin{aligned}x-x' &= x(x'+y'+z') - x'(x+y+z) = \zeta - \eta, \\ y-y' &= \xi - \zeta, \\ z-z' &= \eta - \xi;\end{aligned}$$

and consequently

$$\begin{aligned}\delta^2 = \;& a^2(-\xi^2 - \eta\zeta + \zeta\xi + \xi\eta) \\ & + b^2(-\eta^2 + \eta\zeta - \zeta\xi + \xi\eta) \\ & + c^2(-\zeta^2 + \eta\zeta + \zeta\xi - \xi\eta).\end{aligned}$$

2. Treating x', y', z' as constants and x, y, z as current coordinates, the formula for δ^2 is of course the equation of a circle, centre x', y', z' and radius δ. It thus also appears that the general equation of a circle is

$$-a^2yz - b^2zx - c^2xy + (Lx+My+Nz)(x+y+z) = 0;$$

viz. writing $-a^2yz - b^2zx - c^2xy = U$, and $x+y+z = \Omega$, this is

$$U + (Lx+My+Nz)\,\Omega = 0,$$

where $U=0$ is the circle circumscribed about the triangle ABC, and $\Omega=0$ is the line infinity. Of course the general equation of a circle passing through the points

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(B, C) is $U + Lx\,\Omega = 0$, and similarly those of circles through (C, A) and through (A, B) are $U + My\,\Omega = 0$ and $U + Nz\,\Omega = 0$ respectively. But we require the interpretation of the coefficients L, M, N which enter into these equations.

3. Considering the triangle ABC, if through B, C we have a circle, this is by the side BC divided into two segments, and I consider that lying on the same side with A as the positive segment, and define the angle of the circle to be the angle in this positive segment. It is clear that if we have *within* the triangle a point P, and, through this point and (B, C), (C, A), (A, B) respectively, three circles, then if α, β, γ be the angles of these circles, we have $\alpha + \beta + \gamma = 2\pi$; and conversely, if the circles through (B, C), (C, A), (A, B) are such that their angles α, β, γ satisfy the relation $\alpha + \beta + \gamma = 2\pi$, then the three circles meet in a point. But it is further to be noticed, that if, producing the sides of the triangle so as to divide the plane into seven spaces, the triangle, three trilaterals, and three bilaterals, we take the point P within one of the bilaterals, we still have $\alpha + \beta + \gamma = 2\pi$; but taking it within one of the trilaterals, we have $\alpha + \beta + \gamma = \pi$. And the converse theorem is, that if the three circles (B, C), (C, A), (A, B) are such that $\alpha + \beta + \gamma = \pi$ or 2π, then the circles meet in a point; viz. if the sum is 2π, then this point lies in the triangle or one of the bilaterals; but if the sum is $= \pi$, then this point lies in a trilateral.

4. I seek for the equation of a circle through the points B, C, and containing the angle L. The equation in rectangular coordinates is easily seen to be

$$(X - \alpha_2)(X - \alpha_3) + (Y - \beta_2)(Y - \beta_3) - \cot L\,\{(\beta_2 - \beta_3)X - (\alpha_2 - \alpha_3)Y + \alpha_2\beta_3 - \alpha_3\beta_2\} = 0.$$

In fact this is the equation of a circle through (B, C); and taking for a moment the origin at B, and axis of X to coincide with BC, or writing $\alpha_2, \beta_2 = 0, 0$; $\alpha_3, \beta_3 = a, 0$, the equation is

$$X(X - a) + Y^2 - aY\cot L = 0,$$

viz. the equation of the tangent at B is $-aX - aY\cot L = 0$, that is, $Y = -X\tan L$, or the angle in the positive segment is $= L$.

If for a moment λ, μ, ν are the inclinations of the sides of the triangle ABC to the axis of X, then A, B, C being the angles, we may write

$$\begin{aligned} \mu - \nu &= \pi - A, \\ \nu - \lambda &= \pi - B, \\ \lambda - \mu &= -\pi - C, \end{aligned}$$

and

$$\begin{aligned} X - \alpha_2 &= (\alpha_1 x + \alpha_2 y + \alpha_3 z) - \alpha_2(x + y + z) = c\cos\nu\,.\,x - a\cos\lambda\,.\,z, \\ X - \alpha_3 &= (\alpha_1 x + \alpha_2 y + \alpha_3 z) - \alpha_3(x + y + z) = -b\cos\mu\,.\,x + a\cos\lambda\,.\,y, \\ Y - \beta_2 &= \beta_1 x + \beta_2 y + \beta_3 z - \beta_2(x + y + z) = c\sin\nu\,.\,x - a\sin\lambda\,.\,z, \\ Y - \beta_3 &= \beta_1 x + \beta_2 y + \beta_3 z - \beta_3(x + y + z) = -b\sin\nu\,.\,x + a\sin\lambda\,.\,z; \end{aligned}$$

whence

$$\begin{aligned} &(X - \alpha_2)(X - \alpha_3) + (Y - \beta_2)(Y - \beta_3) \\ &= -a^2yz - b^2zx - c^2xy + bc\cos A\,.\,x^2 + (b^2 - ab\cos C)\,zx + (c^2 - ac\cos B)\,xy; \end{aligned}$$

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viz. this is
$$= -a^2yz - b^2zx - c^2xy + bc\cos A \,.\, x(x+y+z).$$

Moreover, if $\Delta =$ twice the area of the triangle, then
$$(\beta_2 - \beta_3)X - (\alpha_2 - \alpha_3)Y + \alpha_2\beta_3 - \alpha_3\beta_2 = \Delta x(x+y+z) = bc\sin A \,.\, x(x+y+z);$$

so that the equation becomes
$$-a^2yz - b^2zx - c^2xy + bc\sin A(\cot A - \cot L)\,x(x+y+z) = 0,$$

or, what is the same thing,
$$-a^2yz - b^2zx - c^2xy + \Delta(\cot A - \cot L)\,x(x+y+z) = 0,$$

or, if we please,
$$-a^2yz - b^2zx - c^2xy + \Delta(\cot A - \cot L)\,x = 0.$$

Writing as before,
$$-a^2yz - b^2zx - c^2xy = U, \qquad x+y+z = \Omega,$$

the equation is
$$U + \Delta(\cot A - \cot L)\,\Omega x = 0;$$

or forming the like equations of two other similar circles, we have the circles (B, C), (C, A), (A, B) containing the angles L, M, N respectively; and the equations are
$$U + \Delta(\cot A - \cot L)\,\Omega x = 0,$$
$$U + \Delta(\cot B - \cot M)\,\Omega y = 0,$$
$$U + \Delta(\cot C - \cot N)\,\Omega z = 0.$$

Correspondence, A, B, C at P, and A', B', C' at P', subtending equal angles.

5. Consider now the two figures A', B', C', subtending at P' the same angles L, M, N which A, B, C subtend at P; then we have
$$\frac{U}{\Omega\Delta x} + \cot A - \cot L = 0, \quad \frac{U'}{\Omega'\Delta' x'} + \cot A' - \cot L = 0,$$
$$\frac{U}{\Omega\Delta y} + \cot B - \cot M = 0, \quad \frac{U'}{\Omega'\Delta' y'} + \cot B' - \cot M = 0,$$
$$\frac{U}{\Omega\Delta z} + \cot C - \cot N = 0, \quad \frac{U'}{\Omega'\Delta' z'} + \cot C' - \cot N = 0;$$

and thence
$$\frac{U}{\Omega\Delta x} + \cot A = \frac{U'}{\Omega'\Delta' x'} + \cot A',$$
$$\frac{U}{\Omega\Delta y} + \cot B = \frac{U'}{\Omega'\Delta' y'} + \cot B',$$
$$\frac{U}{\Omega\Delta z} + \cot C = \frac{U'}{\Omega'\Delta' z'} + \cot C';$$

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and consequently

$$\frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'} = \frac{U}{\Omega\Delta x} + \cot A - \cot A'$$

$$: \frac{U}{\Omega\Delta y} + \cot B - \cot B'$$

$$: \frac{U}{\Omega\Delta z} + \cot C - \cot C',$$

or, what is the same thing,

$$x' : y' : z' = \frac{x}{U + \Delta(\cot A - \cot A')\,\Omega x}$$

$$: \frac{y}{U + \Delta(\cot B - \cot B')\,\Omega y}$$

$$: \frac{z}{U + \Delta(\cot C - \cot C')\,\Omega z},$$

where observe that the equations

$$U + \Delta(\cot A - \cot A')\,\Omega x = 0,$$
$$U + \Delta(\cot B - \cot B')\,\Omega y = 0,$$
$$U + \Delta(\cot C - \cot C')\,\Omega z = 0,$$

represent circles (B, C), (C, A), (A, B) containing the angles A', B', C'; and since $A' + B' + C' = \pi$, these meet in a point O. We may for convenience write

$$x' : y' : z' = \frac{BC}{BCO} : \frac{CA}{CAO} : \frac{AB}{ABO},$$

where $BC = 0$ denotes $(x = 0)$ the line BC; $BCO = 0$ the circle through B, C, O. And of course, in like manner,

$$x : y : z = \frac{B'C'}{B'C'O'} : \frac{C'A'}{C'A'O'} : \frac{A'B'}{A'B'O'};$$

so that the points P, P' have a rational, or (1, 1), correspondence.

Writing

$$\begin{aligned} x' : y' : z' &= BC \,.\, CAO \,.\, ABO : CA \,.\, ABO \,.\, BCO : AB \,.\, BCO \,.\, CAO \\ &= \quad X \quad : \quad Y \quad : \quad Z \end{aligned}$$

suppose, X, Y, Z are quintic functions of x, y, z, and the curve in the first figure corresponding to the line $\alpha x' + \beta y' + \gamma z' = 0$ of the second figure is

$$\alpha X + \beta Y + \gamma Z = 0;$$

viz. this is a quintic curve having dps. at each of the points A, B, C, O, I, J. In fact, if for BCO we write $BCOIJ$, and so for the other two circles respectively, we have in an algorithm which will be at once understood $X = BC \,.\, CAOIJ \,.\, ABOIJ, = (ABCOIJ)^2$, and similarly $Y = Z, = (ABCOIJ)^2$, or the curve is $(ABCOIJ)^2 = 0$.

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Correspondence, A, B, C, D at P and A′, B′, C′, D′ at P′ subtending equal angles.

6. Consider now *in plano* the points A, B, C, D which at P, and the points A', B', C', D' which at P', subtend equal angles. Let a, b, c, f, g, h denote the perpendicular distances of P from the lines BC, CA, AB, AD, BD, CD respectively; and the like as to a', b', c', f', g', h'. Observe that, neglecting constant factors, a, b, c are what were before represented by x, y, z; we may consider the coordinates of P in regard to the triangles ABC, BCD, CAD, ABD to be (a, b, c), (a, h, g), (b, f, h), (c, g, f) respectively. We have in regard to ABC the point O as before, and in regard to BCD, CAD, ABD the points O_1, O_2, O_3 respectively. Then A, B, C at P and A', B', C' at P' subtending equal angles, we may write

$$a' : b' : c' = \frac{a}{BCO} : \frac{b}{CAO} : \frac{c}{ABO};$$

viz. $BCO = 0$ is here the circle through B, C, O, and the like for CAO and ABO, the expressions being multiplied into the proper constant factors to take account of the constant factors whereby a, b, c and a', b', c' differ from x, y, z and x', y', z' respectively.

We have in like manner

$$a' : h' : g' = \frac{a}{BCO_1} : \frac{h}{CDO_1} : \frac{g}{BDO_1},$$

$$b' : f' : h' = \frac{b}{CAO_2} : \frac{f}{ADO_2} : \frac{h}{CDO_2},$$

$$c' : g' : f' = \frac{c}{ABO_3} : \frac{g}{BDO_3} : \frac{f}{ADO_3}.$$

From the ratios of (f', g', h'), (b', c', f'), (c', a', g'), (a', b', h') respectively we deduce

$$CDO_1 . ADO_2 . BDO_3 - BDO_1 . CDO_2 . ADO_3 = 0,$$

$$CAO \; . ABO_3 . ADO_2 - ABO \; . ADO_3 . CAO_2 = 0,$$

$$ABO \; . BCO_1 . BDO_3 - BCO \; . BDO_1 . ABO_3 = 0,$$

$$BCO \; . CAO_2 . CDO_1 - CAO \; . CDO_2 . BCO_1 = 0,$$

each of which equations represents a sextic curve; and admitting that it can be shown that these pass through O, O_1, O_2, O_3 respectively, the forms are

$$D^3 A\, B\, C\, O\; O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A^3 B\, C\, O\; O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A\, B^3 C\, O\; O_1 O_2 O_3 I^3 J^3 = 0,$$

$$D\, A\, B\, C^3 O\; O_1 O_2 O_3 I^3 J^3 = 0.$$

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7. Now the locus of P is evidently a curve, and this can only happen by reason that the four left-hand functions contain a common factor, and the form of them suggests that this common factor is $ABCDOO_1O_2O_3IJ$, the four extraneous factors being $D^2I^2J^2$, $A^2I^2J^2$, $B^2I^2J^2$, $C^2I^2J^2$; viz. $ABCDOO_1O_2O_3IJ=0$ is a cubic curve passing through the ten points; and $D^2I^2J^2=0$ a cubic curve through each of the points D, I, J twice; viz. it is the triad of lines IJ, DI, DJ; and the like as to the other extraneous factors $A^2I^2J^2$, $B^2I^2J^2$, and $C^2I^2J^2$. I have not worked out the analysis to verify this *à posteriori*; but, the conclusion agreeing with Sturm, I accept it without further investigation, viz. the result is that A, B, C, D at P and A', B', C', D' at P' subtending equal angles, the locus of P is a cubic curve $ABCDOO_1O_2O_3O_4IJ=0$ through the ten points thus represented; and of course the locus of P' is in like manner a cubic curve $A'B'C'D'O'O_1'O_2'O_3'O_4'IJ=0$ through the ten points thus represented.

Correspondence, A, B, C, D, E at P and A', B', C', D', E' at P' subtending equal angles.

8. We may go a step further, and consider A, B, C, D, E at P and A', B', C', D', E' at P' subtending equal angles. Attending only to the points A, B, C, D and A', B', C', D', the locus of P is a cubic curve

$$ABCDOO_1O_2O_3O_4IJ=0;$$

and similarly attending to the points A, B, C, E and A', B', C', E', the locus of P is a cubic curve

$$ABCEOQ_1Q_2Q_3IJ=0.$$

(Observe that O, as depending only on A, B, C, is the same point as before; but that Q_1, Q_2, Q_3, as depending on E instead of D, are not the same as O_1, O_2, O_3.) The two cubic curves have in common the points A, B, C, I, J, O, and they consequently intersect in three other points; that is, there are three positions of the point P, and of course three corresponding positions of P'.

Correspondence, A, B, C at P and A', B', C' at P' subtending equal angles, and AP, $A'P'$ in a given ratio.

9. Consider, as before, A, B, C at P and A', B', C' at P' subtending equal angles, and the points P, P' being moreover such that the distances AP, $A'P'$ are in a given ratio. I write for shortness

$$x' : y' : z' = \frac{x}{L} : \frac{y}{M} : \frac{z}{N},$$

where L, M, N denote

$$U+\Delta(\cot A-\cot A')\,\Omega x,\quad U+\Delta(\cot B-\cot B')\,\Omega y,\quad U+\Delta(\cot C-\cot C')\,\Omega z,$$

respectively. We have

$$\begin{aligned}(AP)^2 &= -a^2yz-b^2z(x-1)-c^2(x-1)y,\\ &= -a^2yz-b^2zx-c^2xy+(b^2z+c^2y)(x+y+z),\\ &= \quad c^2y^2+b^2z^2+(b^2+c^2-a^2)yz;\end{aligned}$$

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or, what is the same thing,

$$(A\,P\,)^2 = c^2y^2 + b^2z^2 + 2bc \cos A \,.\, yz\,;$$

and similarly

$$(A'P')^2 = c'^2y'^2 + b'^2z'^2 + 2b'c' \cos A' \,.\, y'z'.$$

The required relation therefore is

$$\frac{c'^2y'^2 + b'^2z'^2 + 2b'c' \cos A' \,.\, y'z'}{(x'+y'+z')^2} = \theta^2 \,.\, \frac{c^2y^2 + b^2z^2 + 2bc \cos A \,.\, yz}{(x+y+z)^2}\,;$$

viz. substituting for x', y', z' their values, this is

$$L^2 (x+y+z)^2 (b'^2z^2M^2 + c'^2y^2N^2 + 2b'c'yzMN \cos A)$$
$$= \theta^2 (b^2z^2 + c^2y^2 + 2bcyz \cos A)(xMN + yNL + zLM)^2,$$

which is an equation of the 12th order. I say that the points A, B, C, O, I, J are each quadruple. In fact, according to the foregoing algorithm, we may write

$$x+y+z = IJ, \quad zM = AB \,.\, CAOIJ, \text{ \&c.},$$
$$xL = yM = zN = A^2BCOIJ,$$
$$y = z = A, \quad xMN = BC \,.\, CAOIJ \,.\, ABOIJ, \text{ \&c.},$$
$$xMN = yNL = zLM = (ABCOIJ)^2;$$

and the equation is

$$(BCOIJ)^2 (IJ)^2 (A^2BCOIJ)^2 = \theta^2 \,.\, A^2 (ABCOIJ)^4,$$

that is

$$(IJ)^2 (ABCOIJ)^4 = \theta^2 \,.\, A^2 (ABCOIJ)^4;$$

so that the points are each quadruple.

The two Tetrahedra; A, B, C, D at P in ABC and A', B', C', D' at P' in $A'B'C'$ subtending equal angles.

10. I consider now the before-mentioned problem of the two tetrahedra; viz. on the two bases ABC and $A'B'C'$ respectively, letting fall the perpendiculars DK and $D'K'$, then first A, B, C, K at P and A', B', C', K' at P' subtend equal angles; the locus of P is a cubic curve $ABCKOO_1O_2O_3IJ = 0$ through these ten points. ($O = ABC$ is derived from the points A, B, C; and in like manner $O_1 = BCK$, $O_2 = CAK$, $O_3 = ABK$.)

Next, B, C, K at P and B', C', K' at P' subtend equal angles, and moreover the distances KP and $K'P'$ are in a given ratio; the locus of P is a 12-thic curve

$$(BCKO_1IJ)^4 = 0,$$

having each of these six points as a quadruple point. Hence among the 36 intersections of the two curves we have the points B, C, K, O_1, I, J each 4 times, and there remain $36 - 24, = 12$ intersections.

The conclusion is that A, B, C, D at a point P of ABC, and A', B', C', D' at a point P' of $A'B'C'$, subtending equal angles, there are 12 positions of P, and of course 12 corresponding positions of P'.

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