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ON BICURSAL CURVES.

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A CURVE of deficiency 1 may be termed bicursal: there is some distinction according as the order is even or odd, and to fix the ideas I take it to be even.

A bicursal curve of the order n contains

$$\tfrac{1}{2}n(n+3)-\{\tfrac{1}{2}(n-1)(n-2)-1\},=3n \text{ constants};$$

hence, if the order is $=2n$, the number of constants is $=6n$; such a curve is normally represented by a system of equations

$$(x,\ y,\ z)=(1,\ \theta)^n+(1,\ \theta)^{n-2}\sqrt{\Theta},$$

where Θ is a quartic function, which may be taken to be of the form $(1-\theta^2)(1-k^2\theta^2)$, or otherwise to depend on a single constant; viz. (x, y, z) are proportional to n-thic functions involving such a radical: since in the values of (x, y, z) one constant divides out, the number of constants is $3\{(n+1)+(n-1)\}-1+1, =6n$, as it should be.

But the curve of the order $2n$ may be abnormally or improperly represented by a system of equations

$$(x,\ y,\ z)=(1,\ \theta)^{n+k}+(1,\ \theta)^{n+k-2}\sqrt{\Theta},$$

viz. these equations, instead of representing a curve of the order $2n+2k$, will represent a curve of the order $2n$, provided only there exist $2k$ common values of θ for which each of the three functions vanish. The passage to a normal representation is effected by finding θ' a function of θ, $\sqrt{\Theta}$ (viz. an irrational function of θ) such that the foregoing equations become

$$(x,\ y,\ z)=(1,\ \theta')^n+(1,\ \theta')^{n-2}\sqrt{\Theta'};$$

it is shown that such a transformation is possible, and a mode of effecting it, derived from a theorem of Hermite's in relation to the Jacobian H, Θ functions, is given in Clebsch's Memoir "Ueber diejenigen Curven, deren Coordinaten sich als elliptische Functionen eines Parameters darstellen lassen," *Crelle*, t. LXIV. (1865), pp. 210—270. The demonstration is a very interesting one, and I reproduce it at the end of this paper. I remark, in passing, that the analogous reduction in the case of unicursal curves is self-evident; the equations

$$(x,\ y,\ z) = (1,\ \theta)^{n+k}$$

will represent a curve, not of the order $n+k$, but of the order n, provided there exist k common values of θ for which the three functions vanish; in fact, the three functions have then a common factor of the order k, and omitting this, the form is $(x,\ y,\ z) = (1,\ \theta)^n$.

Returning to the curves of deficiency 1, we see that a curve of the order $2m+2n$ contains $6(m+n)$ constants, and is normally represented by a system of equations

$$(x,\ y,\ z) = (1,\ \theta)^{m+n} + (1,\ \theta)^{m+n-2}\sqrt{\Theta}.$$

Such a curve may be otherwise represented: we may derive it by a rational transformation from the curve $(D=1)$ of the order 4 (binodal quartic), the equation of which is

$$(1,\ u)^2(1,\ v)^2 = 0;$$

viz. the coordinates are here connected by a quadriquadric equation; and we then express x, y, z in terms of these by a system of equations

$$(x,\ y,\ z) = (1,\ u)^m(1,\ v)^n.$$

It is, however, to be observed that the form of these functions is not determinate: each of them may be altered by adding to it a term $\{(1,\ u)^{m-2}(1,\ v)^{n-2}\}\{(1,\ u)^2(1,\ v)^2\}$, where the second factor is that belonging to the quadriquadric transformation, and the first factor is arbitrary. Using the arbitrary function to simplify the form, the real number of constants is reduced to $(m+1)(n+1)-(m-1)(n-1),\ = 2(m+n)$; or the three functions contain together $6(m+n)$ constants, one of which divides out. The quadriquadric equation, dividing out one constant, contains eight constants; but reducing by linear transformations on u, v respectively, the number of constants is $8-6,\ =2$. Hence, in the system of equations, the whole number of constants is $6(m+n)-1+2$, $=6(m+n)+1$; viz. this is greater by unity than the number of constants in the curve $(D=1)$ of the order $2(m+n)$. The explanation of the excess is that the same curve of the order $2(m+n)$ may be derived from the different quartic curves $(1,\ u)^2(1,\ v)^2=0$; this will be further examined.

The transition from the one form to the other is not immediately obvious; in fact, if from the quadriquadric equation $(1,\ u)^2(1,\ v)^2=0$ (say this is $A+2Bv+Cv^2=0$, where A, B, C are quadric functions of u) we determine v; this gives $Cv=-B+\sqrt{B^2-AC}$,

$=-B+\sqrt{\Omega}$ suppose; and then substituting in the equations $(x, y, z)=(1, u)^m(1, v)^n$, we find

$$(x, y, z)=(1, u)^m(C, -B+\sqrt{\Omega})^n;$$

viz. we have (x, y, z) proportional to functions of u, involving the quartic radical $\sqrt{\Omega}$; but these functions are of the order (not $m+n$, but) $m+2n$.

In particular, if $n=m$, then, instead of functions of the order $2n$, we have functions of the order $3n$. The reduction in this last case to the form where the order is $2n$ can be effected without difficulty, but in the general case where m and n are unequal, I do not know how it is to be effected except by the general process explained in Clebsch's Memoir.

We may, in fact, by linear transformations on u, v, reduce the quadriquadric relation to

$$\begin{array}{ll} 1 & +u^2 \\ & +2buv \\ +v^2 & +cu^2v^2=0, \end{array}$$

that is

$$1+u^2+2buv+v^2+cu^2v^2=0;$$

or putting herein $u+v=p$, $uv=q$, the relation is

$$1+p^2-2q+2bq+cq^2=0,$$

that is

$$\begin{aligned} p^2 &= -1+(\ \ 2-2b)\,q-cq^2, \\ p^2-4q &= -1+(-2-2b)\,q-cq^2; \end{aligned}$$

viz. extracting the square roots,

$$u+v=\sqrt{Q},\quad u-v=\sqrt{Q'},$$

if for shortness

$$\begin{aligned} Q &= -1+(\ \ 2-2b)\,q-cq^2, \\ Q' &= -1+(-2-2b)\,q-cq^2; \end{aligned}$$

we may then rationalise one of the radicals, for instance, Q; viz. writing

$$-c\{-1+(2-2b)\,q-cq^2\}=\{cq-(1-b)\}^2+c-(1-b)^2,$$

then, if

$$cq-(1-b)=\sqrt{c-(1-b)^2}\,.\,\tfrac{1}{2}\left(\theta-\frac{1}{\theta}\right),$$

this becomes

$$\begin{aligned} -cQ &= \left\{c-(1-b)^2\right\}\left\{\tfrac{1}{4}\left(\theta-\frac{1}{\theta}\right)^2+1\right\} \\ &= \left\{c-(1-b)^2\right\}.\,\tfrac{1}{4}\left(\theta+\frac{1}{\theta}\right)^2, \end{aligned}$$

that is

$$\sqrt{Q} = \tfrac{1}{2}\sqrt{\left(\frac{c-(1-b)^2}{-c}\right)}\left(\theta + \frac{1}{\theta}\right);$$

and the corresponding value of $\sqrt{Q'}$ is

$$\sqrt{Q'} = \sqrt{\left\{\tfrac{1}{4}\,\frac{c-(1-b)^2}{-c}\left(\theta + \frac{1}{\theta}\right)^2 - 4q\right\}},$$

where q stands for its value

$$\frac{1}{c}\left\{1 - b + \sqrt{c-(1-b)^2}\,.\,\tfrac{1}{2}\left(\theta - \frac{1}{\theta}\right)\right\}.$$

We may write these in the form

$$1 : \tfrac{1}{2}\sqrt{Q} : \tfrac{1}{2}\sqrt{Q'} = M\theta : 1 + \theta^2 : \sqrt{\Theta},$$

where M is a constant, and Θ is a quartic function of θ, such that $(1+\theta^2)^2 - \Theta$ is a quadric function only of θ.

The equations

$$(x,\ y,\ z) = (1,\ u)^m\,(1,\ v)^n$$

thus assume the form

$$(x,\ y,\ z) = (M\theta,\ 1 + \theta^2 + \sqrt{\Theta})^m\,(M\theta,\ 1 + \theta^2 - \sqrt{\Theta})^n;$$

and on the right-hand side the term of the highest order in θ is

$$(1 + \theta^2 + \sqrt{\Theta})^m\,(1 + \theta^2 - \sqrt{\Theta})^n,$$

viz. if $n =$ or $> m$, then this is

$$\{(1+\theta^2)^2 - \Theta\}^m\,(1 + \theta^2 - \sqrt{\Theta})^{n-m}.$$

This is

$$= (1,\ \theta)^{2m}\,(1 + \theta^2 - \sqrt{\Theta})^{n-m},$$

which is of the order $2m + 2(n-m)$, $= 2n$ (which, in virtue of $n =$ or $> m$, is $=$ or $> m+n$). In particular, if $n = m$, then the highest order is $= 2n$; or the curve of the order $2n$, as represented by the equations

$$(x,\ y,\ z) = (1,\ u)^n\,(1,\ v)^n,$$

where $(u,\ v)$ are connected by a quadriquadric equation, is also represented by the equations

$$(x,\ y,\ z) = (1,\ \theta)^n + (1,\ \theta)^{n-2}\sqrt{\Theta},$$

which is the required transformation of the original equations.

It is to be noticed that the foregoing form,

$$\begin{array}{lll} 1 & & + u^2 \\ & + 2buv & \\ + v^2 & & + cu^2v^2 = 0, \end{array}$$

is the most special form to which the quadriquadric relation can be reduced by the linear transformations of u, v; in fact, by mere division, the equation is made to have the constant term 1; the number of the coefficients of transformation is then $3+3, =6$; and to reduce the relation to the foregoing form, we have the six conditions, coeff. $u=0$, coeff. $v=0$, coeff. $vu^2=0$, coeff. $v^2u=0$, coeff. $u^2=1$, coeff. $v^2=1$; but in this form, expressing v in terms of u, the radical is $\sqrt{\Omega}$, where

$$\Omega=(1+u^2)(1+cu^2)-b^2u^2,$$

which is not more general than if we had

$$\Omega=(1+u^2)(1+cu^2),$$

viz. there is a superfluous constant b. And we thus see how it is that the system of equations

$$(x,\ y,\ z)=(1,\ u)^n(1,\ v)^n,$$

(u, v) connected by a quadriquadric relation, contains, not $6n$, but $6n+1$ constants, one of these being superfluous.

Clebsch's transformation, above referred to, is as follows:

Starting with the equations

$$(x,\ y,\ z)=(1,\ \theta)^{n+k}+(1,\ \theta)^{n+k-2}\sqrt{\Theta},$$

if the function Θ is not originally of the standard form, we may, by a linear substitution, reduce it to this form, viz. we may write

$$\Theta=\theta(1-\theta)(1-k^2\theta);$$

and then writing $\theta=\operatorname{sn}^2u$, $(\sin^2\operatorname{am}u)$, we have

$$\begin{aligned}\sqrt{\Theta}&=\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\\&=\operatorname{sn}u\operatorname{sn}'u;\end{aligned}$$

so that the formulæ become

$$(x,\ y,\ z)=(1,\ \operatorname{sn}^2u)^{n+k}+(1,\ \operatorname{sn}^2u)^{n+k-2}\operatorname{sn}u\operatorname{sn}'u.$$

Hermite's theorem, used in the demonstration, is that any such function of $\operatorname{sn}u$ is expressible in the form

$$C\frac{H(u-\alpha_1)H(u-\alpha_2)\dots H(u-\alpha_{2n+2k})}{\Theta^{2n+2k}(u)}$$

(H, Θ denoting here the two Jacobian functions), where

$$\alpha_1+\alpha_2\dots+\alpha_{2n+2k}=0.$$

{Observe, in passing, that the equation

$$0=(1,\ \operatorname{sn}^2u)^{n+k}+(1,\ \operatorname{sn}^2u)^{n+k-2}\operatorname{sn}u\operatorname{sn}'u$$

gives, when rationalised, an equation in $\mathrm{sn}^2 u$ of the order $2n+2k$; the roots of this equation are $\mathrm{sn}^2 \alpha_1$, $\mathrm{sn}^2 \alpha_2 \ldots \mathrm{sn}^2 \alpha_{2n+2k}$. Considering the functions $(1, \mathrm{sn}^2 u)^{n+k}$ and $(1, \mathrm{sn}^2 u)^{n+k-2}$ as indeterminate, the coefficients can be found so that all but one of the roots of the equation in $\mathrm{sn}^2 u$ shall have any given values whatever, $\mathrm{sn}^2 \alpha_1$, $\mathrm{sn}^2 \alpha_2, \ldots \mathrm{sn}^2 \alpha_{2n+2k-1}$; the theorem then shows that the remaining root is $\mathrm{sn}^2 \alpha_{2n+2k}$, where

$$-\alpha_{2n+2k} = \alpha_1 + \alpha_2 \ldots + \alpha_{2n+2k-1},$$

which is, in fact, Abel's theorem.}

Now, supposing that the three functions of θ all vanish for $2k$ common values of θ, each of the functions of u will contain the same $2k$ H functions, say these are $H(u-\alpha_{2n+1}) \ldots H(u-\alpha_{2n+2k})$. Omitting these and also the denominator factor $\Theta^{2k}(u)$, we have the set of equations

$$(x,\ y,\ z) = C\frac{H(u-\alpha_1)\,H(u-\alpha_2)\ldots H(u-\alpha_{2n})}{\Theta^{2n}(u)},$$

where, however,

$$\alpha_1 + \alpha_2 \ldots + \alpha_{2n} \neq 0,$$

{the values α_1, $\alpha_2 \ldots \alpha_{2n}$ are of course different for the three coordinates x, a, z respectively}; viz. we have

$$\begin{aligned}\alpha_1 + \alpha_2 \ldots + \alpha_{2n} &= -(\alpha_{2n+1} + \ldots + \alpha_{2n+2k}) \\ &= 2ns \text{ suppose.}\end{aligned}$$

Writing then

$$u = u' + s,\quad \alpha_1 = \alpha_1' + s,\quad \alpha_2 = \alpha_2' + s, \ldots \alpha'_{2n} = \alpha'_{2n} + s,$$

and consequently

$$\alpha_1' + \alpha_2' + \ldots + \alpha'_{2n} = 0,$$

we have

$$(x,\ y,\ z) = C\frac{H(u'-\alpha_1')\ldots H(u'-\alpha'_{2n})}{\Theta^{2n}(u'+s)};$$

or, changing the common denominator,

$$(x,\ y,\ z) = C\frac{H(u'-\alpha_1')\ldots H(u'-\alpha'_{2n})}{\Theta^{2n}(u')},$$

where

$$\alpha_1' + \alpha_2' \ldots + \alpha'_{2n} = 0;$$

or, what is the same thing,

$$(x,\ y,\ z) = (1,\ \mathrm{sn}^2 u')^n + (1,\ \mathrm{sn}^2 u')^{n-2}\,\mathrm{sn}\,u'\ \mathrm{sn}'.\,u';$$

viz. writing $\mathrm{sn}\,u' = \theta'$, and $\Theta' = \theta'(1-\theta')(1-k^2\theta')$, this is

$$(x,\ y,\ z) = (1,\ \theta')^n + (1,\ \theta')^{n-2}\sqrt{\Theta'},$$

a normal representation of the curve of the order $2n$.

The relation between the parameters θ, θ' is given by $\theta' = \operatorname{sn}^2(u-s)$, $\operatorname{sn}^2 u = \theta$, that is, we have

$$\sqrt{\theta'} = \frac{\sqrt{\theta}\operatorname{cn} s \operatorname{dn} s - \operatorname{sn} s \sqrt{(1-\theta)(1-k^2\theta)}}{1 - k^2 \operatorname{sn}^2 s \,.\, \theta};$$

or, writing $\operatorname{sn}^2 s = \sigma$, this is

$$\theta' = \frac{\sqrt{\theta}\sqrt{(1-\sigma)(1-k^2\sigma)} - \sqrt{\sigma}\sqrt{(1-\theta)(1-k^2\theta)}}{1-k^2\sigma\theta};$$

and, conversely, we have

$$\theta = \frac{\sqrt{\theta'}\sqrt{(1-\sigma)(1-k^2\sigma)} + \sqrt{\sigma}\sqrt{(1-\theta')(1-k^2\theta')}}{1-k^2\sigma\theta'},$$

and the theorem, in fact, shows that, substituting this value of θ in the functions of θ which serve to express x, y, z, these become *proportional* to the functions of θ'; viz. they become equal to these functions, each multiplied by an irrational function $A' + B'\sqrt{\Theta'}$, (A', B' rational functions of θ').