500.

ON A THEOREM RELATING TO EIGHT POINTS ON A CONIC.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 344—346.]

THE following is a known theorem:

"In any octagon inscribed in a conic, the two sets of alternate sides intersect in the 8 points of the octagon and in 8 other points lying in a conic."

In fact the two sets of sides are each of them a quartic curve, hence any quartic curve through 13 of the $8+8$ points passes through the remaining 3 points: but the original conic together with a conic through 5 of the 8 new points form together such a quartic curve; and hence the remaining 3 of the new points (inasmuch as obviously they are not situate on the original conic) must be situate on the conic through the 5 new points, that is the 8 new points must lie on a conic.

We may without loss of generality take $(\alpha_1^2, \alpha_1, 1)$, $(\alpha_2^2, \alpha_2, 1)$, ... $(\alpha_8^2, \alpha_8, 1)$, as the coordinates (x, y, z) of the 8 points of the octagon; and obtain hereby an *à posteriori* verification of the theorem, by finding the equation of the conic through the 8 new points: the result contains cyclical expressions of an interesting form.

Calling the points of the octagon 1, 2, 3, 4, 5, 6, 7, 8, the 8 new points are

$$12.45,\quad 23.56,\quad 34.67,\quad 45.78,\quad 56.81,\quad 67.12,\quad 78.23,\quad 81.34,$$

viz. 12.45 is the intersection of the lines 12 and 45; and so on. The 8 points lie on a conic, the equation of which is to be found.

The equation of the line 12 is

$$x-(\alpha_1+\alpha_2)\,y+\alpha_1\alpha_2 z=0,$$

or as it is convenient to write it

$$x-(1+2)\,y+12\,.\,z=0,$$

viz. 1, 2, &c., are for shortness written in place of α_1, α_2, &c. respectively.

The coordinates of the 12.45 are consequently proportional to the terms of

$$1,\ -(1+2),\ 12,$$
$$1,\ -(4+5),\ 45,$$

or say they are as

$$12(4+5)-45(1+2) : 12-45 : 1+2-(4+5).$$

The equation of the line (12.45)(23.56) which joins the points 12.45 and 23.56 thus is

$$\begin{vmatrix} x, & y, & z \\ 12(4+5)-45(1+2), & 12+45, & (1+2)-(4+5) \\ 23(5+6)-56(2+3), & 23-56, & (2+3)-(5+6) \end{vmatrix}=0,$$

where the determinant vanishes identically if $2-5=0$ $(\alpha_2-\alpha_5=0)$; it in fact thereby becomes

$$\begin{vmatrix} x, & y, & z \\ 2^2(1-4), & 2(1-4), & (1-4) \\ 2^2(3-6), & 3(3-6), & (3-6) \end{vmatrix},$$

which is $=0$; the determinant divides therefore by $2-5$; the coefficient of x is easily found to be

$$=(2-5)(12-23+34-45+56-61),$$

and so for the other terms; and omitting the factor $2-5$ the equation is

$$\begin{aligned} &x\{12-23+34-45+56-61\} \\ &-y\{12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)\} \\ &+z\{1234-2345+3456-4561+5612-6123\}=0. \end{aligned}$$

There is now not much difficulty in forming the equation of the required conic; viz. this is

$$\begin{aligned} &(2-8)\{x-(6+7)y+67z\}\times \\ &\left\{\begin{aligned} &x[12-23+34-45+56-61] \\ &-y[12(4+5)-23(5+6)+34(6+1)-45(1+2)+56(2+3)-61(3+4)] \\ &+z[1234-2345+3456-4561+5612-6123] \end{aligned}\right\} \\ &+(6-8)\{x-(1+2)y+12z\}\times \\ &\left\{\begin{aligned} &x[23-34+45-56+67-72] \\ &-y[23(5+6)-34(6+7)+45(7+2)-56(2+3)+67(3+4)-72(4+5)] \\ &+z[2345-3456+4567-5672+6723-7234] \end{aligned}\right\} \\ &=0. \end{aligned}$$

In fact this equation written with an indeterminate coefficient λ, say, for shortness, thus

$$67\,[(12\,.\,45)(23\,.\,56)] = \lambda 12\,.\,[(23\,.\,56)(34\,.\,67)] = 0,$$

is the general equation of the conic through the 4 points 12.67, 34.67, 12.45, and 23.56; and by making the conic pass through 1 of the remaining 4 of the 8 points, I succeeded in finding the value $\lambda = \frac{6-8}{2-8}$, so that the conic in question passes through 5 of the 8 points, and is therefore by the theorem the conic through the 8 points. But as thus written down the equation contains the extraneous factor $2-6$, as appears at once by the observation that the left-hand side on writing therein $6=2\,(\alpha_6=\alpha_2)$ becomes identically $=0$; the value in fact is

$$-(2-8)\,[x-(2+7)\,y+27z]\,(23-34+45-52)\,[x-(1+2)\,y+12z]$$
$$+(2-8)\,[x-(1+2)\,y+12z]\,(23-34+45-52)\,[x-(2+7)\,y+27z]$$

which is $=0$; there is consequently the factor $2-6$ to be rejected, and throwing this out the equation assumes a symmetrical form in regard to the 8 symbols 1, 2, 3, 4, 5, 6, 7, 8. The coefficient of x^2 is very easily found to be

$$=(2-6)\,(12-23+34-45+56-67+78-81),$$

and similarly that of z^2 to be

$$=(2-6)\,\{123456-234567+345678-456781+567812-678123+781234-812345\}:$$

those of the other terms are somewhat more difficult to calculate; but the final result, throwing out the factor $(2-6)$, and introducing an abbreviated notation

$$\Sigma 12 = (12-23+34-45+56-67+78-81),$$

and the like in other cases, is found to be

$$\begin{aligned}
&x^2\,.\;\Sigma 12\\
&+y^2\,.\;[\Sigma 12\,(4+5)\,(6+7)-\tfrac{1}{2}\,\Sigma 1256]\\
&+z^2\,.\;\Sigma 123456\\
&-yz\,.\;\Sigma 16\,(234+235+245+345)\\
&+zx\,.\;[\Sigma 1234 \qquad\qquad +\tfrac{1}{2}\,\Sigma 1256]\\
&-xy\,.\;\Sigma 12\,(4+5+6+7) \qquad\qquad =0,
\end{aligned}$$

where it is to be observed that $\Sigma 1256$ consists of 4 distinct terms each twice repeated: $\frac{1}{2}\Sigma 1256$ consists therefore of these 4 terms; and in the coefficient of y^2 they destroy 4 of the 32 terms of $\Sigma 12\,(4+5)\,(6+7)$ so that the coefficient of y^2 contains $32-4$, $=28$ terms. In the coefficient of zx there is no destruction, and this contains therefore $12+4$, $=16$ terms.