# 503.

## ON THE SURFACES EACH THE LOCUS OF THE VERTEX OF A CONE WHICH PASSES THROUGH $m$ GIVEN POINTS AND TOUCHES $6-m$ GIVEN LINES.

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I CONSIDER the surfaces, each of them the locus of the vertex of a (quadri-)cone which passes through $m$ given points and touches $6-m$ given lines; viz. calling the given points $a, b, c, \ldots$ and the given lines $\alpha, \beta, \gamma, \ldots$, the surfaces in question are:

| | Order |
|---|---|
| $abcdef$ | 4 |
| $abcde\alpha$ | 8 |
| $abcd\alpha\beta$ | 16 |
| $abc\alpha\beta\gamma$ | 24 |
| $ab\alpha\beta\gamma\delta$ | 24 |
| $a\alpha\beta\gamma\delta\epsilon$ | 14 |
| $\alpha\beta\gamma\delta\epsilon\zeta$ | 8 |

I remark that the orders of these several surfaces are in effect determined by the investigations of M. Chasles in regard to the conics in space which satisfy seven conditions. The surface *abcdef* was long ago considered by M. Chasles, and it is treated of in my "Memoir on Quartic Surfaces," [445], and in the same Memoir the surface $\alpha\beta\gamma\delta\epsilon\zeta$ is also referred to: these two surfaces, and also the surfaces $a\alpha\beta\gamma\delta\epsilon$ and $ab\alpha\beta\gamma\delta$ are considered by Dr Hierholzer [1] in his excellent paper "Ueber Kegelschnitte im Raume," *Math. Annalen*, t. II. (1870), pp. 563—586, and to him are due the equations given in the sequel for the surfaces *abcdef* and $\alpha\beta\gamma\delta\epsilon\zeta$: the researches of the present Memoir are in fact a continuation and development of those in the Memoir last referred to.

[1] I was grieved to hear of Dr Hierholzer's death last autumn, at Carlsruhe, at the early age of 30.

*Table of Singularities, and Explanations in regard thereto.*

1. We have on the before-mentioned surfaces respectively certain simple or multiple points, right lines, and curves, as shown in the following Table:

| | | $abcdef$, 4 | $abcdea$, 8 | $abcda\beta$, 16 | $abca\beta\gamma$, 24 | $aba\beta\gamma\delta$, 24 | $a\alpha\beta\gamma\delta\epsilon$, 14 | $\alpha\beta\gamma\delta\epsilon\zeta$, 8 | |
|---|---|---|---|---|---|---|---|---|---|
| Points | $a$ | $6 \times (2)$ | $5 \times (4)$ | $4 \times (8)$ | $3 \times (8)$ | $2 \times (4)$ | $1 \times (2)$ | 0 | (0) |
| Lines | $ab$ | $15 \times (1)$, $C$ | $10 \times (2)$, $C$ | $6 \times (4)$, $C$ | $3 \times (4)$, $C$ | $1 \times (2)$, $C$ | 0 | 0 | (1) |
| | $\alpha$ | 0 | $1 \times (2)$, $C$ | $2 \times (4)$, $C$ | $3 \times (8)$, $C$ | $4 \times (8)$, $C$ | $5 \times (4)$, $C$ | $6 \times (2)$, $C$ | (2) |
| | $[ab, \alpha, \beta, \gamma]$ | 0 | 0 | 0 | $6 \times (4)$, $P$ | (3) $8 \times (2+2)$, $L$ | 0 | 0 | (3) |
| | $[\alpha, \beta, \gamma, \delta]$ | 0 | 0 | 0 | 0 | $2 \times (8)$, $P$ | $10 \times (2)$, $L$ | $30 \times (1)$, $L$ | (4) |
| | $[ab, cd, \alpha, \beta]$ | 0 | 0 | $6 \times (2)$, $P$ | 0 | 0 | 0 | 0 | (5) |
| | $abc$, $def$ | $10 \times (1)$, $P$ | 0 | 0 | 0 | 0 | 0 | 0 | (6) |
| | $abc$, $de$, $\alpha$ | 0 | (1) $10 \times (1)$, $P$ | 0 | 0 | 0 | 0 | 0 | (7) |
| | $abc$, $\alpha$, $\beta$ | 0 | 0 | (2) $4 \times (2+2)$, $P$ | $3 \times (4)$, $P$ | 0 | 0 | 0 | (8) |
| Cubic | $abcdef$ | $1 \times (1)$, $C$ | 0 | 0 | 0 | 0 | 0 | 0 | (9) |
| Quadriquadric | $\alpha\beta\gamma$, $\delta\epsilon\zeta$ | 0 | 0 | 0 | 0 | 0 | 0 | $(10) \times (1)$, $L$ | (10) |
| Excuboquartic | $\alpha\beta\gamma$, $\delta\epsilon$, $a$ | 0 | 0 | 0 | 0 | 0 | (4) $10 \times (1)$, $L$ | 0 | (11) |

1 Tangent plane along line is plane $abc$.
2 Tacnodal line, each sheet touched along line by plane $abc$.
3 Tacnodal line, each sheet of surface touched along line by hyperboloid $\alpha\beta\gamma$.
4 Surface touched along line by hyperboloid $\alpha\beta\gamma$.

2. In the Table, the upper margin refers to the surfaces, and the left-hand margin to the points, lines, and curves situate on these surfaces respectively; the body of the Table showing the number, and in ( ) the multiplicity, of these points, lines, and curves in regard to the several surfaces respectively. Thus, points $a$; for the surface $abcdef$, $6 \times (2)$, there are 6 such points, each of them a 2-conical (ordinary conical) point on the surface: so $abcde\alpha$, $5 \times (4)$, there are 5 such points, each a 4-conical point on the surface (viz. instead of the tangent plane there is a quartic cone); and so on. Similarly, lines $ab$ (viz. these are the lines joining two points $a$, $b$); for the surface $abcdef$, $15 \times (1)$, there are 15 such lines, each a simple line on the surface; surface $abcde\alpha$, $10 \times (2)$, there are 10 such lines, each a double (ordinary nodal) line on the surface; and so on. We have in two places the multiplicity $(2+2)$, which refers to a tacnodal line, as presently explained. The corner letters $C$, $P$, $L$ denote respectively proper cone, plane-pair, and line-pair, as afterwards explained.

3. The lines and curves referred to in the left-hand margin are:

(1) $ab$, line joining the points $a$ and $b$.

(2) $\alpha$, line $\alpha$.

(3) $[ab, \alpha, \beta, \gamma]$, pair of lines meeting each of the four lines, or say the tractors of the four lines $ab$, $\alpha$, $\beta$, $\gamma$. As regards the surface $ab\alpha\beta\gamma\delta$, the multiplicity is given as $(2+2)$, viz. the line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the hyperboloid $\alpha\beta\gamma$.

(4) $[\alpha, \beta, \gamma, \delta]$, tractors of the four lines $\alpha$, $\beta$, $\gamma$, $\delta$.

(5) $[ab, cd, \alpha, \beta]$ tractors of the four lines $ab$, $cd$, $\alpha$, $\beta$.

(6) $abc$, $def$, line of intersection of the planes $abc$ and $def$.

(7) $abc$, $de$, $\alpha$, line in the plane $abc$ joining the intersections of this plane by the lines $de$ and $\alpha$ respectively.

(8) $abc$, $\alpha$, $\beta$, line in the plane $abc$ joining the intersections of this plane by the lines $\alpha$ and $\beta$ respectively. As regards the surface $abcd\alpha\beta$, the multiplicity is given as $(2+2)$, viz. each line is (not an ordinary nodal, but) a tacnodal line, each sheet touching along the whole line the plane $abc$.

(9) Cubic $abcdef$, cubic curve through the six points $a$, $b$, $c$, $d$, $e$, $f$, common intersection of the cones each having its vertex at one of the points and passing through the other five.

(10) Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$, intersection of the quadric surfaces $\alpha\beta\gamma$ and $\delta\epsilon\zeta$, that is, the quadric surfaces through the lines $\alpha$, $\beta$, $\gamma$ and $\delta$, $\epsilon$, $\zeta$ respectively.

(11) Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, $a$, quartic curve generated as follows: viz. taking any line whatever which meets the lines $\alpha$, $\beta$, $\gamma$ (or say any generating line of the quadric $\alpha\beta\gamma$), the plane through this line and the point $a$ meets the lines $\delta$, $\epsilon$ in two points respectively; and the line joining these meets the generating line in a point having for its locus the excuboquartic curve in question (theory further considered in the sequel).

*Special forms of (Quadri-)Cones.*

4. We have to consider the special forms of (quadri-)cones; these are: 1°. The sharp-cone, or plane-pair; that is, a pair of two planes, intersecting in a line called the axis, the vertex being in this case an indeterminate point on the axis. Observe that a plane-pair passes through a given point when either of its planes passes through such point; it touches a given line when its axis meets the given line. 2°. The flat-cone, or line-pair; viz. this is a pair of intersecting lines, their point of intersection being the vertex of the line-pair, and the plane of the two lines being the *diametral* of the line-pair. Observe that the line-pair passes through a given point when its diametral passes through such point; it touches a given line when either of its lines meets the given line. 3°. There is a third kind, the line-pair-plane; viz. the two planes of the plane-pair may come to coincide, retaining, however, a definite line of intersection, or axis: or again, the two lines of a line-pair may come to coincide, retaining a definite plane or diametral; that is, in either case we have a plane passing through a line; and which is to be considered indifferently as two coincident planes intersecting in the line, or as two coincident lines lying in the plane. But there is not, in the present Memoir, any occasion to consider this third kind of special cone.

The letters $C$, $P$, $L$ in the Table denote that the cone is a (proper) cone, plane-pair, or line-pair, as the case may be.

*Singular Lines and Curves on the Surfaces.*

5. We may establish *à priori* the existence, and even to some extent the multiplicity, of the several lines and curves on the surfaces $abcdef, \ldots \alpha\beta\gamma\delta\epsilon\zeta$. Thus:

1°. Lines $ab$: take for the vertex of the cone a point at pleasure on the line $ab$; the cone passing through $b$ will *ipso facto* pass through $a$; and the conditions are thus that the cone shall pass through $b$ and satisfy four other conditions—in all, five conditions: and there is thus a cone with the point in question as vertex; that is, the line $ab$ is situate on the surface. Moreover, for the surfaces $abcdef$, $abcde\alpha$, $abcd\alpha\beta$, $abc\alpha\beta\gamma$, $ab\alpha\beta\gamma\delta$ respectively, for a given position of the vertex on the line $ab$, the number of cones is 1, 2, 4, 4, 2 respectively: and these are the multiplicities of the line $ab$ on the several surfaces respectively.

2°. Lines $\alpha$: take for the vertex of the cone a point at pleasure on the line $\alpha$; then the cone *ipso facto* touches the line $\alpha$, and there are only five other conditions to be satisfied; that is, we have a cone with the vertex in question; or the line $\alpha$ is situate on the surface. Moreover, for the surfaces $abcde\alpha$, $abcd\alpha\beta$, $abc\alpha\beta\gamma$, $ab\alpha\beta\gamma\delta$, $a\alpha\beta\gamma\delta\epsilon$, $\alpha\beta\gamma\delta\epsilon\zeta$ respectively, the number of cones is 1, 2, 4, 4, 2, 1 respectively: and it may be seen that the multiplicities of the line $\alpha$ are the doubles of these numbers, or are $=2$, 4, 8, 8, 4, 2 for the several surfaces respectively.

3°. Lines [$ab$, $\alpha$, $\beta$, $\gamma$]: taking the vertex in one of these tractors, the cone cannot be a proper cone, but (if it exist) it must be either a line-pair having the tractor for one of its lines, or else a plane-pair having the tractor for its axis. The two cases are:

Surface $abc\alpha\beta\gamma$. Cone is a plane-pair, the two planes intersecting in the tractor, and passing, the one of them through the points $a$, $b$, the other through the point $c$. The vertex being an indeterminate point on the tractor, the tractor is situate on the surface.

Surface $ab\alpha\beta\gamma\delta$. Cone is a line-pair, one line being the tractor, the other a line drawn in the plane of the tractor and $ab$ to meet $\delta$, and which meets the tractor in an arbitrary point thereof: the tractor is thus a line on the surface.

4°. Lines [$\alpha$, $\beta$, $\gamma$, $\delta$]: taking the vertex in one of these tractors, then, as in the last case, the cone is either a line-pair having the tractor for one of its lines or a plane-pair having the tractor for its axis. The three cases are:

Surface $ab\alpha\beta\gamma\delta$. Cone is a plane-pair, the two planes intersecting in the tractor and passing through the points $a$, $b$ respectively.

Surface $a\alpha\beta\gamma\delta\epsilon$. Cone is a line-pair, one line being the tractor, the other a line in the plane of the tractor and $a$, meeting the line $\epsilon$ and meeting the tractor in an indeterminate point.

Surface $\alpha\beta\gamma\delta\epsilon\zeta$. Cone is a line-pair, one line being the tractor, the other a line drawn from an indeterminate point of the tractor to meet the lines $\epsilon$ and $\zeta$.

5°. Lines [$ab$, $cd$, $\alpha$, $\beta$]. Cone is a plane-pair, the two planes intersecting in the tractor, and passing through the points $a$, $b$ and the points $c$, $d$ respectively.

6°. Line $abc$, $def$. Cone is a plane-pair, consisting of the two planes $abc$ and $def$.

7°. Line $abc$, $de$, $\alpha$. Cone is a plane-pair, the two planes intersecting in the line; one plane being $abc$, the other a plane through the line $de$.

8°. Line $abc$, $\alpha$, $\beta$. There are two cases:

Surface $abcd\alpha\beta$. Cone is a plane-pair, the two planes intersecting in the line; the one being $abc$, and the other passing through the point $d$.

Surface $abc\alpha\beta\gamma$. Cone is a line-pair; one line being $abc$, $\alpha$, $\beta$, the other a line in the plane $abc$ meeting the line $\delta$, and meeting the line $abc$, $\alpha$, $\beta$ in an indeterminate point.

9°. Cubic $abcdef$. Each point of the cubic is the vertex of a proper cone passing through the cubic, and therefore through the six points; that is, the cubic is a line on the surface $abcdef$.

10°. Quadriquadric $\alpha\beta\gamma$, $\delta\epsilon\zeta$. Cone is a line-pair; viz. it is composed of the lines drawn from any point of the curve, one of them to meet the lines $\alpha$, $\beta$, $\gamma$, and the other to meet the lines $\delta$, $\epsilon$, $\zeta$.

11°. Excuboquartic $\alpha\beta\gamma$, $\delta\epsilon$, $a$. Cone is a line-pair; the two lines being, one of them a line at pleasure meeting $\alpha$, $\beta$, $\gamma$, the other the line which, in the plane of the other line and the point $a$, meets the lines $\delta$, $\epsilon$.

*Mode of obtaining the several Equations: Notations and Formulæ.*

6. The equations of the several surfaces are obtained by taking as centre of projection an assumed position of the vertex, and projecting everything upon an arbitrary plane; the projections of the given points and lines are points and lines in the arbitrary plane, and the section of the cone by this plane is a conic; the equation of the surface is thus obtained as the condition that there shall be a conic passing through $m$ given points and touching $6-m$ given lines.

7. We take as current coordinates $(X, Y, Z, W)$, or when plane-coordinates are employed $(\xi, \eta, \zeta, \omega)$: the coordinates of the vertex are throughout represented by $(x, y, z, w)$; but in explanations &c., these are also used as current coordinates. The plane of projection is taken to be $W=0$. The coordinates of the given points $a$, &c., are taken to be $(x_a, y_a, z_a, w_a)$, &c. There is no confusion occasioned by so doing, and I retain the ordinary letters $(a, b, c, f, g, h)$ for the six coordinates of a line, it being understood that these letters so used have no reference whatever to the given points $a$, $b$, &c.; viz. the coordinates of the given lines $\alpha$, &c., are $(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, &c.; there is sometimes occasion to consider the coordinates of other lines $ab$, &c., but the notation will always be explained.

8. I write $l, m, n, p, q, r$ for the coordinates of the line joining the vertex $(x, y, z, w)$ with a point $(x', y', z', w')$; viz.

$$\begin{aligned} l &= yz' - y'z, \quad p = xw' - x'w, \\ m &= zx' - z'x, \quad q = yw' - y'w, \\ n &= xy' - x'y, \quad r = zw' - z'w, \end{aligned}$$

$(l_a = yz_a - y_a z$, &c., this being explained when necessary); and also

$$\begin{aligned} P &= \quad . \quad hy - gz + aw, \\ Q &= -hx \quad . \quad + fz + bw, \\ R &= \quad gx - fy \quad . \quad + cw, \\ S &= -ax - by - cz \quad . \quad , \end{aligned}$$

$(P_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c., this being explained when necessary).

This being so, then projecting from the vertex $(x, y, z, w)$, say on the plane $W=0$, the $x, y, z$ coordinates of the projection of a point $a$ are as $p_a : q_a : r_a\,(p_a = xw_a - x_a w$, &c.); and the equation of the projection of a line $\alpha$ is

$$P_\alpha X + Q_\alpha Y + R_\alpha Z = 0,$$

$(P_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c.). We thus have, in the projection on the plane $W=0$, the $m$ points and $6-m$ lines situate in and touched by the conic.

The following notations and formulæ are convenient:

9. $pabc=0$ is the equation of the plane through the points $a$, $b$, $c$; viz.

$$pabc = \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \\ x_c, & y_c, & z_c, & w_c \end{vmatrix}.$$

Of course $pbac=-pabc$, &c. Observe that here, and in the notations which follow, the letter $p$ is used as referring to the coordinates $(x, y, z, w)$, and that the index of $p$ $(=1$ when no index is expressed) shows the degree in these coordinates.

10. $pa\alpha=0$ is the equation of the plane through the point $a$ and the line $\alpha$; viz. $pa\alpha$ is the foregoing determinant, if for a moment $b$, $c$ are any two points on the line $\alpha$; or, what is the same thing,

$$pa\alpha = P_a x + Q_a y + R_a z + S_a w,$$

where

$$\begin{aligned} P_a &= \quad . \quad hy_a - gz_a + aw_a, \\ Q_a &= -hx_a \quad . \quad + fz_a + bw_a, \\ R_a &= gx_a - fy_a \quad . \quad + cw_a, \\ S_a &= -ax_a - by_a - cz_a \quad . \quad ; \end{aligned}$$

and $(a, b, c, f, g, h)$ are the coordinates of the line $\alpha$: observe that $pa\alpha = p\alpha a$.

11. $p^2\alpha\beta\gamma=0$ is the equation of the quadric surface through the lines $\alpha$, $\beta$, $\gamma$; viz. we have

$$\begin{aligned} p^2\alpha\beta\gamma = (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 & \\ + [(abg) - (cah)]\,xw & \\ + [(bch) - (abf)]\,yw & \\ + [(caf) - (bcg)]\,zw & \\ + [(bfg) + (chf)]\,yz & \\ + [(cgh) + (afg)]\,zx & \\ + [(ahf) + (bgh)]\,xy, & \end{aligned}$$

where

$$agh = \begin{vmatrix} a_\alpha, & g_\alpha, & h_\alpha \\ a_\beta, & g_\beta, & h_\beta \\ a_\gamma, & g_\gamma, & h_\gamma \end{vmatrix} \text{ \&c.}$$

$(a_\alpha, b_\alpha, c_\alpha, f_\alpha, g_\alpha, h_\alpha)$, $(a_\beta, \ldots)$, $(a_\gamma, \ldots)$ being the coordinates of the given lines $\alpha$, $\beta$, $\gamma$. Observe that $p^2\beta\alpha\gamma = -p^2\alpha\beta\gamma$, &c.

12. It is to be noticed that, writing

$$\begin{aligned} P &= \quad\;\; \cdot \quad\;\; hy - gz + aw, \\ Q &= -hx \quad \cdot \quad + fz + bw, \\ R &= \;\; gx - fy \quad \cdot \quad + cw, \\ S &= -ax - by - cz \quad \cdot \;, \end{aligned}$$

viz. $P_\alpha = h_\alpha y - g_\alpha z + a_\alpha w$, &c., then that we have identically

$$\begin{vmatrix} \lambda\,, & \mu\,, & \nu\,, & \rho \\ P_\alpha, & Q_\alpha, & R_\alpha, & S_\alpha \\ P_\beta, & Q_\beta, & R_\beta, & S_\beta \\ P_\gamma, & Q_\gamma, & R_\gamma, & S_\gamma \end{vmatrix} = -(\lambda x + \mu y + \nu z + \rho w)\,.\,p^2\alpha\beta\gamma,$$

and further that we have identically

$$-p^2\alpha\beta\gamma = L_{\alpha\beta}P_\gamma + M_{\alpha\beta}Q_\gamma + N_{\alpha\beta}R_\gamma + \Omega_{\alpha\beta}S_\gamma,$$

where

$$\begin{aligned} L &= (af' - a'f)\,x + (bf' - b'f)\,y + (cf' - c'f)\,z - (bc' - b'c)\,w, \\ M &= (ag' - a'g)\,x + (bg' - b'g)\,y + (cg' - c'g)\,z - (ca' - c'a)\,w, \\ N &= (ah' - a'h)\,x + (bh' - b'h)\,y + (ch' - c'h)\,z - (ab' - a'b)\,w, \\ \Omega &= (gh' - g'h)\,x + (hf' - h'f)\,y + (fg' - f'g)\,z + (af' - a'f + bg' - b'g + ch' - c'h)\,w; \end{aligned}$$

and $L_{\alpha\beta}$, &c. are the values of $L$, &c. on substituting therein $(a_\alpha, \ldots)$ and $(a_\beta, \ldots)$ for the unaccented and accented letters respectively.

13. Observe that we have

$$\begin{aligned} L + (a'f + b'g + c'h)\,x &= \quad\;\; \cdot \quad - c'Q + b'R - f'S, \\ M + (\quad\;\; ,, \quad\;\;)\,y &= \;\; c'P \quad \cdot \quad - a'R - g'S, \\ N + (\quad\;\; ,, \quad\;\;)\,z &= -b'P + a'Q \quad \cdot \quad - h'S, \\ \Omega + (\quad\;\; ,, \quad\;\;)\,w &= \;\; f'P + g'Q + h'R \quad \cdot \;; \end{aligned}$$

and similarly

$$\begin{aligned} -L + (af' + bg' + ch')\,x &= \quad\;\; \cdot \quad - cQ' + bR' - fS', \\ -M + (\quad\;\; ,, \quad\;\;)\,y &= \;\; cP' \quad \cdot \quad - aR' - gS', \\ -N + (\quad\;\; ,, \quad\;\;)\,z &= -bP' + aQ' \quad \cdot \quad - hS', \\ -\Omega + (\quad\;\; ,, \quad\;\;)\,w &= \;\; fP' + gQ' + hR' \quad \cdot \;; \end{aligned}$$

whence also

$$\begin{aligned} \cdot \quad h'M - g'N + a'\Omega &= -(a'f + b'g + c'h)\,P', \\ -h'L \quad \cdot \quad + f'N + b'\Omega &= -(\quad\;\; ,, \quad\;\;)\,Q', \\ g'L - f'M \quad \cdot \quad + c'\Omega &= -(\quad\;\; ,, \quad\;\;)\,R', \\ -a'L - b'M - c'N \quad \cdot \quad &= -(\quad\;\; ,, \quad\;\;)\,S'; \end{aligned}$$

and

$$\begin{array}{rrrrcl} hM & -gN & +a\Omega & = & (af'+bg'+ch') & P, \\ -hL & . & +fN & +b\Omega & = & (\quad ,, \quad) Q, \\ gL & -fM & . & +c\Omega & = & (\quad ,, \quad) R, \\ -aL & -bM & -cN & . & = & (\quad ,, \quad) S. \end{array}$$

14. $p^3a\,.\,a\alpha\beta\,.\,\gamma\delta = 0$ is the equation of the cubic surface through the lines $\alpha$, $\beta$, $\gamma$, $\delta$ and $a\alpha\beta$, $a\gamma\delta$ (viz. $a\alpha\beta$ is the line from $a$ to meet $\alpha$, $\beta$, and so $a\gamma\delta$ is the line from $a$ to meet $\gamma$, $\delta$). Observe that the conditions which determine this cubic surface thus are that the cubic shall pass through

$a$; the points of $a\alpha\beta$ on $\alpha$ and $\beta$ respectively, 3 other points on $\alpha$, 3 on $\beta$, and 1 on $a\alpha\beta$;

also the points of $a\gamma\delta$ on $\gamma$ and $\delta$ respectively, 3 other points on $\gamma$, 3 on $\delta$, and 1 on $a\gamma\delta$; in all, $1+9+9=19$ points;

viz. the conditions completely determine the surface.

15. We have

$$p^3a\,.\,a\beta\,.\,\gamma\delta = \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L_{\alpha\beta}, & M_{\alpha\beta}, & N_{\alpha\beta}, & \Omega_{\alpha\beta} \\ L_{\gamma\delta}, & M_{\gamma\delta}, & N_{\gamma\delta}, & \Omega_{\gamma\delta} \end{vmatrix},$$

viz. this determinant, equated to zero, gives the equation of the surface.

To prove this, take as before the unaccented letters ($a$, $b$, $c$, $f$, $g$, $h$) to refer to the line $\alpha$, and the letters with one, two, and three accents to refer to the lines $\beta$, $\gamma$, $\delta$ respectively; write also $L$, $M$, $N$, $\Omega$ and $L'$, $M'$, $N'$, $\Omega'$ for $L_{\alpha\beta}$, &c. and $L_{\gamma\delta}$, &c., respectively. Referring to the foregoing expressions for $L$, $M$, $N$, $\Omega$, and observing that for a point on the line $\alpha$, the values of $P$, $Q$, $R$, $S$ are each $=0$, then for such a point we have $L+(a'f+b'g+c'h)\,x=0$, &c., that is, $L:M:N:\Omega = x:y:z:w$, and these values satisfy the equation of the surface, which is thus a surface passing through the line $\alpha$; and similarly it passes through the lines $\beta$, $\gamma$, $\delta$.

To show that the surface passes through the line $a\alpha\beta$, take the coordinates of the point $a$ to be 0, 0, 0, 1; then the line $a\alpha\beta$ is given as the intersection of the planes $ax+by+cz=0$ and $a'x+b'y+c'z=0$, that is, $S=0$ and $S'=0$. And the equation of the surface, writing therein $x_a$, $y_a$, $z_a$, $w_a = 0, 0, 0, 1$, becomes

$$\begin{vmatrix} x, & y, & z \\ L, & M, & N \\ L', & M', & N' \end{vmatrix} = 0,$$

or, as this may be written,

$$\begin{vmatrix} S & , S' & , z \\ aL+bM+cN, & a'L+b'M+c'N, & N \\ aL'+bM'+cN', & a'L'+b'M'+c'N', & N' \end{vmatrix}=0;$$

and, for a point on the line $a\alpha\beta$, this is

$$\begin{vmatrix} aL+bM+cN, & a'L+b'M+c'N \\ aL'+bM'+cN', & a'L'+b'M'+c'N' \end{vmatrix}=0.$$

But in the equations $-a'L-b'M-c'N=-(a'f+b'g+c'h)S'$, and $-aL-bM-cN=(af'+bg'+ch')S$, writing $S=0$ and $S'=0$, we have $aL+bM+cN=0$ and $a'L+b'M+c'N=0$, and the equation is satisfied; that is, the surface passes through the line $a\alpha\beta$, and similarly it passes through the line $a\gamma\delta$.

*Surface abcdef.*

16. The equation may be written

$$pabe\,.\,pcde\,.\,pacf\,.\,pdbf-pabf\,.\,pcdf\,.\,pace\,.\,pdbe=0,$$

where $pabe=0$ is the equation of the plane through the points $a$, $b$, $e$; and the like for the other symbols. The form is one out of 45 like forms, depending on the partitionment

$$\left.\begin{array}{c} ab\,.\,cd \\ ac\,.\,db \\ ad\,.\,bc \end{array}\right\}(ef),$$

of the six letters.

17. *Investigation.* In the projection, the six points $(p_a,\ q_a,\ r_a)$ are situate on a conic; the condition for this is

$$(p,\ q,\ r)^2=0,$$

where the left-hand side represents the determinant obtained by writing successively $(p_a,\ q_a,\ r_a)$, &c., for $(p,\ q,\ r)$. The equation in question may be written

$$abe\,.\,cde\,.\,acf\,.\,dbf-abf\,.\,cdf\,.\,ace\,.\,dbe=0;$$

where

$$abe=\begin{vmatrix} p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \\ p_e, & q_e, & r_e \end{vmatrix}, \text{ \&c.};$$

and substituting for $p_a, \ldots$, their values, we have $abe=w^2\,.\,pabe$, whence the foregoing result.

{Surface *abcdef.*}

18. *Singularities.* The form of the equation shows at once that [1]

(0)[2] The point $a$ is a 2-conical point; in fact, for this point we have $pabe = 0$, $pacf = 0$, $pabf = 0$, $pace = 0$.

(1) The line $ab$ a simple line; in fact, for any point of this line we have $pabe = 0$, $pabf = 0$.

(2) The line $abe \,.\, cdf$ a simple line; in fact, for any point of this line we have $pabe = 0$, $pcdf = 0$.

(9) To show analytically that the cubic curve $abcdef$ is a line on the surface, observe that the equation of the surface is satisfied if we have simultaneously ($\lambda$ being arbitrary)

$$pabe \,.\, pacf - \lambda \,.\, pabf \,.\, pace = 0,$$

$$\lambda \,.\, pcde \,.\, pdbf - \quad pcdf \,.\, pdbe = 0.$$

The first of these equations is a cone, vertex $a$, which passes through the points $b, e$ $c, f$, and which, if $\lambda$ is properly determined, will pass through the point $d$; the second is a cone, vertex $d$, which passes through the points $b, e, c, f$ and which, if $\lambda$ is properly determined, will pass through the point $a$; the two determinations of $\lambda$ are

$$dabe \,.\, dacf - \lambda \,.\, dabf \,.\, dace = 0,$$

$$\lambda \,.\, acde \,.\, adbf - \quad acdf \,.\, adbe = 0;$$

giving the same value of $\lambda$; and the equations then represent cones, the first having $a$ for its vertex, and passing through $d, b, e, c, f$; the second having $d$ for its vertex, and passing through $a, b, e, c, f$; the two intersect in the line $ad$, and in the cubic curve $abcdef$, which is thus a curve on the surface.

## *Surface abcdeα.*

19. The equation may be written

$$(pabe \,.\, pcde \,.\, p^2\alpha ac \,.\, db - pace \,.\, pdbe \,.\, p^2\alpha ab \,.\, cd)^2$$
$$+ 4 pabe \,.\, pcde \,.\, pace \,.\, pdbe \,.\, pabc \,.\, pdbc \,.\, pa\alpha \,.\, pd\alpha = 0,$$

or, what is the same thing,

$$(pabe \,.\, pcde \,.\, p^2\alpha ac \,.\, db + pace \,.\, pdbe \,.\, p^2\alpha ab \,.\, cd)^2$$
$$+ 4 pabe \,.\, pcde \,.\, pace \,.\, pdbe \,.\, pbad \,.\, pcad \,.\, pb\alpha \,.\, pc\alpha = 0,$$

(the equivalence of the two depending on the identity

$$-p^2\alpha ab \,.\, cd \,.\, p^2\alpha ac \,.\, db + pabc \,.\, pdbc \,.\, pa\alpha \,.\, pd\alpha - pbad \,.\, pcad \,.\, pb\alpha \,.\, pc\alpha = 0)$$

[1] Or course, as regards the present surface and the other surfaces for which the equation is given in an unsymmetrical form, the conclusion obtained in regard to any point or line of the surface applies to every point or line of the same kind. Thus $ab$ being a simple line, we have also $ad$ a simple line, although the equation, as written down, does not put this in evidence.

[2] The bracketed numbers refer to the lines of the Table.

{Surface *abcdeα.*}

where, as before, $pabe=0$ is the equation of the plane through the points $a$, $b$, $e$; $p^2\alpha acdb=0$ the equation of the quadric surface through the lines $\alpha$, $ac$, $db$; and $pa\alpha=0$ the equation of the plane through the point $a$ and the line $\alpha$.

The above forms are 2 out of 30 like forms, as appears by the partitionment

$$\left.\begin{Bmatrix} ab, & cd \\ ac, & db \\ ad, & bc \end{Bmatrix}\right. e\alpha.$$

20. *Investigation.* In the projection, the equation of the conic through the five points may be written

$$\left|\begin{matrix} (X, & Y, & Z)^2 \\ (p, & q, & r)^2 \end{matrix}\right| = 0,$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to $(p, q, r)$ the suffixes $a, b, c, d, e$ respectively. This is at once transformed into

$$abe\,.\,cde\,.\,ac\Delta\,.\,db\Delta - ace\,.\,dbe\,.\,ab\Delta\,.\,cd\Delta = 0,$$

or, what is the same thing,

$$pabe\,.\,pcde\,.\,ac\Delta\,.\,db\Delta - pace\,.\,pdbe\,.\,ab\Delta\,.\,cd\Delta = 0,$$

or say,

$$pabe\,.\,pcde\,(A''X+B''Y+C''Z)\,(A'''X+B'''Y+C'''Z)$$
$$-pace\,.\,pdbe\,(AX+BY+CZ)\,(A'X+B'Y+C'Z),$$

where $pabe$, &c. signify as before, and

$$AX+BY+CZ = \left|\begin{matrix} X, & Y, & Z \\ p_a, & q_a, & r_a \\ p_b, & q_b, & r_b \end{matrix}\right|$$

and so for $A'X+B'Y+C'Z$, &c., the suffixes for $A'$, $B'$, $C'$ being $(c, d)$, and those for $A''$, $B''$, $C''$ and $A'''$, $B'''$, $C'''$ being $(a, c)$ and $(d, b)$ respectively.

21. Passing to the reciprocal equation, and making the conic touch the line $\alpha$, we obtain the equation of the surface in the form

$$\left\{pabe\,.\,pcde \left|\begin{matrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{matrix}\right| - pace\,.\,pdbe \left|\begin{matrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A', & B', & C' \end{matrix}\right|\right\}^2$$

$$+4\,pace\,.\,pdbe\,.\,pabe\,.\,pcde \left|\begin{matrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A, & B, & C \\ A'', & B'', & C'' \end{matrix}\right| \left|\begin{matrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A', & B', & C' \\ A''', & B''', & C''' \end{matrix}\right| = 0,$$

{Surface *abcdea*.}

(where $P_a = h_a y - g_a z + a_a w = 0$) or in the equivalent form wherein we have in the first term + instead of −, and in the second term the determinants

$$\begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A''', & B''', & C''' \end{vmatrix}, \quad \begin{vmatrix} P_a, & Q_a, & R_a \\ A', & B', & C' \\ A'', & B'', & C'' \end{vmatrix}.$$

22. {The question, in fact, is to find the reciprocal of the form

$$\lambda (ax+by+cz)(a'x+b'y+c'z) - \mu (a''x+b''y+c''z)(a'''x+b'''y+c'''z) = 0;$$

taking $\xi$, $\eta$, $\zeta$ for the reciprocal variables, the coefficient of $\xi^2$ is

$$\{\lambda (bc'+b'c) - \mu (b''c'''+b'''c'')\}^2 - (2\lambda bb' - 2\mu b''b''')(2\lambda cc' - 2\mu c''c'''),$$

viz. this is

$$\lambda^2 (bc'-b'c)^2 + \mu^2 (b''c'''-b'''c'')^2 + 2\lambda\mu \{2bb'c''c''' + 2b''b'''cc' - (bc'+b'c)(b''c'''+b'''c'')\},$$

or, as it may be written,

$$\{\lambda (bc'-b'c) \pm \mu (b''c'''-b'''c'')\}^2 + 2\lambda\mu \left\{\begin{matrix} 2bb'c''c''' + 2b''b'''cc' \\ \mp (bc-b'c)(b''c''-b'''c'') \\ -(bc'+b'c)(b''c''+b'''c'') \end{matrix}\right\}.$$

Taking the upper signs, this is

$$\{\lambda (bc'-b'c) + \mu (b''c'''-b'''c'')\}^2 + 4\lambda\mu \left(\begin{matrix} bb'c''c''' + b''b'''cc' \\ -bc'b''c''' - b'cb'''c'' \end{matrix}\right);$$

viz. the term in $\lambda\mu$ is

$$= +4\lambda\mu (bc'''-b'''c)(b'c''-b''c').$$

Taking the lower signs, it is

$$\{\lambda (bc'-b'c) - \mu (b''c'''-b'''c'')\}^2 + 4\lambda\mu \left(\begin{matrix} bb'c''c''' + b''b'''cc' \\ -bc'b'''c'' - b'cb''c''' \end{matrix}\right);$$

viz. the term in $\lambda\mu$ is

$$4\lambda\mu (bc''-b''c)(b'c'''-b'''c');$$

and it is thence easy to infer the forms of the other coefficients, and to obtain the reciprocal equation in the two equivalent forms

$$\left\{\lambda \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} + \mu \begin{vmatrix} \xi, & \eta, & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\right\}^2 + 4\lambda\mu \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a''', & b''', & c''' \end{vmatrix} \begin{vmatrix} \xi, & \eta, & \zeta \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0,$$

$$\left\{\lambda \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} - \mu \begin{vmatrix} \xi, & \eta, & \zeta \\ a'', & b'', & c'' \\ a''', & b''', & c''' \end{vmatrix}\right\}^2 + 4\lambda\mu \begin{vmatrix} \xi, & \eta, & \zeta \\ a, & b, & c \\ a'', & b'', & c'' \end{vmatrix} \begin{vmatrix} \xi, & \eta, & \zeta \\ a', & b', & c' \\ a''', & b''', & c''' \end{vmatrix} = 0,$$

which are the required auxiliary formulæ.}

{Surface *abcdea*.}

23. To reduce the foregoing result, we have

$$A,\ B,\ C = \left\| \begin{matrix} xw_a - wx_a, & yw_a - wy_a, & zw_a - wz_a \\ xw_b - wx_b, & yw_b - wy_b, & zw_b - wz_b \end{matrix} \right\|$$

proportional to the three determinants which contain $w$, of the set

$$\left\| \begin{matrix} x\ , & y\ , & z\ , & w \\ x_a, & y_a, & z_a, & w_a \\ x_b, & y_b, & z_b, & w_b \end{matrix} \right\|, \text{ viz. } A = w \begin{vmatrix} y\ , & z\ , & w \\ y_a, & z_a, & w_a \\ y_b, & z_b, & w_b \end{vmatrix}, \text{ \&c.};$$

and similarly $A'$, $B'$, $C'$ are proportional to the three determinants which contain $w$, of the set

$$\left\| \begin{matrix} x\ , & y\ , & z\ , & w \\ x_c, & y_c, & z_c, & w_c \\ x_d, & y_d, & z_d, & w_d \end{matrix} \right\|, \text{ viz. } A' = w \begin{vmatrix} y\ , & z\ , & w \\ y_c, & z_c, & w_c \\ y_d, & z_d, & w_d \end{vmatrix}, \text{ \&c.}$$

Hence, omitting the factor $w$, and writing (a, b, c, f, g, h) and (a′, b′, c′, f′, g′, h′) for the coordinates of the lines $ab$ and $cd$ respectively, we have

$$\begin{aligned} A &= \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w, & A' &= \quad \mathrm{h}'y - \mathrm{g}'z + \mathrm{a}'w, \\ B &= -\mathrm{h}x \quad + \mathrm{f}z + \mathrm{b}w, & B' &= -\mathrm{h}'x \quad + \mathrm{f}'z + \mathrm{b}'w, \\ C &= \quad \mathrm{g}x - \mathrm{f}y \quad + \mathrm{c}w, & C' &= \quad \mathrm{g}'x - \mathrm{f}'y \quad + \mathrm{c}'w; \end{aligned}$$

and thence

$$\begin{aligned} BC' - B'C &= \Omega x - Lw, \\ CA' - C'A &= \Omega y - Mw, \\ AB' - A'B &= \Omega z - Nw, \end{aligned}$$

where

$$\begin{aligned} L &= (\mathrm{af}' - \mathrm{a}'\mathrm{f})\,x + (\mathrm{bf}' - \mathrm{b}'\mathrm{f})\,y + (\mathrm{cf}' - \mathrm{c}'\mathrm{f})\,z - (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,w, \\ M &= (\mathrm{ag}' - \mathrm{a}'\mathrm{g})\,x + (\mathrm{bg}' - \mathrm{b}'\mathrm{g})\,y + (\mathrm{cg}' - \mathrm{c}'\mathrm{g})\,z - (\mathrm{ca}' - \mathrm{c}'\mathrm{a})\,w, \\ N &= (\mathrm{ah}' - \mathrm{a}'\mathrm{h})\,x + (\mathrm{bh}' - \mathrm{b}'\mathrm{h})\,y + (\mathrm{ch}' - \mathrm{c}'\mathrm{h})\,z - (\mathrm{ab}' - \mathrm{a}'\mathrm{b})\,w, \\ \Omega &= (\mathrm{gh}' - \mathrm{g}'\mathrm{h})\,x + (\mathrm{hf}' - \mathrm{h}'\mathrm{f})\,y + (\mathrm{fg}' - \mathrm{f}'\mathrm{g})\,z - (\mathrm{af}' - \mathrm{a}'\mathrm{f} + \mathrm{bg}' - \mathrm{b}'\mathrm{g} + \mathrm{ch}' - \mathrm{c}'\mathrm{h})\,w; \end{aligned}$$

and consequently

$$\begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ A\ , & B\ , & C \\ A', & B', & C' \end{vmatrix} = \Omega\,(xP_\alpha + yQ_\alpha + zR_\alpha) - w\,(LP_\alpha + MQ_\alpha + NR_\alpha)$$

$$= -\,w\,(LP_\alpha + MQ_\alpha + NR_\alpha + \Omega S_\alpha);$$

or omitting the factor $-w$, say it is $= LP_\alpha + MQ_\alpha + NR_\alpha + \Omega S_\alpha$, viz. this is $= p^2\alpha\, ab\,.\,cd$.

{Surface *abcdea*.}

We have similarly

$$\begin{vmatrix} P_a, & Q_a, & R_a \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{vmatrix}$$

taken to be $=p^2\alpha\, ac \,.\, db$.

24. We have in like manner the other two determinants

$$\begin{vmatrix} P_a, & Q_a, & R_a \\ A, & B, & C \\ A'', & B'', & C'' \end{vmatrix} \text{ and } \begin{vmatrix} P_a, & Q_a, & R_a \\ A', & B', & C' \\ A''', & B''', & C''' \end{vmatrix}$$

taken to be $=p^2\alpha\, ab \cdot ac$ and $p^2\alpha\, cd \,.\, db$ respectively.

But we have

$$p^2\alpha\, ab \,.\, ac = pa\alpha \,.\, pabc,$$

(viz. geometrically the hyperboloid through the lines $\alpha$, $ab$, $ac$ breaks up into the plane $pa\alpha$ through the line $\alpha$ and point $a$, and the plane $pabc$ through the points $a$, $b$, $c$).

And similarly

$$p^2\alpha\, cd \,.\, db = -p^2\alpha\, dc \,.\, db = +p^2\alpha\, db \,.\, dc = pad \,.\, pdbc\,;$$

whence, substituting for the several determinants, we have the foregoing equation of the surface.

25. *Singularities.* The form of the equation shows that

(0) The point $a$ is a 4-conical point: in fact, for this point we have $pabe=0$, $p^2\alpha\, ac \,.\, db=0$, $pace=0$, $p^2\alpha\, ab \,.\, cd=0$.

(1) The line $ab$ is a double line: in fact, for any point of the line we have $pabe=0$, $p^2\alpha\, ab \,.\, cd=0$, $pabc=0$.

(2) The line $\alpha$ is a double line: in fact, for any point of the line we have $p^2\alpha\, ac \,.\, db=0$, $p^2\alpha\, ab \,.\, cd=0$, $pa\alpha=0$, $pd\alpha=0$.

(7) The line $abe \,.\, cd \,.\, \alpha$ is a simple line: in fact, for any point of the line we have $pabe=0$, $p^2\alpha\, ab \,.\, cd=0$. Observe that, on writing in the equation $pabe=0$ the equation becomes $(p^2\alpha\, ab \,.\, cd)^2=0$; so that the surface along the line in question touches the plane $pabe$.

*Surface abcdaβ.*

26. The equation of the surface is

$$\text{Norm}\,\{\sqrt{pa\alpha \,.\, pa\beta} \,.\, pbcd - \sqrt{pb\alpha \,.\, pb\beta} \,.\, pcda + \sqrt{pc\alpha \,.\, pc\beta} \,.\, pdab - \sqrt{pd\alpha \,.\, pd\beta} \,.\, pabc\} = 0,$$

where the norm is the product of 8 factors.

As before, $pa\alpha=0$ is the equation of the plane through the point $a$ and the line $\alpha$; and $pbcd=0$ the equation of the plane through the points $b$, $c$, $d$. The form is unique.

{Surface *abcdaβ.*}

27. *Investigation.* In the projection, the equation of the conic touching the projections of the lines $\alpha$, $\beta$ is

$$\sqrt{(P_\alpha X+Q_\alpha Y+R_\alpha Z)(P_\beta X+Q_\beta Y+R_\beta Z)}+AX+BY+CZ=0,$$

where $A$, $B$, $C$ are arbitrary coefficients. To make this pass through the projection of the point $a$, we must write $X : Y : Z = p_a : q_a : r_a$; viz. we thus have

$$\begin{aligned} P_\alpha X+Q_\alpha Y+R_\alpha Z &= w_a(x P_\alpha + y Q_\alpha + z R_\alpha) \\ &\quad - w\ (x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha), \\ &= -w\ (x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha + w_a S_\alpha), \\ &= -w \,.\, pa\alpha; \end{aligned}$$

and similarly

$$P_\beta X+Q_\beta Y+R_\beta Z=-w\,.\,pa\beta.$$

We thus have

$$w\sqrt{pa\alpha\,.\,pa\beta}+Ap_a+Bq_a+Cr_a=0.$$

Or, forming the like equations for the points $b$, $c$, $d$ respectively and eliminating, the equation is

$$\begin{vmatrix} \sqrt{pa\alpha\,.\,pa\beta}, & p_a, & q_a, & r_a \\ \sqrt{pb\alpha\,.\,pb\beta}, & p_b, & q_b, & r_b \\ \sqrt{pc\alpha\,.\,pc\beta}, & p_c, & q_c, & r_c \\ \sqrt{pd\alpha\,.\,pd\beta}, & p_d, & q_d, & r_d \end{vmatrix}=0;$$

which, substituting for $(p_a, q_a, r_a)$, &c., their values, viz. $p_a = xw_a - x_a w$, &c., is readily converted into

$$\begin{vmatrix} & x, & y, & z, & w \\ \sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & w_a \\ \sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & w_b \\ \sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & w_c \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{vmatrix}=0.$$

or, what is the same thing,

$$\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd-\sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda+\sqrt{pc\alpha\,.\,pc\beta}\,.\,pdab-\sqrt{pd\alpha\,.\,pd\beta}\,.\,pabc=0;$$

viz. taking the norm, we have the form mentioned above.

28. *Singularities.* The equation shows that

(0) The point $a$ is an 8-conical point; in fact, for the point in question $pa\alpha=0$, $pa\beta=0$, $pcda=0$, $pdab=0$, $pabc=0$; each factor is of the form $0^1$, and the norm is $0^8$.

{Surface $abcd\alpha\beta$.}

(1) The line $ab$ is a 4-tuple line. To show this, observe in the first instance, that we may obtain the 8 factors of the norm by giving to the radical $\sqrt{pa\alpha\,.\,pa\beta}$ the sign $+$, and to the other three radicals the signs $+, -$, at pleasure. For a point on the line in question, we have $pdab=0$, $pabc=0$; hence the norm is the product of the four equal factors

$$\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd-\sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda,$$

and the other four equal factors obtained by writing herein $+$ instead of $-$.

Now for a point on the line $ab$, we may write for $x, y, z, w$ the values $ux_a+vx_b$, $uy_a+vy_b$, $uz_a+vz_b$, $uw_a+vw_b$, where $u, v$ are arbitrary coefficients. We have

$$\begin{aligned}
pa\alpha &= u\,.\,aa\alpha+v\,.\,ba\alpha=v\,.\,ba\alpha=-v\,.\,ab\alpha,\\
pa\beta &= v\,.\,ba\beta=-v\,.\,ab\beta,\\
pb\alpha &= u\,.\,ab\alpha+v\,.\,bb\alpha=u\,.\,ab\alpha,\\
pb\beta &= u\,.\,ab\beta,\\
pbcd &= u\,.\,abcd+v\,.\,bbcd=u\,.\,abcd,\\
pcda &= u\,.\,acda+v\,.\,bcda=v\,.\,bcda=-v\,.\,abcd,
\end{aligned}$$

where $ab\alpha=0$ is the condition that the points $a$, $b$ and the line $\alpha$ may be in the same plane (or, what is the same thing, that the lines $ab$ and $\alpha$ may intersect), viz. $ba\alpha$ is $=P_ax_b+Q_ay_b+R_az_b+S_aw_b$. And similarly $abcd=0$ is the condition that the four points $a$, $b$, $c$, $d$ may be in a plane; viz. we have

$$abcd=\begin{vmatrix} x_a, & y_a, & z_a, & w_a\\ x_b, & y_b, & z_b, & w_b\\ x_c, & y_c, & z_c, & w_c\\ x_d, & y_d, & z_d, & w_d\end{vmatrix}.$$

Substituting, we have $\sqrt{pa\alpha\,.\,pa\beta}\,.\,pbcd$ and $\sqrt{pb\alpha\,.\,pb\beta}\,.\,pcda$, each equal (save as to sign) to $uv\sqrt{ab\alpha\,.\,ab\beta}\,.\,abcd$; that is, the four equal factors of one set will vanish. The vanishing factors are of the form $0^1$, and the norm is $0^4$, that is, the line in question, $ab$, is a 4-tuple line.

(2) The line $\alpha$ is a 4-tuple line; in fact, for any point of the line we have $pa\alpha=0$, $pb\alpha=0$, $pc\alpha=0$, $pd\alpha=0$; each factor of the norm is therefore evanescent, of the form $0^{\frac{1}{2}}$, and the norm itself is thus $=0^4$.

29. (5) The line $(ab, cd, \alpha, \beta)$ is a double line. To show this, take $z=0$, $w=0$ as the equations of the line in question; then we have $h_\alpha=0$, $h_\beta=0$, $z_aw_b-z_bw_a=0$; or say $w_a=\lambda z_a$, $w_b=\lambda z_b$: and $z_cw_d-z_dw_c=0$; or say $w_c=\mu z_c$, $w_d=\mu z_d$ ($\lambda$ and $\mu$ arbitrary coefficients). Putting for shortness

$$I=(g-\lambda a)\,x-(f+\lambda b)\,y,\quad J=(g-\mu a)\,x-(f+\mu b)\,y;$$

viz. $I_\alpha = (g_\alpha - \lambda a_\alpha)x - (f_\alpha + \lambda b_\alpha)y$, &c., and writing $z=0$, $w=0$, we have $pa\alpha \,.\, pa\beta = z_a{}^2 I_\alpha I_\beta$, $pb\alpha \,.\, pb\beta = z_b{}^2 I_\alpha I_\beta$, $pc\alpha \,.\, pc\beta = z_c{}^2 J_\alpha J_\beta$, $pd\alpha \,.\, pd\beta = z_d{}^2 J_\alpha J_\beta$; and the factor of the norm (reverting to the expression thereof as a determinant) is

$$\left|\begin{array}{ccccc} & x\,, & y & & \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & \lambda z_a \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b\,, & z_b\,, & \lambda z_b \\ z_c\sqrt{J_\alpha J_\beta}, & x_c, & y_c\,, & z_c\,, & \mu z_c \\ z_d\sqrt{J_\alpha J_\beta}, & x_d, & y_d, & z_d, & \mu z_d \end{array}\right|$$

which vanishes. In fact, resolving the determinant into a set of products of the form $\pm 2\,.\,13\,.\,45$, where the single symbol denotes a term of the top line, and the binary symbols refer to the second and third lines, and the fourth and fifth lines respectively (denoting minors composed with the terms in these pairs of lines respectively); then each product will contain a term 14, 15, or 45, and the minor so designated (to whichever of the two pairs of lines it belongs) is $=0$. The factor is thus evanescent, being, as it is easy to see, $=0^1$. There are two factors which vanish; viz. taking the first radical to be $+$, the second radical must be also $+$, but the third and fourth radicals may be either both $+$ or both $-$; the norm is thus $=0^2$, viz. the line $(ab, cd, \alpha, \beta)$ is a double line.

30. (8) The line $abc$, $\alpha$, $\beta$ is a double line. To prove this, take $w=0$ for the equation of the plane $abc$, and $(z=0,\ w=0)$ for those of the line in question; we have $h_\alpha = 0$, $h_\beta = 0$, $w_a = 0$, $w_b = 0$, $w_c = 0$; and writing $I_\alpha = -g_\alpha x + f_\alpha y$, $I_\beta = -g_\beta x + f_\alpha y$, then for $z=0$, $w=0$, the factor expressed as a determinant is

$$\left|\begin{array}{ccccc} & x\,, & y & . & . \\ z_a\sqrt{I_\alpha I_\beta}, & x_a, & y_a, & z_a, & . \\ z_b\sqrt{I_\alpha I_\beta}, & x_b, & y_b, & z_b\,, & . \\ z_c\sqrt{I_\alpha I_\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{array}\right|$$

which is

$$= w_d\sqrt{I_\alpha I_\beta}\left|\begin{array}{cccc} . & x\,, & y & . \\ z_a, & x_a, & y_a, & z_a \\ z_b\,, & x_b, & y_b, & z_b \\ z_c\,, & x_c, & y_c, & z_c \end{array}\right|$$

and consequently vanishes, the form being $0^1$. There are two such factors, viz. the radical $\sqrt{pd\alpha\,.\,pd\beta}$ may be either $+$ or $-$, hence the norm is $=0^2$.

31. But it is to be further shown that the line is tacnodal, each sheet of the surface being touched along the line by the plane $w=0$: we have to show that the

{Surface $abcda\beta$.}

factor operated upon by $\Delta = X\delta_x + Y\delta_y + Z\delta_z + W\delta_w$, reduces itself for $z=0$, $w=0$ to a multiple of $W$. Considering the factor in the form of a determinant, the result of the operation is

$$\left|\begin{array}{ccccc} & X, & Y, & Z, & W \\ \sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & . \\ \sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & . \\ \sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{array}\right| + \left|\begin{array}{ccccc} & x\,, & y\,, & . & . \\ \Delta\sqrt{pa\alpha\,.\,pa\beta}, & x_a, & y_a, & z_a, & . \\ \Delta\sqrt{pb\alpha\,.\,pb\beta}, & x_b, & y_b, & z_b, & . \\ \Delta\sqrt{pc\alpha\,.\,pc\beta}, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{array}\right|;$$

the first term is

$$\left|\begin{array}{ccccc} & X, & Y, & Z, & W \\ z_a\sqrt{I_\alpha I_\beta}\,, & x_a, & y_a, & z_a, & . \\ z_b\sqrt{I_\alpha I_\beta}\,, & x_b, & y_b, & z_b, & . \\ z_c\sqrt{I_\alpha I_\beta}\,, & x_c, & y_c, & z_c, & . \\ \sqrt{pd\alpha\,.\,pd\beta}, & x_d, & y_d, & z_d, & w_d \end{array}\right|,$$

where the first column may be replaced by

$$\begin{array}{l} -Z\sqrt{I_\alpha I_\beta} \\ \qquad\vdots \\ \sqrt{pd\alpha\,.\,pd\beta} - z_d\sqrt{I_\alpha I_\beta}, \end{array}$$

and the term in question thus becomes

$$\{w_d Z\sqrt{I_\alpha I_\beta} + W(-z_d\sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha\,.\,pd\beta})\}\,.\,abc,$$

if for shortness

$$\left|\begin{array}{ccc} x_a, & y_a, & z_a \\ x_b, & y_b, & z_b \\ x_c, & y_c, & z_c \end{array}\right| = abc.$$

As regards the second term, we have

$$\Delta\sqrt{pa\alpha\,.\,pa\beta} = \frac{pa\alpha\,.\,\Delta pa\beta + pa\beta\,.\,\Delta pa\alpha}{2\sqrt{pa\alpha\,.\,pa\beta}},$$

which is

$$= \frac{I_\alpha\Delta pa\beta + I_\beta\Delta pa\alpha}{2\sqrt{I_\alpha I_\beta}}.$$

But

$$\begin{aligned} pa\alpha &= x\,(-g_\alpha z_a) + y\,(f_\alpha z_a) + z\,(g_\alpha x_a - f_\alpha y_a) + w\,(a_\alpha x_a - b_\alpha y_a - c_\alpha z_a), \\ &= x_a\,(g_\alpha z - a_\alpha w) + y_a\,(-f_\alpha z - b_\alpha w) + z_a\,(-g_\alpha x + f_\alpha y - c_\alpha w); \end{aligned}$$

and thence

$$\Delta pa\alpha = x_a\,(g_\alpha Z - a_\alpha W) + y_a\,(-f_\alpha Z - b_\alpha W) + z_a\,(-g_\alpha X + f_\alpha Y - c_\alpha W),$$

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with the like formula for $\Delta pa\beta$; hence

$$\frac{I_\alpha \Delta pa\beta + I_\beta \Delta pa\alpha}{2\sqrt{I_\alpha I_\beta}} = Ax_a + By_a + Cz_a,$$

where

$$A = \frac{1}{2\sqrt{I_\alpha I_\beta}} \{I_\beta (\ g_\alpha Z - a_\alpha W) + I_\alpha (\ g_\beta Z - a_\beta W)\},$$

$$B = \frac{1}{2\sqrt{I_\alpha I_\beta}} \{I_\beta (-f_\alpha Z - b_\alpha W) + I_\alpha (-f_\beta Z - b_\beta W)\},$$

$$C = \frac{1}{2\sqrt{I_\alpha I_\beta}} \{I_\beta (-g_\alpha X + f_\alpha Y - c_\alpha W) + I_\alpha (-g_\beta X + f_\beta Y - c_\beta W)\}.$$

The term in question is thus

$$\begin{vmatrix} & x, & y, & . & . \\ Ax_a + By_a + Cz_a, & x_a, & y_a, & z_a & . \\ Ax_b + By_b + Cz_b, & x_b, & y_b, & z_b, & . \\ Ax_c + By_c + Cz_c, & x_c, & y_c, & z_c, & . \\ \Delta\sqrt{pd\alpha\,.\,pd\beta} \quad , & x_d, & y_d, & z_d, & w_d \end{vmatrix}$$

viz. replacing the first column by

$$-Ax - By$$
$$:$$
$$\Delta\sqrt{pd\alpha\,.\,pd\beta} - Ax_d - By_d - Cz_d;$$

this is

$$= (Ax + By)\, w_d\,.\, abc;$$

and we have

$$Ax + By = \frac{1}{2\sqrt{I_\alpha I_\beta}} \begin{matrix} [\ \ I_\beta (\ \ g_\alpha x - f_\alpha y) + I_\alpha (\ \ g_\beta x - f_\beta y)]\, Z \\ [+ I_\beta (-a_\alpha x - b_\alpha y) + I_\alpha (-a_\beta x - b_\beta y)]\, W, \end{matrix}$$

$$= \frac{1}{2\sqrt{I_\alpha I_\beta}} (-2I_\alpha I_\beta Z - MW),$$

if for shortness

$$M = (-g_\beta x + f_\beta y)(a_\alpha x + b_\alpha y) + (-g_\alpha x + f_\alpha y)(a_\beta x + b_\beta y);$$

viz. the whole term is

$$w_d \left\{-\sqrt{I_\alpha I_\beta}\, Z - \frac{\frac{1}{2} M}{\sqrt{I_\alpha I_\beta}}\, W\right\} abc.$$

Hence the first and second terms together are

$$= W\left\{-z_d\sqrt{I_\alpha I_\beta} + \sqrt{pd\alpha\,.\,pd\beta} - \frac{\frac{1}{2} M}{\sqrt{I_\alpha I_\beta}}\, w_d\right\} abc;$$

viz. this is a multiple of $W$, which was the theorem to be proved.

{Surface $abcd\alpha\beta$.}

*Surface* $abc\alpha\beta\gamma$.

32. The equation is

$$\text{Norm}\begin{vmatrix} \sqrt{pa\alpha}, & \sqrt{pb\alpha}, & \sqrt{pc\alpha} \\ \sqrt{pa\beta}, & \sqrt{pb\beta}, & \sqrt{pc\beta} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{vmatrix} = 0,$$

where the norm is a product of 16 factors, each of the order $\frac{3}{2}$. As before, $pa\alpha = 0$ is the equation of the plane through the point $a$ and the line $\alpha$; viz. $pa\alpha$ has the value already mentioned.

33. *Investigation.* In the projection, the equation of the conic touching the projections of the lines $\alpha$, $\beta$, $\gamma$ is

$$A\sqrt{P_\alpha X + Q_\alpha Y + R_\alpha Z} + B\sqrt{P_\beta X + Q_\beta Y + R_\beta Z} + C\sqrt{P_\gamma X + Q_\gamma Y + R_\gamma Z} = 0;$$

and to make this pass through the projection of the point $a$, we must write herein $X : Y : Z = p_a : q_a : r_a$. As before, we have

$$\begin{aligned} P_\alpha X + Q_\alpha Y + R_\alpha Z = &\ \ w_a(x\, P_\alpha + y\, Q_\alpha + z\, R_\alpha) \\ &- w\,(x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha), \\ = &- w\,(x_a P_\alpha + y_a Q_\alpha + z_a R_\alpha + w_a S_\alpha), \\ = &- w\,.\, pa\alpha\,; \end{aligned}$$

and so for the other terms; the equation thus is

$$A\sqrt{pa\alpha} + B\sqrt{pa\beta} + C\sqrt{pa\gamma} = 0;$$

or forming the like equations in regard to the points $b$, $c$ respectively, and eliminating we have a determinant $=0$, and then, taking the norm, we obtain the above-written equation of the surface.

34. *Singularities.* The equation of the surface shows that

(0) The point $a$ is 8-conical: in fact, for the point in question we have $pa\alpha = 0$, $pa\beta = 0$, $pa\gamma = 0$; each factor is $0^{\frac{1}{2}}$, and the norm is $0^8$.

(1) The line $ab$ is 4-tuple. To prove this, observe that the sixteen factors are obtained by attributing at pleasure the signs $+$, $-$ to the radicals $\sqrt{pb\beta}$, $\sqrt{pc\beta}$, $\sqrt{pb\gamma}$, $\sqrt{pc\gamma}$; hence there are four factors in which $\sqrt{pb\beta}$, $\sqrt{pb\gamma}$ have determinate signs, but in which we attribute to the radicals $\sqrt{pc\beta}$, $\sqrt{pc\gamma}$ the signs $+$ or $-$ at pleasure. It is to be shown that the four factors each vanish for a point on the line $ab$; that is, on writing therein for $x$, $y$, $z$, $w$ the values $ux_a + vx_b$, $uy_a + vy_b$, &c. But we thus

{Surface $abc\alpha\beta\gamma$.}

have, as before, $pa\alpha = -v \,.\, ab\alpha$ and $pb\alpha = u \,.\, ab\alpha$, with the like formulæ with $\beta$ and $\gamma$ in place of $\alpha$. The factor thus becomes

$$\sqrt{-uv}\left|\begin{array}{lll} \sqrt{ab\alpha}, & \sqrt{ab\alpha}, & \sqrt{pc\alpha} \\ \sqrt{ab\beta}, & \sqrt{ab\beta}, & \sqrt{pc\beta} \\ \sqrt{ab\gamma}, & \sqrt{ab\gamma}, & \sqrt{pc\gamma} \end{array}\right|,$$

which vanishes, being $= 0^1$; and the norm is thus $= 0^4$, viz. the line is 4-tuple.

(2) The line $\alpha$ is 8-tuple: in fact, for a point on the line we have $pa\alpha = 0$, $pb\alpha = 0$, $pc\alpha = 0$, whence each factor vanishes, being $= 0^{\frac{1}{2}}$, and the norm is therefore $0^8$.

(3) The line $(ab,\ \alpha,\ \beta,\ \gamma)$ is 4-tuple: in fact, writing $z = 0$, $w = 0$ for the equations of the line, we have $h_\alpha = 0$, $h_\beta = 0$, $h_\gamma = 0$, and $z_a w_b - z_b w_a = 0$, or say $w_a = \lambda z_a$, $w_b = \lambda z_b$. Hence, writing

$$I = (g - \lambda a)\, x - (f + \lambda b)\, y,$$

viz. $I_\alpha = (g_\alpha - \lambda a_\alpha)\, x - (f_\alpha + \lambda b_\alpha)\, y$, &c., for $z = 0$, $w = 0$, we have $pa\alpha = z_a I_\alpha$, $pb\alpha = z_b I_\alpha$; and similarly $va\beta = z_a I_\beta$, $pb\beta = z_b I_\beta$, and $pa\gamma = z_a I_\gamma$, $pb\gamma = z_b I_\gamma$. The factor thus is

$$\sqrt{z_a z_b}\left|\begin{array}{lll} \sqrt{I_\alpha}, & \sqrt{I_\alpha}, & \sqrt{pc\alpha} \\ \sqrt{I_\beta}, & \sqrt{I_\beta}, & \sqrt{pc\beta} \\ \sqrt{I_\gamma}, & \sqrt{I_\gamma}, & \sqrt{pc\gamma} \end{array}\right|,$$

which vanishes, being $= 0^1$; there are four such factors, or the norm is $0^4$; whence the line is 4-tuple.

(8) The line $abc \,.\, \alpha \,.\, \beta$ is a 4-tuple line. To prove it, take as before $w = 0$ for the equation of the plane $abc$, and $(z = 0,\ w = 0)$ for the equations of the line in question. We have $h_\alpha = 0$, $h_\beta = 0$, $w_a = 0$, $w_b = 0$, $w_c = 0$; whence (if $z = 0$, $w = 0$), writing for shortness $I = gx - fy$ (viz. $I_\alpha = g_\alpha x - f_\alpha y$, $I_\beta = g_\beta x - f_\beta y$), we have $pa\alpha$, $pb\alpha$, $pc\alpha = I_\alpha z_a$, $I_\alpha z_b$, $I_\alpha z_c$, and similarly $pa\beta$, $pb\beta$, $pc\beta = I_\beta z_a$, $I_\beta z_b$, $I_\beta z_c$: the factor thus is

$$\left|\begin{array}{lll} \sqrt{I_\alpha z_a}, & \sqrt{I_\alpha z_b}, & \sqrt{I_\alpha z_c} \\ \sqrt{I_\beta z_a}, & \sqrt{I_\beta z_b}, & \sqrt{I_\beta z_c} \\ \sqrt{pa\gamma}, & \sqrt{pb\gamma}, & \sqrt{pc\gamma} \end{array}\right|$$

which vanishes, being $= 0^1$: and there are four such factors, obtained by giving to the radicals the signs $+, -$ at pleasure: hence the norm is $= 0^4$.

{Surface $abc\alpha\beta\gamma$.}

*Surface abαβγδ.*

35. The equation is

$$\text{Norm }\{\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta-\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma\}=0,$$

where the norm is the product of 8 factors each of the order 3. As before, $pa\alpha=0$ is the equation of the plane through the point $a$ and the line $\alpha$; viz. $pa\alpha$ has the value previously mentioned: and $p^2\beta\gamma\delta=0$ is the equation of the quadric surface through the lines $\beta$, $\gamma$, $\delta$.

36. *Investigation.* In the projection, taking $\xi$, $\eta$, $\zeta$ as current line-coordinates, the equation of the conic passing through the projections of the points $a$, $b$ is

$$\sqrt{(p_a\xi+q_a\eta+r_a\zeta)(p_b\xi+q_b\eta+r_b\zeta)}+A\xi+F;+C\zeta=0,$$

where $A$, $B$, $C$ are arbitrary coefficients. To make this touch the projection of the line $\alpha$, we must write $\xi:\eta:\zeta=P_\alpha:Q_\alpha:R_\alpha$; and then

$$\begin{aligned}p_a\xi+q_a\eta+r_a\zeta &= p_aP_\alpha+q_aQ_\alpha+r_aR_\alpha,\\ &= w_a(x\,P_\alpha+y\,Q_\alpha+z\,R_\alpha)\\ &\quad -w\,(x_aP_\alpha+y_aQ_\alpha+z_aR_\alpha),\\ &= -w\,(x_aP_\alpha+y_aQ_\alpha+z_aR_\alpha+w_aS_\alpha),\\ &= -w\,.\,pa\alpha,\end{aligned}$$

and similarly

$$p_b\xi+q_b\eta+r_b\zeta=-w\,.\,pb\alpha.$$

Hence the equation is

$$w\sqrt{pa\alpha\,.\,pb\alpha}+AP_\alpha+BQ_\alpha+CR_\alpha=0;$$

and forming the like equations for the lines $\beta$, $\gamma$, $\delta$ respectively, and eliminating, we have

$$\begin{vmatrix}\sqrt{pa\alpha\,.\,pb\alpha}, & P_\alpha, & Q_\alpha, & R_\alpha\\ \sqrt{pa\beta\,.\,pb\beta}, & P_\beta, & Q_\beta, & R_\beta\\ \sqrt{pa\gamma\,.\,pb\gamma}, & P_\gamma, & Q_\gamma, & R_\gamma\\ \sqrt{pa\delta\,.\,pb\delta}, & P_\delta, & Q_\delta, & R_\delta\end{vmatrix}=0;$$

which, throwing out a factor $w$, becomes

$$\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta-\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma=0;$$

or, taking the norm, we have the above written equation.

37. *Singularities.* The equation shows that

(0) The point $a$ is a 4-conical point; in fact, for the point in question we have $pa\alpha=0$, $pa\beta=0$, $pa\gamma=0$, $pa\delta=0$; each factor is $=0^{\frac{1}{2}}$, and the norm is $=0^4$.

{Surface $ab\alpha\beta\gamma\delta$.}

(1) The line $ab$ is a 2-tuple line. To prove this, we have for the coordinates of a point on the line in question $ux_a + vx_b$, $uy_a + vy_b$, &c.; the values of $pa\alpha$, $pb\alpha$ become as before $-v\,.\,ab\alpha$, $+u\,.\,ab\alpha$, and similarly for $pa\beta$, $pb\beta$, &c.; so that, omitting the constant factor $\sqrt{-uv}$, the value of the factor is

$$ab\alpha\,.\,p^2\beta\gamma\delta - ab\beta\,.\,p^2\gamma\delta\alpha + ab\gamma\,.\,p^2\delta\alpha\beta - ab\delta\,.\,p^2\alpha\beta\gamma.$$

Taking (a, b, c, f, g, h) for the coordinates of the line $ab$, we have

$$ab\alpha = \mathrm{a}f_\alpha + \mathrm{b}g_\alpha + \mathrm{c}h_\alpha + \mathrm{f}a_\alpha + \mathrm{g}b_\alpha + \mathrm{h}c_\alpha,$$

with the like expressions for $ab\beta$, &c.; and substituting for $p^2\beta\gamma\delta$, &c., their values, the factor is

| | $x^2$ | $y^2$ | $z^2$ | $w^2$ | $xw$ | $yw$ | $zw$ | $yz$ | $zx$ | $xy$ |
|---|---|---|---|---|---|---|---|---|---|---|
| a | $fagh$ | | | $fabc$ | $fabg - fach$ | $fbch$ | $-fbcg$ | | $fcgh$ | $fbgh$ |
| b | | $gbhf$ | | $gabc$ | $-gach$ | $gbch - gabf$ | $gcaf$ | $gchf$ | $hafg$ | $gahf$ |
| c | | | $hcfg$ | $habc$ | $habg$ | $-habf$ | $hcaf - hbcg$ | $hbfg$ | | |
| f | | $abhf$ | $acfg$ | | | $abch$ | $-abcg$ | $abfg + achf$ | $acgh$ | $abgh$ |
| g | $bagh$ | | $bcfg$ | | $-bach$ | | $bcaf$ | $bchf$ | $bcgh + bafg$ | $bahf$ |
| h | $cagh$ | $cbhf$ | | | $cabg$ | $-cabf$ | | $cbfg$ | $cafg$ | $cahf + cbgh$ |

viz. the value of the factor is $\{\mathrm{a}\,(fagh) + \mathrm{g}\,(bagh) + \mathrm{h}\,(cagh)\}\,x^2 +$ &c., where $fagh = f_\alpha a_\beta g_\gamma h_\delta$ is the determinant

$$\begin{vmatrix} f, & a, & g, & h \\ \vdots & & & \end{vmatrix},$$

the suffixes in the four lines being $\alpha$, $\beta$, $\gamma$, $\delta$ respectively.

Collecting, this is

$$\begin{aligned}
&(\quad . \qquad\ \ cbhfy - bcfgz + fabcw)\,(\quad . \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w)\\
&(-caghx \quad . \qquad +acfgz + gabcw)\,(-\mathrm{h}x \quad . \quad +\mathrm{f}z + \mathrm{b}w)\\
&(+baghx - abhfy \quad . \qquad +habcw)\,(\ \mathrm{g}x - \mathrm{f}y \quad . \quad +\mathrm{c}w)\\
&(-afghx - bfghy - cfghz \qquad . \quad)\,(\ \mathrm{a}x + \mathrm{b}y + \mathrm{c}z \quad . \ )\\
&+ bcgh\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - x\,(\quad . \quad \mathrm{h}y - \mathrm{g}z + \mathrm{a}w)]\\
&+ cahf\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - y\,(-\mathrm{h}x \quad . \quad +\mathrm{f}z + \mathrm{b}w)]\\
&+ abfg\,[w\,(\mathrm{a}x + \mathrm{b}y + \mathrm{c}z) - z\,(\ \mathrm{g}x - \mathrm{f}y \quad . \quad +\mathrm{c}w)] = 0\,;
\end{aligned}$$

or, what is the same thing,

$$AP + BQ + CR + DS = 0,$$

{Surface $abcd\alpha\beta$.}

where

$$\begin{aligned}
A &= \{-bcghx + cbhfy - bcfgz + fabcw\},\\
B &= \{-caghx + cahfy + acfgz + gabcw\},\\
C &= \{+baghx - abhfy + abfgz + habcw\},\\
D &= \{-afghx - bfghy - cfghz + (bcgh + cahf + abfg)\,w\},
\end{aligned}$$

$$\begin{array}{rlllll}
P = ( & . & hy & -\,gz & +\,aw & ),\\
Q = ( & -\,hx & . & +\,fz & +\,bw & ),\\
R = ( & gx & -\,fy & . & +\,cw & ),\\
S = ( & ax & +\,by & +\,cz & & ) = 0,
\end{array}$$

the right-hand factors vanishing for the values $ux_a + vx_b$ of the coordinates.

38. It thus appears so far that the factor is $=0^1$; it is, in fact, $=0^2$, viz. we can show that, operating upon it with

$$\Delta = Xd_x + Yd_y + Zd_z + Wd_w,$$

the value (for any point of the line $ab$) is $=0$. We have

$$\Delta\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta = \frac{pa\alpha\,.\,lb\alpha + pb\alpha\,.\,la\alpha}{2\sqrt{pa\alpha\,.\,pb\alpha}}\,p^2\beta\gamma\delta + \sqrt{pa\alpha\,.\,pb\alpha}\,.\,\Delta\,.\,p^2\beta\gamma\delta,$$

where $lb\alpha\ (=\Delta pb\alpha)$ is what $pb\alpha$ becomes on writing therein $(X, Y, Z, W)$ in place of $(x, y, z, w)$. Writing, as before, for $x, y, z, w$ the values $ux_a + vx_b$, &c., we have $pa\alpha = -v\,.\,ab\alpha$, $pb\alpha = u\,.\,ab\alpha$; and putting for shortness

$$-v\,.\,lb\alpha + u\,.\,la\alpha = lk\alpha, \text{ \&c.},$$

the expression in question, divided by $\sqrt{-uv}$, is

$$\begin{aligned}
&= -2vu\,\{ab\alpha\,.\,\Delta p^2\beta\gamma\delta - \text{\&c.}\}\\
&\quad + \{lk\alpha\,.\ \ p^2\beta\gamma\delta - \text{\&c.}\},
\end{aligned}$$

where, denoting the determinants

$$\left|\begin{array}{cccc} X & Y & Z & W\\ ux_a - vx_b, & uy_a - vy_b, & uz_a - vz_b, & uw_a - vw_b \end{array}\right|$$

by (a′, b′, c′, f′, g′, h′), we have

$$lk\alpha = \mathrm{a}'\mathrm{f}_a + \mathrm{b}'\mathrm{f}_a + \mathrm{c}'\mathrm{g}_a + \mathrm{f}'a_a + \mathrm{g}'b_a + \mathrm{h}'c_a.$$

But $ab\alpha\,.\,\Delta p^2\beta\gamma\delta = \Delta ab\alpha\,.\,p^2\beta\gamma\delta$, since $ab\alpha$ is independent of $(x, y, z, w)$; and the expression is

$$\begin{aligned}
&= -2vu\Delta\,(AP + BQ + CR + DS)\\
&\quad + AP' + BQ' + CR' + DS',
\end{aligned}$$

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where $P'$, $Q'$, $R'$, $S'$ denote $\mathrm{h}'y-\mathrm{g}'z+\mathrm{a}'w$, &c., and where, finally, $x$, $y$, $z$, $w$ are to be replaced by $ux_a+vx_b$, &c. Since for these values $P$, $Q$, $R$, $S$ vanish, the expression becomes

$$=-2vu\,(A\Delta P+B\Delta Q+C\Delta R+D\Delta S)$$
$$+AP'+BQ'+CR'+DS';$$

that is

$$=A\,(P'-2uv\Delta P)+B\,(Q'-2uv\Delta Q)+C\,(R'-2uv\Delta R)+D\,(S'-2uv\Delta S)$$

and we have, in fact, $P'-2uv\Delta P=0$, &c. For, writing for a moment

$$x,\ y,\ z,\ w=ux_a+vx_b,\ uy_a+vy_b,\ uz_a+vz_b,\ uw_a+vw_b,$$
$$x',\ y',\ z',\ w'=ux_a-vx_b,\ uy_a-vy_b,\ uz_a-vz_b,\ uw_a-vw_b;$$

then, for instance,

$$S'=\mathrm{a}'x+\mathrm{b}'y+\mathrm{c}'z,$$

where

$$\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}'=Yz'-Zy',\ Zx'-Xz',\ Xy'-Yx';$$

and thence

$$S'=-\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ x', & y', & z' \end{vmatrix}$$
$$=\ 2uv\,(\mathrm{a}X+\mathrm{b}Y+\mathrm{c}Z)$$
$$=\ 2uv\Delta S;$$

and similarly for the other equations. The factor is thus $=0^2$; there is only one such factor, and the line $ab$ is double.

(2) The line $\alpha$ is an 8-tuple line: in fact, for a point on the line we have $pa\alpha=0$, $pb\alpha=0$, $p^2\gamma\delta\alpha=0$, $p^2\delta\alpha\beta=0$, $p^2\alpha\beta\gamma=0$; and the factor vanishes, being $=0^1$. Each of the factors is $0^1$, and the norm is $=0^8$.

39. (3) The line $[ab,\ \alpha,\ \beta,\ \gamma]$ is a double line. To prove this, observe first that for a point on this line we have $p^2\alpha\beta\gamma=0$.

Taking as before $z=0$, $w=0$ for the equation of the line $ab$, $\alpha$, $\beta$, $\gamma$, we have $h_\alpha=0$, $h_\beta=0$, $h_\gamma=0$, and $z_aw_b-z_bw_a=0$; or say $w_a=\lambda z_a$, $w_b=\lambda z_b$; whence, writing for shortness $I=-(g-\lambda a)\,x+(f+\lambda b)\,y$, viz. $I_\alpha=-(g_\alpha-\lambda a_\alpha)\,x+(f_\alpha+\lambda b_\alpha)\,y$, we have (when $z=0$, $w=0$) $pa\alpha=z_aI_\alpha$, $pb\alpha=z_bI_\alpha$, or omitting the factor $\sqrt{z_az_b}$, $\sqrt{pa\alpha\,.\,pb\alpha}=I_\alpha$; and so for $\sqrt{pa\beta\,.\,pb\beta}$ and $\sqrt{pa\gamma\,.\,pb\gamma}$. The factor thus is

$$I_\alpha\,.\,p^2\beta\gamma\delta-I_\beta\,.\,p^2\gamma\delta\alpha+I_\gamma\,.\,p^2\delta\alpha\beta;$$

viz. writing $z=0$, $w=0$ in the expressions of $p^2\beta\gamma\delta$, &c., this may be written

$$\Sigma\,[(g-\lambda a)\,x-(f+\lambda b)\,y]\,\{(agh)\,x^2+[(ahf)+(bgh)]\,xy+(bhf)\,y^2\},$$

where observe that $\Sigma$ denotes a sum of three terms of the form

$$\alpha\,.\,\beta\gamma\delta-\beta\,.\,\gamma\delta\alpha+\gamma\,.\,\delta\alpha\beta.$$

{Surface $abcda\beta$.}

Adding thereto a fourth term $-\delta\,.\,\alpha\beta\gamma$, the value of the sum would be $=\alpha\beta\gamma\delta$, or the sum of the three terms is $=\alpha\beta\gamma\delta+\delta\,.\,\alpha\beta\gamma$, where the symbols represent determinants. But in each case the determinant $\alpha\beta\gamma$ is $=0$, as containing the column $h_\alpha$, $h_\beta$, $h_\gamma$, the terms of which are each $=0$: thus $\Sigma g\,.\,agh$ is $=gagh-g_\delta\,.\,agh$, where in $gagh$ the suffixes are $\alpha$, $\beta$, $\gamma$, $\delta$, and in $agh$ they are $\alpha$, $\beta$, $\gamma$: that is, we have $\Sigma g\,.\,agh=gagh$. And the whole expression thus is

$$\begin{aligned}
= \;& x^3\;(gagh-\lambda aagh)\\
&+x^2y\,(gahf-\lambda aahf+gbgh-\lambda abgh-fagh-\lambda bagh)\\
&+xy^2\,(gbhf-\lambda abhf\qquad\qquad -fahf-\lambda bahf-f\,bgh-\lambda bbgh)\\
&+y^3\;(\qquad\qquad\qquad\qquad\qquad -f\,hbf-\lambda bbhf),
\end{aligned}$$

where $gahf$ denotes the determinant $\left|\begin{matrix} g,\ a,\ h,\ f\\ \vdots \end{matrix}\right|$, with the suffixes $\alpha$, $\beta$, $\gamma$, $\delta$, in the four lines respectively, and so in other cases: the terms, such as $gagh$, which contain a twice-repeated letter, vanish of themselves; and in the coefficients of $x^2y$ and $xy^2$, the terms which do not separately vanish destroy each other in pairs, $gahf-fagh=0$, &c.; whence the factor vanishes, being $=0^1$; there are two such factors (viz. the zero term $\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma$ may be taken with the sign $+$ or $-$ at pleasure), and the norm is thus $=0^2$.

40. But the line is tacnodal, each sheet of the surface touching along the line in question the hyperboloid $p^2\alpha\beta\gamma$. To prove this, write

$$\Delta=X\delta_x+Y\delta_y+Z\delta_z+W\delta_w\,;$$

we have for the hyperboloid, writing $z=0$, $w=0$,

$$\Delta p^2\alpha\beta\gamma=(afg\,.\,x+bfg\,.\,y)\,Z+(abg\,.\,x-abf\,.\,y)\,W\,;$$

and it is to be shown that

$$\Delta\,(\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta\mp\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma)$$

each contain the factor $\Delta p^2\alpha\beta\gamma$; or, what is the same thing, that

$$\Delta\Sigma\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta$$

contains the factor in question, $\Sigma$ denoting the sum of the first three terms of the original expression. The value is

$$=\Sigma\left(\frac{pa\alpha\,.\,Pb\alpha+pb\alpha\,.\,Pa\alpha}{2\sqrt{pa\alpha\,.\,pb\alpha}}\,p^2\beta\gamma\delta+\sqrt{pa\alpha\,.\,pb\alpha}\,.\,\Delta p^2\beta\gamma\delta\right);$$

where $Pa\alpha$, $=\Delta pa\alpha$, denotes what $pa\alpha$ becomes on writing therein $X$, $Y$, $Z$, $W$ for $x$, $y$, $z$, $w$; and the like as to $Pb\alpha$. Substituting for $pa\alpha$ and $pb\alpha$ their values $z_aI_\alpha$ and $z_bI_\alpha$, and multiplying by $\sqrt{z_az_b}$, the expression is

$$=\Sigma\,\{(z_aPb\alpha+z_bPa\alpha)\,p^2\beta\gamma\delta+2z_az_bI_\alpha\,\Delta p^2\beta\gamma\delta\},$$

{Surface $abcda\beta$.}

where we have

$$z_a P b\alpha + z_b P a\alpha$$

$$= z_b\{x_a(Zg_a - Wa_a) + y_a(-Zf_a - Wb_a) + z_a[X(-g_a+\lambda a_a) + Y(f_a+\lambda b_a) + (\lambda Z - W)c_a]\}$$

$$+ z_a\{x_b(Zg_a - Wa_a) + y_b(-Zf_a - Wb_a) + z_b[X(-g_a+\lambda a_a) + Y(f_a+\lambda b_a) + (\lambda Z - W)c_a]\},$$

$$= (z_b x_a + z_a x_b)(\quad Zg_a - Wa_a)$$

$$+ (z_b y_a + z_a y_b)(-Zf_a - Wb_a)$$

$$+ 2z_a z_b\{X(-g_a+\lambda a_a) + Y(f_a+\lambda b_a) + (\lambda Z - W)c_a\}.$$

Also

$$\begin{aligned} z_a z_b I_a &= \quad z_a z_b\{(-g_a+\lambda a_a)x + (f_a+\lambda b_a)y\},\\ p^2\beta\gamma\delta &= \quad x^2.agh + xy(ahf+bgh) + y^2.hbf,\\ \Delta p^2\beta\gamma\delta &= \quad X.2x.agh + \quad y(ahf+bgh)\\ &\quad + Y.x(ahf+bgh) + 2y.hbf\\ &\quad + Z.x(cgh+afg) + \quad y(bfg+chf)\\ &\quad + W.x(abg-cah) + \quad y(bch-abf). \end{aligned}$$

41. The whole expression is a linear function of $X, Y, Z, W$, and it is easy to see *à priori*, or to verify, that the coefficients of $X, Y$, each of them vanish. The coefficient of $Z$ is

$$= \Sigma\{(z_b x_a + z_a x_b)g_a - (z_b y_a + z_a y_b)f_a + 2\lambda z_a z_b c_a\}\, p^2\beta\gamma\delta$$

$$+ \Sigma z_a z_b[(-g_a+\lambda a_a)x + (f_a+\lambda b_a)y][x(cgh+afg) + y(bfg+chf)],$$

with a like expression for the coefficient of $W$.

The foregoing expression may be written

$$\begin{aligned} &(z_b x_a + z_a x_b)\,\Sigma g[agh.x^2 + (ahf+bgh)xy + bhf.y^2]\\ &-(z_b y_a + z_a y_b)\,\Sigma f[agh.x^2 + (ahf+bgh)xy + bhf.y^2]\\ &+ 2\lambda z_a z_b\,\Sigma\{c[agh.x^2 + (ahf+bgh)xy + bhf.y^2]\\ &\qquad + (ax+by)[(cgh+afg)x + (bfg+chf)y]\}\\ &+ 2z_a z_b\Sigma(-gx+fy)[(cgh+afg)x + (bfg+chf)y]. \end{aligned}$$

The first sum is

$$\begin{aligned} &x^2.gagh + xy(gahf+gbgh) + y^2.gbhf,\\ &= -xy\;.afgh - y^2.bfgh,\\ &= -h_\delta y(afg.x + bfg.y); \end{aligned}$$

where $afg$, $bfg$ denote determinants with the suffixes $\alpha, \beta, \gamma$. Similarly the second sum is

$$= -h_\delta x(afg.x + bfg.y);$$

the third sum is

$$(a_\delta x + b_\delta y)(afg.x + bfg.y),$$

and the fourth sum is

$$(-g_\delta x + f_\delta y)(afg.x + bfg.y).$$

{Surface $abcda\beta$.}

The whole coefficient of $Z$ thus contains the factor $(afg\,.\,x+bfg\,.\,y)$; and similarly it would appear that the whole coefficient of $W$ contains the factor $(abg\,.\,x-abf\,.\,y)$, the other factor being the same in each case; viz. the two terms together are

$$\left\{\begin{aligned}&-(z_bx_a+z_ax_b)\,h_\delta y\\&+(z_by_a+z_ay_b)\,h^\delta x\\&+2\lambda z_az_b(a_\delta x+b_\delta y)\\&+2z_az_b(-g_\delta x+f_\delta y)\end{aligned}\right\}\{Z\,(afg\,.\,x+bfg\,.\,y)+W\,(abg\,.\,x-abf\,.\,y)\};$$

where the second factor is $\Delta p^2\alpha\beta\gamma$, which is the required result. See *post*, Nos. 59 *et seq.*

42. (4) The line $[\alpha, \beta, \gamma, \delta]$ is an 8-tuple line; in fact, for any point of the line in question we have $p^2\beta\gamma\delta=0$, $p^2\gamma\delta\alpha=0$, $p^2\delta\alpha\beta=0$, $p^2\alpha\beta\gamma=0$; whence each factor is $0^1$, or the norm is $0^8$.

I notice that the surface meets the quadric $p^2\alpha\beta\gamma$ in

| | | | | |
|---|---|---|---|---|
| lines | $\alpha, \beta, \gamma$ | each 8 | times | 24 |
| „ | $(\alpha, \beta, \gamma, \delta)$ | „ | „ | 16 |
| „ | $(ab, \alpha, \beta, \gamma)$ | „ 4 | „ | 8 |
| | | | $24\times 2=$ | 48 |

*Surface $a\alpha\beta\gamma\delta\epsilon$.*

43. The equation is

$$(p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3a\alpha\gamma\,.\,\delta\beta+p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3a\alpha\beta\,.\,\gamma\delta)^2$$
$$-4p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\alpha\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,p\alpha a\,.\,p\delta a=0;$$

or, what is the same thing,

$$(p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^3a\alpha\gamma\,.\,\delta\beta-p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^3a\alpha\beta\,.\,\gamma\delta)^2$$
$$-4p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\beta\alpha\delta\,.\,p^2\gamma\alpha\delta\,.\,p\beta a\,.\,p\gamma a=0;$$

the equivalence of the two depending on the identity

$$\begin{aligned}&p^3a\alpha\beta\,.\,\gamma\delta\,.\,p^3a\alpha\gamma\,.\,\delta\beta\\&-p^2\alpha\beta\gamma\,.\,p^2\delta\beta\gamma\,.\,p\alpha a\,.\,p\delta a\\&+p^2\beta\alpha\delta\,.\,p^2\gamma\alpha\delta\,.\,p\beta a\,.\,p\gamma a=0;\end{aligned}$$

where, as before, $p^2\alpha\beta\epsilon=0$ is the equation of the quadric through the lines $\alpha, \beta, \epsilon$, and $p\alpha a=0$ is the equation of the plane through the line $\alpha$ and the point $a$; viz. $p^2\alpha\beta\epsilon$, &c., and $p\alpha a$, &c., have the values already mentioned: $p^3a\alpha\beta\,.\,\gamma\delta=0$ as already mentioned is the cubic surface through the lines $\alpha, \beta, \gamma, \delta$ and $a\alpha\beta$, $a\gamma\delta$.

{Surface $a\alpha\beta\gamma\delta\epsilon$.}

44. *Investigation.* In the projection, using line-coordinates, the equation of the conic touching the five lines may be written

$$\left| \begin{matrix} (\xi, & \eta, & \zeta)^2 \\ (P, & Q, & R)^2 \end{matrix} \right| = 0;$$

where the symbol denotes a determinant the last five lines of which are obtained by giving to $(P, Q, R)$ the suffixes $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ respectively. This is at once transformed into

$$\alpha\beta\epsilon\,.\,\gamma\delta\epsilon\,.\,\alpha\gamma\Delta\,.\,\delta\beta\Delta - \alpha\gamma\epsilon\,.\,\delta\beta\epsilon\,.\,\alpha\beta\Delta\,.\,\gamma\delta\Delta = 0,$$

or, what is the same thing,

$$p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,.\,\alpha\gamma\Delta\,.\,\delta\beta\Delta - p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,\alpha\beta\Delta\,.\,\gamma\delta\Delta = 0;$$

or say

$$\begin{aligned} p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon\,(A''\xi + B''\eta + C''\zeta)\,(A'''\xi + B'''\eta + C'''\zeta) & \\ - p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,(A\xi + B\eta + C\zeta)\,(A'\xi + B'\eta + C'\zeta) &= 0; \end{aligned}$$

where $p^2\alpha\beta\epsilon$, &c., signify as before; and

$$A\xi + B\eta + C\zeta = \left| \begin{matrix} \xi, & \eta, & \zeta \\ P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \end{matrix} \right|,$$

and so for $A'\xi + B'\eta + C'\zeta$, &c., the suffixes for $A'$, $B'$, $C'$ being $(\gamma, \delta)$; and those for $A''\xi + B''\eta + C''\zeta$ and $A'''\xi + B'''\eta + C'''\zeta$ being $(\alpha, \gamma)$ and $(\delta, \beta)$ respectively.

45. Passing to the reciprocal equation, and making the conic pass through the point $a$, we obtain the equation of the surface in the form

$$\begin{aligned} &\left\{ p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon \left| \begin{matrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{matrix} \right| - p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \left| \begin{matrix} p_a, & q_a, & r_a \\ A'', & B'', & C'' \\ A''', & B''', & C''' \end{matrix} \right| \right\}^2 \\ &+ 4p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \left| \begin{matrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A'', & B'', & C'' \end{matrix} \right| \left| \begin{matrix} p_a, & q_a, & r_a \\ A', & B', & C' \\ A''', & B''', & C''' \end{matrix} \right| = 0; \end{aligned}$$

or in the equivalent form, where in the first term we have + instead of −, and in the second term the determinants are

$$\left| \begin{matrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A''', & B''', & C''' \end{matrix} \right|, \quad \left| \begin{matrix} p_a, & q_a, & r_a \\ A', & B', & C' \\ A'', & B'', & C'' \end{matrix} \right|.$$

46. To reduce this result, observe that we have

$$A, B, C = \left\| \begin{matrix} hy - gz + aw, & -hx + fz + bw, & gx - fy + cw \\ h'y - g'z + a'w, & -h'x + f'z + b'w, & g'x - f'y + c'w \end{matrix} \right\|$$

where, for convenience, I retain the unaccented and accented letters $(a,\ldots)$, $(a',\ldots)$ instead of these letters with the suffixes $\alpha$ and $\beta$ respectively. Writing as before

$$\begin{aligned} L &= (af' - a'f)\,x + \ldots \\ M &= (ag' - a'g)\,x + \ldots \\ N &= (ah' - a'h)\,x + \ldots \\ \Omega &= (gh' - g'h)\,x + \ldots \end{aligned}$$

then

$$\begin{aligned} A &= \Omega x - Lw, \\ B &= \Omega y - Mw, \\ C &= \Omega z - Nw, \end{aligned}$$

and similarly

$$\begin{aligned} A' &= \Omega' x - L'w, \\ B' &= \Omega' y - M'w, \\ C' &= \Omega' z - N'w; \end{aligned}$$

where for $L'$, $M'$, $N'$, $\Omega'$ we have $(a'',\ldots)$ and $(a''',\ldots)$. Hence

$$BC' - B'C = w \begin{vmatrix} y, & z, & w \\ M, & N, & \Omega \\ M', & N', & \Omega' \end{vmatrix}$$

with like expressions for $CA' - C'A$ and $AB' - A'B$; and substituting, we have

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = w \begin{vmatrix} p_a, & q_a, & r_a, & \\ x, & y, & z, & w \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

or substituting for $p_a$, $q_a$, $r_a$ their values $xw_a - wx_a$, $yw_a - wy_a$, $zw_a - wz_a$, this is

$$\begin{vmatrix} p_a, & q_a, & r_a \\ A, & B, & C \\ A', & B', & C' \end{vmatrix} = -w^2 \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix}$$

whence, omitting the factors $w^2$, the equation is

$$\left\{p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L', & M', & N', & \Omega' \end{vmatrix} - p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L'', & M'', & N'', & \Omega'' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix}\right\}^2$$

$$+\,4p^2\alpha\gamma\epsilon\,.\,p^2\delta\beta\epsilon\,.\,p^2\alpha\beta\epsilon\,.\,p^2\gamma\delta\epsilon \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L, & M, & N, & \Omega \\ L'', & M'', & N'', & \Omega'' \end{vmatrix} \begin{vmatrix} x, & y, & z, & w \\ x_a, & y_a, & z_a, & w_a \\ L', & M', & N', & \Omega' \\ L''', & M''', & N''', & \Omega''' \end{vmatrix} = 0,$$

{Surface $\alpha\alpha\beta\gamma\delta\epsilon$.}

where I recall that for $(L, \ldots)$, $(L', \ldots)$, $(L'', \ldots)$, $(L''', \ldots)$ the suffixes are $(\alpha, \beta)$, $(\gamma, \delta)$, $(\alpha, \gamma)$, and $(\delta, \beta)$ respectively. The values of the first two determinants thus are $p^3a\alpha\beta \,.\, \gamma\delta$ and $p^3a\alpha\gamma \,.\, \delta\beta$ respectively: that of the third is $p^3a\alpha\beta \,.\, \alpha\gamma$; viz. this is $= p^2\alpha\beta\gamma \,.\, pa\alpha$; similarly, that of the fourth is $p^3a\gamma\delta \,.\, \delta\beta$, which is $= -p^3a\delta\gamma \,.\, \delta\beta = +p^3a\delta\beta \,.\, \delta\gamma$; or finally this is $= p^2\delta\beta\gamma \,.\, pa\delta$. And we have thus the before-mentioned equation of the surface.

47. *Singularities.* The equation of the surface shows that

(0) The point $a$ is a 2-conical point: in fact, we have for this point $p^3a\alpha\beta \,.\, \gamma\delta = 0$, $p^3a\alpha\gamma \,.\, \delta\beta = 0$, $pa\alpha = 0$, $pa\delta = 0$.

(2) The line $\alpha$ is a 4-tuple line: in fact, for any point on this line $p^2\alpha\beta\epsilon = 0$, $p^3a\alpha\beta \,.\, \gamma\delta = 0$, $p^2\alpha\gamma\epsilon = 0$, $p^3a\alpha\gamma \,.\, \delta\beta = 0$, $p^2\alpha\beta\gamma = 0$, $p^2a\alpha = 0$.

(4) The line $(\alpha, \beta, \gamma, \epsilon)$ is a 2-tuple line: in fact, for any point on the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\gamma\epsilon = 0$.

(10) The excuboquartic $\alpha\beta\epsilon \,.\, \gamma\delta \,.\, a$ is a simple curve: in fact, for any point of this curve we have $p^2\alpha\beta\epsilon = 0$, $p^3a\alpha\beta \,.\, \gamma\delta = 0$, these two surfaces intersecting in the lines $\alpha$, $\beta$ and the curve. It is, moreover, obvious that the surface is touched along the curve by the hyperboloid $p^2\alpha\beta\epsilon$.

I notice that the surface meets the quadric $p^2\alpha\beta\gamma$ in

| | | | |
|---|---|---|---|
| lines $(\alpha, \beta, \gamma)$ | each | 4 times, | 12 |
| „ $(\alpha, \beta, \gamma, \delta)$ | „ | twice, | 4 |
| „ $(\alpha, \beta, \gamma, \epsilon)$ | „ | „ | 4 |
| curve $a\alpha\beta\gamma \,.\, \delta\epsilon$ | „ | „ | 8 |
| | | | $14 \times 2 = 28$ |

*Surface $\alpha\beta\gamma\delta\epsilon\zeta$.*

48. The equation of the surface may be written

$$p^2\alpha\beta\epsilon \,.\, p^2\gamma\delta\epsilon \,.\, p^2\alpha\gamma\zeta \,.\, p^2\delta\beta\zeta - p^2\alpha\beta\zeta \,.\, p^2\gamma\delta\zeta \,.\, p^2\alpha\gamma\epsilon \,.\, p^2\delta\beta\epsilon = 0,$$

where $p^2\alpha\beta\epsilon = 0$ is the equation of the quadric through the lines $\alpha$, $\beta$, $\epsilon$; viz. $p^2\alpha\beta\epsilon$ has the value already mentioned.

The form is one of 45 like forms depending on the partitionment

$$\left\{\begin{matrix} \alpha\beta \,.\, \gamma\delta \\ \alpha\gamma \,.\, \delta\beta \\ \alpha\delta \,.\, \beta\gamma \end{matrix}\right\} (\epsilon, \zeta)$$

of the six letters.

{Surface $\alpha\beta\gamma\delta\epsilon\zeta$.}

49. *Investigation.* The projections of the six lines are tangents to a conic: the condition for this is $(P, Q, R)^2 = 0$, where the left-hand side represents the determinant obtained by writing successively $(P_\alpha, Q_\alpha, R_\alpha)$, &c. for $(P, Q, R)$. The equation may be written

$$\alpha\beta\epsilon \,.\, \gamma\delta\epsilon \,.\, \alpha\gamma\zeta \,.\, \delta\beta\zeta - \alpha\beta\zeta \,.\, \gamma\delta\zeta \,.\, \alpha\gamma\epsilon \,.\, \delta\beta\gamma = 0,$$

where

$$\alpha\beta\epsilon = \begin{vmatrix} P_\alpha, & Q_\alpha, & R_\alpha \\ P_\beta, & Q_\beta, & R_\beta \\ P_\epsilon, & Q_\epsilon, & R_\epsilon \end{vmatrix}$$

and substituting for $P_\alpha$, &c., their values, we have $\alpha\beta\epsilon = w \,.\, p^2\alpha\beta\epsilon$; whence the foregoing result.

50. *Singularities.* The equation shows that

(2) The line $\alpha$ is a 2-tuple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\gamma\zeta = 0$, $p^2\alpha\beta\zeta = 0$, $p^2\alpha\gamma\epsilon = 0$.

(4) The line $(\alpha, \beta, \epsilon, \zeta)$ is a simple line: in fact, for each point of the line we have $p^2\alpha\beta\epsilon = 0$, $p^2\alpha\beta\zeta = 0$.

(9) The quadriquadric $\alpha\beta\epsilon \,.\, \gamma\delta\zeta = 0$ is a simple curve on the surface: in fact, for each point of the curve we have $p^2\alpha\beta\epsilon = 0$, $p^2\gamma\delta\zeta = 0$.

It may be remarked that the surface meets the hyperboloid $p^2\alpha\beta\epsilon$ in

| | | | |
|---|---|---|---|
| lines | $(\alpha, \beta, \epsilon)$ | each twice, | 6 |
| „ | $(\alpha, \beta, \epsilon, \gamma)$ | „ once, | 2 |
| „ | $(\alpha, \beta, \epsilon, \delta)$ | „ „ | 2 |
| „ | $(\alpha, \beta, \epsilon, \zeta)$ | „ „ | 2 |
| curve | $\alpha\beta\epsilon \,.\, \gamma\delta\zeta$ | „ „ | 4 |
| | | | $2 \times 8 = 16$ |

51. It might be thought that there should be on the surface some curve $\alpha\beta\gamma\delta\epsilon\zeta$, such as the cubic *abcdef* on the surface *abcdef*; but I cannot find that this is so. The equation of the surface is satisfied if we have simultaneously ($\lambda$ being arbitrary)

$$p^2\alpha\beta\epsilon \,.\, p^2\alpha\gamma\zeta - \lambda p^2\alpha\beta\zeta \,.\, p^2\alpha\gamma\epsilon = 0,$$

$$\lambda p^2\gamma\delta\epsilon \,.\, p^2\delta\beta\zeta - \;p^2\gamma\delta\zeta \,.\, p^2\delta\beta\epsilon = 0\,;$$

which equations represent quartic surfaces, the first of them having $\alpha$ for a double line, and passing through the lines $\beta, \gamma, \epsilon, \zeta$ $(13 + 4 \times 5 = 33$ conditions, so that the equation of such a surface contains only an arbitrary parameter $\lambda)$; and the second having $\delta$ for a double line, and passing through the lines $\beta, \gamma, \epsilon, \zeta$. But I see no condition by which $\lambda$ can be determined so as to have the same value in the two equations respectively. Of course, leaving it arbitrary, the two quartic surfaces intersect in the lines $\beta, \gamma, \epsilon, \zeta$ and in a curve of the order 12 depending on the arbitrary value of $\lambda$, which curve lies on the surface $\alpha\beta\gamma\delta\epsilon\zeta$.

{Surface $\alpha\beta\gamma\delta\epsilon\zeta$.}

*The Excuboquartic* $\alpha\beta\gamma$, $\delta\epsilon$, $a$.

52. The notion is, that we have a fixed point $a$, two fixed lines $\delta$, $\epsilon$, and a singly infinite series of lines, or say the generating lines of a skew surface: each generating line determines, with the point $a$, a plane; and if in this plane we draw, meeting the lines $\delta$, $\epsilon$, a line to meet the generating line in a point $P$, then the locus of this point $P$ is the curve about to be considered.

53. In the case in question, the singly infinite series of lines is that of the lines which meet each of the lines $\alpha$, $\beta$, $\gamma$, or say these are the generatrices of the hyperboloid $\alpha\beta\gamma$: the locus, or curve $\alpha\beta\gamma$, $\delta\epsilon$, $a$, is (as mentioned above) an excuboquartic. It is not necessary for the purpose of the memoir, but it is interesting to consider in conjunction therewith the excuboquartic arising in like manner from the directrices of the hyperboloid; it will appear that the two curves are the complete intersection of the quadric $\alpha\beta\gamma$ by a quartic surface. Observe that the two curves are given as follows: viz. considering for the quadric $\alpha\beta\gamma$ any tangent-plane through the point $a$, and drawing in this plane, to meet the lines $\delta$ and $\epsilon$, a line, this meets the section of the quadric surface by the tangent-plane in two points, the locus of which is the aggregate of the two curves: viz. the section being a line-pair, the two points belong, one of them to a generatrix and the other to a directrix of the quadric surface.

54. It is convenient to take $x=0$, $y=0$ for the equations of the line $\delta$; $z=0$, $w=0$ for those of the line $\epsilon$: for then, for any plane $Ax+By+Cz+Dw=0$, the line in this plane and meeting the lines $\delta$ and $\epsilon$, has for its equations $Ax+By=0$, $Cz+Dw=0$; or, what is the same thing, for the plane $P=0$ the equations of the line are $P_{xy}=0$, $P_{zw}=0$, where $P_{xy}$, $P_{zw}$ denote the terms in $x$, $y$ and in $z$, $w$ respectively.

I take also $x_0$, $y_0$, $z_0$, $w_0$ for the coordinates of the point $a$, and $PS-QR=0$ for the equation of the quadric surface, $P$, $Q$, $R$, $S$ being given linear functions of $(x, y, z, w)$: we have then say $P-\theta R=0$, $Q-\theta S=0$ for the equations of any generatrix, and $P-\phi Q=0$, $R-\phi S=0$ for the equations of any directrix of the hyperboloid.

The equation of the plane through the point $a$ and the generatrix $P-\theta R=0$, $Q-\theta S=0$, is clearly

$$(Q_0-\theta S_0)(P\ \ -\theta R\ \ )-(P_0-\theta R_0)(Q\ \ -\theta S\ \ )=0;$$

so that for the line in this plane, meeting the lines $\delta$ and $\epsilon$, we have

$$(Q_0-\theta S_0)(P_{xy}-\theta R_{xy})-(P_0-\theta R_0)(Q_{xy}-\theta S_{xy})=0,$$
$$(Q_0-\theta S_0)(P_{zw}-\theta R_{zw})-(P_0-\theta R_0)(Q_{zw}-\theta S_{zw})=0;$$

and joining thereto the equations

$$\theta=\frac{P}{R}=\frac{Q}{S}=\frac{P_{xy}+P_{zw}}{R_{xy}+R_{zw}}=\frac{Q_{xy}+Q_{zw}}{R_{xy}+R_{zw}},$$

(equivalent in all to three equations,) the elimination of $\theta$ gives the required curve: the equations thus are

$$PS-QR=0,$$
$$(Q_0S-QS_0)(P_{xy}R-PR_{xy})-(P_0R-PR_0)(Q_{xy}S-QS_{xy})=0,$$

or, as the second equation may also be written,

$$(Q_0 S - QS_0)(P_{xy}R_{zw} - P_{zw}R_{xy}) - (P_0 R - PR_0)(Q_{xy}S_{zw} - Q_{zw}S_{xy}) = 0\,;$$

viz. the second equation represents a cubic surface having upon it the lines $(P=0, R=0)$ and $(Q=0, S=0)$: it therefore intersects the quadric $PS - QR = 0$ in these two lines, and besides in an excuboquartic curve, which is the required locus.

55. Representing the determinants

$$\left|\begin{matrix} P\ , & Q\ , & R\ , & S \\ P_0\ , & Q_0\ , & R_0\ , & S_0 \end{matrix}\right| \text{ by } (\text{a}', \text{b}', \text{c}', \text{f}', \text{g}', \text{h}'), \text{ viz. } \text{a}' = QR_0 - Q_0R, \ldots \quad \text{f}' = PS_0 - P_0S, \ldots;$$

$$\left|\begin{matrix} P_{xy}, & Q_{xy}, & R_{xy}, & S_{xy} \\ P_{zw}, & Q_{zw}, & R_{zw}, & S_{zw} \end{matrix}\right| \text{ by } (\text{a}, \text{b}, \text{c}, \text{f}, \text{g}, \text{h}), \text{ viz. } \text{a} = Q_{xy}R_{zw} - Q_{zw}R_{xy}, \ldots;$$

so that $(\text{a}', \ldots)$ are linear functions, $(\text{a}, \ldots)$ quadric functions, of the coordinates; the equation of the cubic surface is $\text{gb}' - \text{bg}' = 0$, viz. the excuboquartic arising from the generatrices is the partial intersection of the quadric $PS - QR = 0$ and the cubic $\text{gb}' - \text{g}'\text{b} = 0$; the two surfaces besides intersecting in the lines $(P=0, R=0)$ and $(Q=0, S=0)$.

It appears, in the same manner, that the excuboquartic arising from the directrices is the partial intersection of the quadric $PS - QR = 0$ and the cubic $\text{hc}' - \text{ch}' = 0$; the two surfaces besides intersecting in the lines $(P=0, Q=0)$ and $(R=0, S=0)$.

56. But the elimination may be performed in a different manner, as follows: from the first two equations in $\theta$, multiplying by $P_{zw}, -P_{xy}$ and adding, and so with $Q_{zw}, -Q_{xy}$, &c., we obtain

$$\begin{aligned}
(Q_0 - \theta S_0)(\quad\ - \theta\text{b}) &- (P_0 - \theta R_0)(-\text{c} + \theta\text{f}) = 0, \\
(Q_0 - \theta S_0)(\ \text{c} + \theta\text{a}) &- (P_0 - \theta R_0)(\quad\ \theta\text{g}) = 0, \\
(Q_0 - \theta S_0)(-\text{b} \quad\ ) &- (P_0 - \theta R_0)(\ \text{a} + \theta\text{h}) = 0, \\
(Q_0 - \theta S_0)(\ \text{f} - \theta\text{h}) &- (P_0 - \theta R_0)(\ \text{g} \quad\ ) = 0.
\end{aligned}$$

We then have

$$\theta = \frac{-\text{c} + \theta\text{f}}{\text{a} + \theta\text{h}} = \frac{\text{c} + \theta\text{a}}{\text{f} - \theta\text{h}},$$

or, what is the same thing,

$$\text{h}\theta^2 + (\text{a} - \text{f})\,\theta + \text{c} = 0.$$

Using this equation, written in the form $(\text{a} + \theta\text{h})\,\theta = -\text{c} + \theta\text{f}$, to transform the first or third of the four equations in $\theta$, we obtain

$$-\text{a}P_0 - \text{b}Q_0 - \text{c}R_0 + \theta\,(-\text{h}P_0 \quad . \ + \text{f}R_0 + \text{b}S_0) = 0\,;$$

and using the same equation, written in the form $(\text{f} - \theta\text{h})\,\theta = \text{c} + \theta\text{a}$, to transform the second or fourth equation, we obtain

$$\text{g}P_0 - \text{f}Q_0 \ + \text{c}S_0 + \theta\,(\quad\ \text{h}Q_0 - \text{g}R_0 + \text{a}S_0) = 0\,;$$

and hence, eliminating $\theta$, we obtain

$$(\mathrm{h}Q_0 - \mathrm{g}R_0 + \mathrm{a}S_0)(-\mathrm{a}P_0 - \mathrm{b}Q_0 - \mathrm{c}R_0) - (-\mathrm{h}P_0 + \mathrm{f}R_0 + \mathrm{b}S_0)(\mathrm{g}P_0 - \mathrm{f}Q_0 . + \mathrm{c}S_0) = 0,$$

which, as being of the second order in (a, ...), represents a quartic surface. The equation remains unaltered by the interchange of $Q$, $R$, and the consequent interchanges among (a, b, c, f, g, h): hence the quartic surface contains not only the excuboquartic arising from the generatrices, but also that arising from the directrices; and these two curves are the complete intersection of the quartic by the quadric $PS - QR = 0$.

57. I obtain this same result also as follows. Consider a point $(P_1, Q_1, R_1, S_1)$ on the quadric surface; $P_1S_1 - Q_1R_1 = 0$; the tangent plane at the point is

$$PS_1 - QR_1 - RQ_1 + SP_1 = 0;$$

and if this passes through the point $a$, then

$$P_0S_1 - Q_0R_1 - R_0Q_1 + S_0P_1 = 0.$$

The line which in the tangent-plane meets the lines $\delta$, $\epsilon$ is given, as before, by the equations

$$P_{xy}S_1 - Q_{xy}R_1 - R_{xy}Q_1 + S_{xy}P_1 = 0,$$
$$P_{zw}S_1 - Q_{zw}R_1 - R_{zw}Q_1 + S_{zw}P_1 = 0.$$

Remembering the significations of (a, ...), the last three equations give

$$\begin{aligned} S_1 : R_1 : -Q_1 : -P_1 = & \quad . \quad \mathrm{h}Q_0 - \mathrm{g}R_0 + \mathrm{a}S_0 \\ & : -\mathrm{h}P_0 \quad . \quad + \mathrm{f}R_0 + \mathrm{b}S_0 \\ & : \mathrm{g}P_0 - \mathrm{f}Q_0 \quad . \quad + \mathrm{c}S_0 \\ & : -\mathrm{a}P_0 - \mathrm{b}Q_0 - \mathrm{c}R_0 \quad . \quad ; \end{aligned}$$

and substituting these values in $S_1P_1 - Q_1R_1 = 0$, we have the above equation of the quadric surface.

58. Or again, changing the notation, I take the equation of the quadric surface to be

$$(a, b, c, d, f, g, h, l, m, n \between x, y, z, w)^2 = 0.$$

A tangent-plane hereof is

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

where $\xi$, $\eta$, $\zeta$, $\omega$ are any quantities satisfying the relation

$$(A, B, C, D, F, G, H, L, M, N \between \xi, \eta, \zeta, \omega)^2 = 0,$$

the capitals denoting the inverse coefficients.

Supposing that the tangent-plane passes through a fixed point $a$, coordinates $(\alpha, \beta, \gamma, \delta)$, we have

$$\alpha\xi + \beta\eta + \gamma\zeta + \delta\omega = 0;$$

and if the equations of the lines $\delta$, $\epsilon$ are as before $(x=0,\ y=0)$ and $(z=0,\ w=0)$; then for the line in the tangent-plane meeting the lines $\delta$, $\epsilon$, we have

$$\xi x+\eta y=0,\quad \zeta z+\omega w=0.$$

These last equations may be represented by

$$\xi=ly,\quad \eta=-lx,\quad \zeta=mw,\quad \omega=-mz;$$

and, substituting these values, we have

$$(A,\ldots\between ly,\ -lx,\ mw.\ -mz)^2=0,$$
$$(\alpha\ ,\ldots\between ly,\ -lx,\ mw,\ -mz)^1=0,$$

that is

$$(Ay^2-2Hxy+Bx^2,\ -Fxw+Gyw-Lyz+Mxz,\ Cw^2-2Nwz+Dz^2\between l,\ m)^2=0,$$

and

$$(\alpha y-\beta x,\ \gamma w-\delta z\between l,\ m)=0.$$

Whence, eliminating $l$, $m$, we have the quartic equation

$$(Ay^2-2Hxy+Bx^2,\quad -Fxw+Gyw-Lyz+Mxz,\ Cw^2-2Nzw+Dz^2\between \gamma w-\delta z,\ \beta x-\alpha y)^2=0.$$

*Further Investigation as to the Surface $ab\alpha\beta\gamma\delta$.*

59. The theorem that in the surface $ab\alpha\beta\gamma\delta$, the equation of which is

$$\text{Norm}\ \{\sqrt{pa\alpha\,.\,pb\alpha}\,.\,p^2\beta\gamma\delta-\sqrt{pa\beta\,.\,pb\beta}\,.\,p^2\gamma\delta\alpha+\sqrt{pa\gamma\,.\,pb\gamma}\,.\,p^2\delta\alpha\beta-\sqrt{pa\delta\,.\,pb\delta}\,.\,p^2\alpha\beta\gamma\}=0;$$

the lines $(ab,\ \alpha,\ \beta,\ \gamma)$ are tacnodal, each sheet touching along the line the quadric $p^2\alpha\beta\gamma$, may be proved in a different manner by investigating the intersection of the surface with the quadric $p^2\alpha\beta\gamma$.

For this purpose take the equation of the quadric to be $yz-xw=0$; the equations of the lines $\alpha$, $\beta$, $\gamma$ will be

$$\begin{pmatrix} z-\lambda_\alpha w=0\\ x-\lambda_\alpha y=0\end{pmatrix},\ \begin{pmatrix} z-\lambda_\beta w=0\\ x-\lambda_\beta y=0\end{pmatrix},\ \begin{pmatrix} z-\lambda_\gamma w=0\\ x-\lambda_\gamma y=0\end{pmatrix};$$

and we may write $(a,\ b,\ c,\ f,\ g,\ h)$ for the coordinates of the line $\delta$. The equation of the surface will be

$$\text{Norm}\left\{\Sigma\left[\pm\sqrt{pa\alpha\,.\,pb\alpha}\,(\lambda_\beta-\lambda_\gamma)\left\{\begin{aligned}&(a-f)\,xz-(\lambda_\beta+\lambda_\gamma)\,yz+\lambda_\beta\lambda_\gamma yw\\ &+(b-g)\,\lambda_\beta\lambda_\gamma\,(yz-xw)\\ &\qquad+c\,(z-\lambda_\beta w)\,(z-\lambda_\gamma w)\\ &\qquad+h\,(x-\lambda_\beta y)\,(x-\lambda_\gamma z)\end{aligned}\right\}\right]\right.$$

$$\left.-\sqrt{pa\delta\,.\,pb\delta}\,(\lambda_\beta-\lambda_\gamma)\,(\lambda_\gamma-\lambda_\alpha)\,(\lambda_\alpha-\lambda_\beta)\,(yz-xw)\right\};$$

where $\Sigma$ denotes the sum of the three terms obtained by the cyclical interchange of $\alpha$, $\beta$, $\gamma$; and

$$pa\alpha = (z_a - \lambda w_a)(x - \lambda y) - (x_a - \lambda y_a)(z - \lambda w),$$

$$pb\alpha = (z_b - \lambda w_b)(x - \lambda y) - (x_b - \lambda y_b)(z - \lambda w);$$

$\lambda$ here standing for $\lambda_\alpha$; and similarly for $pa\beta$, &c.

60. To obtain the intersection with $xw - yz = 0$, writing $w = \frac{yz}{x}$, then

$$pa\alpha = [z_a - \lambda w_a - \frac{z}{x}(x_a - \lambda y_a)](x - \lambda y), \quad (\lambda = \lambda_\alpha),$$

$$pb\alpha = [z_b - \lambda w_b - \frac{z}{x}(x_b - \lambda y_b)](x - \lambda y);$$

or say

$$\sqrt{pa\alpha \,.\, pb\alpha} = \sqrt{M_\alpha}\,(x - \lambda_\alpha y);$$

also the expression in { } becomes

$$= \{(a - f)\frac{z}{x} + c\frac{z^2}{x^2} + h\}(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

so that the norm in question is

$$\text{Norm}\ \Sigma\sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma)\{(a - f)\frac{z}{x} + c\frac{z^2}{x^2} + h\}(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

or say

$$\text{Norm}\ \Sigma\sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma)\{hx^2 + (a - f)zx + cz^2\}(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y);$$

where $M_\alpha$ is now considered to stand for

$$\{(z_a x - zx_a) - \lambda(w_a x - y_a z)\}\{(z_b x - zx_b) - \lambda(w_b x - y_b z)\}.$$

Observing that the norm was originally the product of 8 factors, this breaks up into

$$\{hx^2 + (a - f)zx + cz^2\}^8\{(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y)\}^8 = 0,$$

and

$$\text{Norm}^2\ \sqrt{M_\alpha}\,(\lambda_\beta - \lambda_\gamma) = 0,$$

where the new norm is the product of 4 factors.

61. Writing for greater convenience $\lambda$, $\mu$, $\nu$ in place of $\lambda_\alpha$, $\lambda_\beta$, $\lambda_\gamma$, and observing that $M_\alpha$ is a quadric function of $\lambda_\alpha$, that is of $\lambda$, the last-mentioned norm is

$$\text{Norm}\ \sqrt{A + B\lambda + C\lambda^2}\,(\mu - \nu),$$

which is easily seen to be

$$= (4AC - B^2)(\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2;$$

or writing for a moment

$$(A + B\lambda + C\lambda^2) = (P - Q\lambda)(P' - Q'\lambda),$$

whence

$$A = PP', \quad B = -(PQ' + P'Q), \quad C = QQ';$$

then

$$4AC - B^2 = -(PQ' - P'Q)^2;$$

and we have

$$P, Q = z_a x - z x_a, \quad w_a x - y_a z,$$
$$P', Q' = z_b x - z x_b, \quad w_b x - y_b z,$$

whence

$$\begin{aligned} PQ' - P'Q = \ & (z_a w_b - z_b w_a)\, x^2 \\ & + [y_a z_b - y_b z_a - (x_a w_b - x_b w_a)]\, xz \\ & + (x_a y_b - x_b y_a)\, z^2; \end{aligned}$$

viz. if (a, b, c, f, g, h) are the coordinates of the line $ab$, this is

$$= \mathrm{h}x^2 + (\mathrm{a} - \mathrm{f})\, xz + \mathrm{c}z^2.$$

Hence, omitting the constant factor $(\mu - \nu)^4 (\nu - \lambda)^4 (\lambda - \mu)^4$ {that is $(\lambda_\beta - \lambda_\gamma)^4 (\lambda_\gamma - \lambda_\alpha)^4 (\lambda_\alpha - \lambda_\beta)^4$}, the foregoing equation $\text{norm}^2 = 0$ becomes

$$[\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f})\, xz + \mathrm{c}z^2]^4 = 0,$$

and the intersections of the quadric with the surface are obtained by combining the equation $xw - yz = 0$ with the several equations

$$\{\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f})\, zx + \mathrm{c}z^2\}^8 = 0,$$
$$\{(x - \lambda_\alpha y)(x - \lambda_\beta y)(x - \lambda_\gamma y)\}^8 = 0,$$
$$\{\mathrm{h}x^2 + (\mathrm{a} - \mathrm{f})\, zx + \mathrm{c}z^2\}^4 = 0;$$

viz. these are

| | | | | |
|---|---|---|---|---|
| lines $(\alpha, \beta, \gamma, \delta)$ | each 8 times | 16 | | |
| line $(x = 0, z = 0)$ | 16 | „ | | 16 |
| lines $\alpha, \beta, \gamma$ | each 8 | „ | 24 | |
| line $(x = 0, y = 0)$ | 24 | „ | | 24 |
| lines $[ab, \alpha, \beta, \gamma]$ | each 4 | „ | 8 | |
| line $(x = 0, z = 0)$ | 8 | „ | | 8 |
| | | $(16 + 24 + 8) \times 2 =$ | $48$ | $+ 48$ |

But it is clear that the lines $(x = 0, y = 0)$ and $(x = 0, z = 0)$ are introduced by the process of elimination, and are no part of the intersection. The complete intersection consists of the lines $(\alpha, \beta, \gamma, \delta)$ each 8 times, the lines $(\alpha, \beta, \gamma)$ each 8 times, and the lines $[ab, \alpha, \beta, \gamma]$ each 4 times. But the last-mentioned lines being only double lines on the surface, this means that the two sheets each touch the quadric surface, or that the lines are tacnodal.