499.

ON THE THEORY OF THE CURVE AND TORSE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 294—317.]

THE fundamental relations in the theory of the Curve and Torse were first established in my "Mémoire sur les Courbes à double courbure et sur les Surfaces developpables," *Liouv.* t. X. (1845), [30] see also *Camb. and Dubl. Math. Jour.*, t. IV. (1850), [83], viz. I showed that the systems (m, r, β, h, n, y) and (r, n, m, x, α, g), (the notation is subsequently explained), each of them satisfied the Plückerian relations. An additional set of equations giving the values of $(\gamma, t, k, q, \gamma', t', k', q')$ was furnished by Dr Salmon's "Theory of Reciprocal Surfaces," *Trans. R. I. Acad.*, t. XXIII. (1857), see also the *Solid Geometry*. The theory as thus established is complete in itself, but it does not take account of certain singularities υ, ω, H, G; the singularity υ was first considered in my paper "On a special sextic Developable," *Quart. Math. Jour.* t. VII. (1865), [373], (there called θ), and I afterwards endeavoured to take account of the remaining singularities ω, H, G. I was in correspondence on the subject with Prof. Cremona, and the discovery of the complete forms of several of the formulæ is due to him.

There has recently appeared a very valuable memoir by M. Zeuthen, "Sur les singularités ordinaires d'une courbe gauche et d'une surface developpable," *Annali di Matem.* t. III. (1869); he excludes, however, from consideration the singularity ω, and does not throughout attend to H, G.

I propose in the present memoir to reproduce and develope the whole theory.

Explanations and Notation.

1. We have a singly infinite series of points, lines, and planes; viz. each line passes through two consecutive points and lies in two consecutive planes; each plane passes through three consecutive points and contains two consecutive lines; each point

is the intersection of three consecutive planes and of two consecutive lines. The points describe and the lines envelope a *curve*; the lines describe and the planes envelope a *torse*; the entire system of points, lines, and planes is thus the system of the curve and torse. The curve is the edge of regression or cuspidal curve of the torse; in regard to the curve the points are points (or ineunts), the lines tangents, and the planes osculating planes: in regard to the torse the points are points on the edge of regression, the lines generating lines, and the planes tangent planes.

2. Each line of the system is met by a certain number of non-consecutive lines, and the locus of the points of intersection (or say the locus of the intersection of two intersecting non-consecutive lines) is a nodal curve on the torse, or say simply it is the *nodal curve.* The plane containing the two intersecting non-consecutive lines envelopes a torse which is called the *nexal torse.*

3. There is occasion to consider

m, order of the system; this is the number o points of the system which lie in a given plane; or it is the order of the curve.

r, rank of the system; this is the number of lines of the system which meet a given line. It is thus the class of the curve; and the order of the torse.

n, class of the system; this is the number of planes of the system which pass through a given point. It is thus the class of the torse.

α, number of stationary planes; that is, planes each passing through four consecutive points of the system.

β, number of stationary points, that is, points each of them the intersection of four consecutive planes of the system.

g, number of *lines in two planes* (that is, lines each of them the intersection of two non-consecutive planes of the system) contained in a given plane; or say, number of apparent double planes of the torse.

G, number of double planes, or *tropes*, of the torse; viz. considering the torse as the envelope of a variable plane, if the plane in the course of its motion comes twice into the same position, we have then a double plane or trope.

h, number of *lines through two points* (that is, lines each through two non-consecutive points of the system) passing through a given point; or say, number of apparent double points of the curve.

H, number of double points, or *nodes* of the curve; viz. considering the curve as described by a variable point, if the point in the course of its motion comes twice into the same position we have then a double point or node of the curve.

x, number of *points in two lines* (that is, points each of them the intersection of two non-consecutive lines of the system) contained in a given plane; or what is the same thing, order of nodal curve.

y, number of *planes through two lines* (that is, planes each containing two non-consecutive lines of the system) passing through a given point; or what is the same thing, class of the nexal torse.

ν, number of stationary lines of the system, that is, lines each containing three consecutive points of the system.

ω, number of double lines of the system, that is, lines each containing two pairs of consecutive points of the system.

t, number of *points on three lines* (that is, points each of them the common intersection of three non-consecutive lines of the system): these are also triple points on the curve.

γ, number of points of the system, through each of which passes a non-consecutive line of the system: these are intersections of the curve with the nodal curve, stationary points on the latter curve.

k, number of apparent double points of nodal curve.

q, class of nodal curve.

t', number of *planes through three lines* (that is, planes each of them through three non-consecutive lines of the system): these are also triple tangent planes of the torse.

γ', number of planes of the system each of them passing through a non-consecutive line of the system: these are common tangent planes of the torse and nexal torse, stationary planes of the latter torse.

k', number of apparent double planes of nexal torse.

q', order of nexal torse.

4. The formulæ thus contains in all the 21 quantities

$$m, r, n, \alpha, \beta, g, G, h, H, x, y, \nu, \omega \parallel t, \gamma, k, q \mid t', \gamma', k', q'.$$

My own Plückerian equations, or, say, the Plücker-Cayley equations, establish in all 6 relations between the first 13 quantities, and thus enable the expression of them in terms of any seven, say of

$$m, r, n, G, H, \nu, \omega,$$

and the Salmon-Cremona equations then lead to the expressions in terms of these, of the remaining eight quantities $t, \gamma, k, q, t', \gamma', k', q'$.

I also consider

$$D_m, \text{ the deficiency of the curve,}$$

$$D_x, \text{ the deficiency of the nodal curve.}$$

I will first consider the equations themselves, and the mere algebraical transformations thereof; and afterwards the geometrical theory.

The Plücker-Cayley Equations.

5. These are found (as will be further explained) by considering first the cone, vertex an arbitrary point, which passes through the curve; and secondly, the section of the torse by an arbitrary plane. We have in the two figures

Cone.	*Section.*
m, order,	n, class,
r, class,	r, order,
$h+H$, double lines,	$g+G$, double tangents,
β, stationary lines,	α, stationary tangents,
$y+\omega$, double planes,	$x+\omega$, double points,
$n+v$, stationary planes.	$m+v$, stationary points.

And hence the two sets of quantities respectively are connected by the Plückerian relations, viz. these are

$$r = m(m-1) - 2(h+H) - 3\beta,$$
$$n+v = 3m(m-2) - 6(h+H) - 8\beta,$$
$$y+\omega = \tfrac{1}{2}m(m-2)(m^2-9) - (m^2-m-6)\{2(h+H)+3\beta\} + 2(h+H)(h+H-1) + 6(h+H)\beta + \tfrac{9}{2}\beta(\beta-1),$$
$$m = r(r-1) - 2(y+\omega) - 3(n+v),$$
$$\beta = 3r(r-2) - 6(y+\omega) - 8(n+v),$$
$$h+H = \tfrac{1}{2}r(r-2)(r^2-9) - (r^2-r-6)\{2(y+\omega)+3(n+v)\} + 2(y+\omega)(y+\omega-1) + 6(y+\omega)(n+v) + \tfrac{9}{2}(n+v)(n+v-1),$$
$$n+v-\beta = 3(r-m),$$
$$y+\omega-h-H = \tfrac{1}{2}(r-m)(r+m-9),$$
$$\tfrac{1}{2}(r-1)(r-2) - (y+\omega) - (n+v) = \tfrac{1}{2}(m-1)(m-2) - (h+H) - \beta,$$

$$n = r(r-1) - 2(x+\omega) - 3(m+v),$$
$$\alpha = 3r(r-2) - 6(x+\omega) - 8(m+v),$$
$$g+G = \tfrac{1}{2}r(r-2)(r^2-9) - (r^2-r-6)\{2(x+\omega)+3(m+v)\} + 2(x+\omega)(x+\omega-1) + 6(x+\omega)(m+v) + \tfrac{9}{2}(m+v)(m+v-1),$$
$$r = n(n-1) - 2(g+G) - 3\alpha,$$
$$m+v = 3n(n-2) - 6(g+G) - 8\alpha,$$
$$x+\omega = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)\{2(g+G)+3\alpha\} + 2(g+G)(g+G-1) + 6(g+G)\alpha + \tfrac{9}{2}\alpha(\alpha-1),$$
$$\alpha-(m+v) = 3(n-r),$$
$$g+G-(x+\omega) = \tfrac{1}{2}(n-r)(n+r-9),$$
$$\tfrac{1}{2}(r-1)(r-2) - (x+\omega) - (m+v) = \tfrac{1}{2}(n-1)(n-2) - (g+G) - \alpha,$$

and combining the two systems

$$\beta = \alpha + 2(m-n),$$
$$y = x + \ (m-n),$$
$$h + H = g + G + \tfrac{1}{2}(m-n)(m+n-7),$$
$$\tfrac{1}{2}(m-1)(m-2) - (h+H) - \beta \ = \tfrac{1}{2}(n-1)(n-2) - (g+G) - \alpha,$$
$$\tfrac{1}{2}(r-1)(r-2) - (x+\omega) - (m+\upsilon) = \tfrac{1}{2}(r-1)(r-2) - (y+\omega) - (n+\upsilon).$$

6. Taking as data r, m, n, G, H, υ, ω, we find very easily

$$h = \tfrac{1}{2}(m^2 - 10m - 3n + 8r - 3\upsilon - 2H),$$
$$g = \tfrac{1}{2}(n^2 - 10n - 3m + 8r - 3\upsilon - 2G),$$
$$x = \tfrac{1}{2}(r^2 - \ r - \ n - 3m - 3\upsilon - 2\omega),$$
$$y = \tfrac{1}{2}(y^2 - \ r - \ m - 3n - 3\upsilon - 2\omega),$$
$$\alpha = \ m - \ 3r + 3n + \ \upsilon \ ,$$
$$\beta = \ n - \ 3r + 3m + \ \upsilon \ .$$

The Salmon-Cremona Equations.

7. These are

$$m(r-2) = 2n + 4\beta + \gamma + 4\upsilon + 4\omega + 4H,$$
$$x(r-2) = n(r-4) + 2\beta + 3\gamma + 3t + \upsilon(3r-14) + \omega(2r-10) + 12H,$$
$$x(r-2)(r-3) = n(x-2r+8) + 3mx + 4k - 3\alpha - 9\beta - 6\gamma$$
$$+ \upsilon(3x - 6r + 18) + \omega(2x - 4r + 8) - 12H,$$
$$q = x^2 - x - 2k - 3\gamma - 6t - 3\upsilon(r-6) - 2\omega(r-8) - 2G - 18H,$$
$$\{= r(n-3) - 3\alpha - 2G\},$$

and

$$n(r-2) = 2m + 4\alpha + \gamma' + 4\upsilon + 4\omega + 4G,$$
$$y\ (r-2) = m(r-4) + 2\alpha + 3\gamma' + 3t' + \upsilon(3r-14) + \omega(2r-10) + 12G,$$
$$y(r-2)(r-3) = m(y-2r+8) + 3ny + 4k' - 3\beta - 9\alpha - 6\gamma'$$
$$+ \upsilon(3y - 6r + 18) + \omega(2y - 4r + 8) - 12G,$$
$$q' = y^2 - y - 2k' - 3\gamma' - 6t' - 3\upsilon(r-6) - 2\omega(r-8) - 2H - 18G,$$
$$\{= r(m-3) - 3\beta - 2H\},$$

where the second values of q, q' respectively are reduced forms obtained by the aid of the foregoing Plückerian relations.

8. Expressing these in terms of r, m, n, G, H, υ, ω, we obtain

$$\gamma = rm + 12r - 14m - 6n - 8\upsilon - 4\omega - 4H,$$
$$t = \tfrac{1}{6}[r^3 - 3r^2 - 58r - 3r(n + 3m + 3\upsilon + 2\omega) + 42n + 78m + 78\upsilon + 48\omega],$$
$$k = \tfrac{1}{8}[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)(n + 3m + 3\upsilon + 2\omega)$$
$$+ (n + 3m + 3\upsilon + 2\omega)^2 - 58n - 126m - 126\upsilon - 76\omega - 24H],$$
$$q \ = rn + 6r - 3m - 9n - 3\upsilon - 2G,$$

and

$$\begin{aligned}
\gamma' &= rn + 12r - 14n - 6m - 8\upsilon - 4\omega - 4G,\\
t' &= \tfrac{1}{6}\,[r^3 - 3r^2 - 58r - 3r\,(m + 3n + 3\upsilon + 2\omega) + 42m + 78n + 78\upsilon + 48\omega],\\
k' &= \tfrac{1}{8}\,[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)\,(m + 3n + 3\upsilon + 2\omega)\\
&\qquad + (m + 3n + 3\upsilon + 2\omega)^2 - 58m - 126n - 126\upsilon - 76\omega - 24G],\\
q' &= rm + 6r - 3n - 9m - 3\upsilon - 2H.
\end{aligned}$$

9. We have thence

$$\begin{aligned}
\gamma' - \gamma &= -(r-8)\,(m-n) - 4\,(G-H),\\
t' - t &= \quad (r-6)\,(m-n),\\
q' - q &= \quad (r-6)\,(m-n) + 2\,(G-H),\\
k' - k &= \quad \tfrac{1}{2}\,(m-n)\,(r^2 - 5r - 2m - 2n - 3\upsilon - 2\omega + 17) - 3\,(G-H)\\
&= \quad \tfrac{1}{2}\,(m-n)\,(x + y - 4r + 17) - 3\,(G-H).
\end{aligned}$$

10. Instead of obtaining the above values of γ, t, k, q directly it is convenient to verify them by substitution in the equations from which they were obtained; viz. writing for shortness $n + 3m + 3\upsilon + 2\omega = P$, these may be written

$$\begin{aligned}
&-m\,(r-2) + 2n + 4\beta + 4\upsilon + 4\omega + 4H + \gamma &&= 0,\\
&-2x\,(r-2) + 2r\,(P - 3m) - 8n + 4\beta - 28\upsilon - 20\omega + 6\gamma + 24H + 6t &&= 0,\\
&-2x\,(r^2 - 5r + 6 - P) - 4\,(r-4)\,n - 18\beta - 12\gamma - 6\alpha + 36\upsilon + 16\omega - 24H &&\\
&\qquad\qquad - 4r\,(-n - 3m + P) + 8k &&= 0,\\
&-4q + 2x\,(2x-2) - 8k - 8G - 4r\,(-n - 3m + P) + 72\upsilon + 64\omega - 12\gamma - 24t - 72H &&= 0,
\end{aligned}$$

which are to be satisfied by

$$\begin{aligned}
\gamma &= rm + 12r - 14m - 6n - 8\upsilon - 4\omega - 4H,\\
t &= \tfrac{1}{6}\,[r^3 - 3r^2 - 58r - 3rP + 42n + 78m + 78\upsilon + 48\omega],\\
k &= \tfrac{1}{8}\,[r^4 - 6r^3 + 11r^2 + 66r - (2r^2 - 10r)\,P + P^2 - 58n - 126m - 126\upsilon - 76\omega - 24H],\\
q &= rn + 6r - 3m - 9n - 3\upsilon - 2G,\\
x &= \tfrac{1}{2}\,(r^2 - r - P),\\
\alpha &= m - 3r + 3n + \upsilon,\\
\beta &= n - 3r + 3m + \upsilon,\\
P &= n + 3m + 3\upsilon + 2\omega.
\end{aligned}$$

11. We have, in fact, first

$$\left.\begin{aligned}
&-mr + 2m\\
&\qquad\quad + 2n\\
&\qquad\quad + 12m + 4n - 12r + 4\upsilon\\
&\qquad\qquad\qquad\qquad\quad + 4\upsilon\\
&\qquad\qquad\qquad\qquad\qquad + 4\omega\\
&\qquad\qquad\qquad\qquad\qquad\quad + 4H\\
&+ mr - 14m - 6n + 12r - 8\upsilon - 4\omega - 4H
\end{aligned}\right\} = 0,$$

secondly

$$\begin{array}{l}
-r^3+r^2+rP\\
+2r^2-2r-2P\\
+2rP-6mr\\
-8n\\
-12r+4n+12m+4\upsilon\\
-28\upsilon\\
-20\omega\\
+72r+6mr-36n-84m-48\upsilon-24\omega-24H\\
+24H\\
+r^3-3r^2-58r-3rP+42n+78m+78\upsilon+48\omega\\
\hline
\end{array}$$

that is $\quad -2P+2n+6m+6\upsilon=0,$

thirdly

$$\begin{array}{l}
-r^4+5r^3-6r^2\\
+r^3-5r^2+6r+P(2r^2-6r+6)-P^2\\
-4rn+16n\\
+54r-18n-54m-18\upsilon\\
-144r-12rm+72n+168m+96\upsilon+48\omega+48H\\
+18r-18n-6m-6\upsilon\\
+36\upsilon\\
+16\omega\\
-24H\\
+P(-4r)+12rm+4rn\\
+r^4-6r^3+11r^2+66r+P(-4r^2+10r)+P^2-58n-126m-126\upsilon-76\omega-24H\\
\hline
\end{array}$$

that is $\quad 6P-6n-18m-18\upsilon-12\omega-24H=0,$

and lastly

$$\begin{array}{l}
-24r-4rn+12m+36n+12\upsilon+8G\\
+r^4-2r^3-r^2+2r+P(-2r^2+2r+2)+P^2+126m+58n+126\upsilon+76\omega+24H\\
-r^4+6r^3-11r^2-66r+P(2r^2-10r)-P^2-8G\\
+P(-4r)+4rn+12rm+72\upsilon+64\omega\\
-144r-12rm+168m+72n+96\upsilon+48\omega+48H\\
-4r^3+12r^2+232r+P(12r)-312m-168n-312\upsilon-192\omega-72H\\
\hline
\end{array}$$

that is $\quad 2P-6m-2n-6\upsilon-4\omega=0$

which completes the verification.

12. The deficiency of the curve m is given by the equation

$$2D_m = r - 2m + 2 + \beta,$$

or substituting for β its value $= n - 3r + 3m + \upsilon$, this is

$$2D_m = m + n - 2r + \upsilon + 2.$$

The deficiency of the nodal curve x is given by

$$2D_x = q - 2x + 2 + \gamma + \upsilon(r-6) + 2H,$$

which, substituting for q, x, and γ, the values

$$\begin{aligned} &rn + 6r - 3m - 9n - 3\upsilon - 2G, \\ &\tfrac{1}{2}(r^2 - r - n - 3m - 3\upsilon - 2\omega), \\ &rm + 12r - 14m - 6n - 8\upsilon - 4\omega - 4H, \end{aligned}$$

respectively, becomes

$$2D_x = (r-14)(m+n+\upsilon) - r^2 + 19r + 2 - 2\omega - 2G - 2H;$$

whence, also, writing herein $m + n + \upsilon = 2D_m + 2r - 2$, we have

$$2D_x - (r-14) \,.\, 2D_m = (r-5)(r-6) - 2\omega - 2G - 2H,$$

a relation between the two deficiencies.

Geometrical Theory of the foregoing Relations.

13. In considering the geometrical theory, we have to speak of the original curve, or curve of the system, and also of the nodal curve; it will be convenient to call them the curve m and the curve x respectively. I speak of the torse absolutely, to signify the torse of the system, as in what follows there is not the like occasion to speak of the nexal torse. I speak also of a plane α, meaning thereby any one of the stationary planes, the number of which is $=\alpha$; and so of a line α, meaning the line in the stationary plane α; and a point α, meaning the point of contact of such line with the curve m; or in the plural, the planes α, lines α, &c. And so in other cases; thus we have the stationary tangents υ, and the points υ, which are the points of contact hereof with the curve m. As regards a double tangent ω, we have here two points of contact; one of these separately would be a point ω; and we may speak of the points (or pair) 2ω, meaning thereby the two points of contact of the same tangent ω; or of the points 2ω, meaning the system of the 2ω points of contact of the tangents ω.

14. Observe that the expressions, the planes α, lines α, &c., have an absolute signification; there are other such expressions which have only a relative signification, in regard to the system considered in connexion with a given point, line, or plane, as the case may require. Thus the expression, *the lines g*, must be understood of the system in connexion with *a given plane,* to signify the lines in two planes contained in the *given plane*; the planes g, points g, would of course mean the planes or points of the system belonging to the lines g.

In particular the points m are the points of the system which lie in a given plane, the lines r are the lines which meet a given line; the planes n are the planes which pass through a given point. There will be occasion to speak, not only of the planes n, but also of the lines n and points n; these are of course the lines and points in the planes n.

15. It is to be remarked that, considering the torse as a surface, the nodal curve thereof is made up of the curve x and of the double lines ω (or its order is $=x+\omega$); the cuspidal curve is made up of the curve m, and of the stationary lines v (or its order is $=m+v$).

The Plücker-Cayley Equations.

16. The mode of obtaining these equations has already been indicated. We in fact consider the system in connexion with an arbitrary point, and with an arbitrary plane. The point is the vertex of a cone passing through the curve m, and this cone is of the order m, the class r, with $h+H$ double lines, β stationary lines, $y+\omega$ double tangent planes, and $n+v$ stationary tangent planes; viz. the order of the cone is equal to the number of lines in which this is intersected by a plane through the vertex; but each of these is determined as the line joining the vertex with an intersection of the plane by the curve m, and the number of them is thus $=m$. The class of the cone is equal to the number of the tangent planes which can be drawn through an arbitrary line through the vertex; but this is in fact the number of lines of the system which meet the arbitrary line, viz. it is $=r$. Again, any line drawn from the vertex to meet the curve twice, and also any line drawn to one of the points H, is a double line of the cone, that is, the whole number of double lines is $=h+H$. A line from the vertex to one of the points β is a stationary line of the cone; the number of these is $=\beta$. A plane through the vertex, and containing two tangents of the curve m, or containing a double tangent ω, is a double tangent plane of the cone, the number of these is thus $=y+\omega$. A plane through the vertex, which is also a plane of the system, is a stationary tangent plane; in fact, we have here on the curve m three consecutive points lying in a plane with the vertex, the tangent plane of the cone is the plane through the vertex, and the first and second of the points on the curve; but this is also the plane through the vertex, and the second and third points, or the plane is a stationary tangent plane. But the plane through the vertex and the tangent v is also a stationary tangent plane of the cone; and the number of stationary tangent planes is thus $=n+v$.

17. Similarly for the section by the arbitrary plane; this is a curve of the order r and class n with $x+\omega$ double points, $m+v$ stationary points, $g+G$ double tangents, and α stationary tangents. In fact, the order of the curve is equal to that of the torse; that is, to the number of lines which meet an arbitrary line, or $=r$. The class of the curve is equal to the number of tangents which pass through an arbitrary point of the plane; or, what is the same thing, the number of planes of the system which pass through this same arbitrary point, viz. it is $=n$. Each point of the plane which is the intersection of two lines of the system, and also each intersection of the plane by a line ω, is a double point of the curve; viz. the number of these is $=x+\omega$. Each intersection with the curve m, and also each intersection with the tangent v, is a stationary point of the curve; viz. the number is $=m+v$. Each line

in the plane, which is the intersection of two planes of the system, and also each intersection of the plane with a double tangent plane G, is a double tangent of the curve; viz. the number is $=g+G$. And, finally, each intersection of the plane with a stationary plane α is a stationary tangent of the curve; viz. the number of these is $=\alpha$.

We have thus the Plückerian relations as above between

$$m,\ r,\ h+H,\ \beta,\ y+\omega,\ n+v,$$

and between

$$n,\ r,\ g+G,\ \alpha,\ x+\omega,\ n+v.$$

Zeuthen's Tables.

18. The vertex of the cone and the plane of the section may occupy special positions. We have a table given by Zeuthen, but which I have completed so far as regards the double lines ω as follows:

Cone through the Curve.

	Vertex	Order	Class	Double lines	Stationary lines	Double tangent planes	Stationary tangent planes
1	Arbitrary	m	r	h $+H$	β	y $+\omega$	n $+v$
2	On a tangent	m	$r-1$	$h-1$ $+H$	β $+1$	$y-r+4$ $+\omega$	$n-2$ $+v$
3	On the curve	$m-1$	$r-2$	$h-m+2$ $+H$	β	$y-2r+8$ $+\omega$	$n-3$ $+v$
4	At a point H	$m-2$	$r-4$	$h-2m+6$ $+H-1$	β	$y-4r+20$ $+\omega$	$n-6$ $+v$
5	At a point β	$m-2$	$r-3$	$h-2m+6$ $+H$	$\beta-1$	$y-3r+13$ $+\omega$	$n-4$ $+v$
6	On stationary tangent v	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+9$ $+\omega$	$n-3$ $+v-1$
7	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+14$ $+\omega$	$n-4$ $+v-1$
8	On double tangent ω	m	$r-2$	$h-2$ $+H$	β $+2$	$y-2r+10$ $+\omega-1$	$n-4$ $+v$
9	At point of contact of ditto	$m-1$	$r-3$	$h-m+1$ $+H$	β $+1$	$y-3r+15$ $+\omega-1$	$n-5$ $+v$

Plane Section of the Torse.

	Plane	Class	Order	Double tangents	Stationary tangents	Double points	Stationary points
1	Arbitrary	n	r	g $+G$	α	x $+\omega$	m $+v$
2	Through a tangent	n	$r-1$	$g-1$ $+G$	α $+1$	$x-r+4$ $+\omega$	$m-2$ $+v$
3	A tangent plane	$n-1$	$r-2$	$g-n+2$ $+G$	α	$x-2r+8$ $+\omega$	$m-3$ $+v$
4	A double tangent plane G	$n-2$	$r-4$	$g-2n+6$ $+G-1$	α	$x-4r+20$ $+\omega$	$m-6$ $+v$
5	A stationary tangent plane α	$n-2$	$r-3$	$g-2n+6$ $+G$	$\alpha-1$	$x-3r+13$ $+\omega$	$m-4$ $+v$
6	Through stationary tangent v	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+9$ $+\omega$	$m-3$ $+v-1$
7	Tangent plane at contact of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+14$ $+\omega$	$m-4$ $+v-1$
8	Through double tangent ω	n	$r-2$	$g-2$ $+G$	α $+2$	$x-2r+10$ $+\omega-1$	$m-4$ $+v$
9	A tangent plane at one of contacts of ditto	$n-1$	$r-3$	$g-n+1$ $+G$	α $+1$	$x-3r+15$ $+\omega-1$	$m-5$ $+v$

19. To avoid confusion with the geometrical term line, I will speak (not of the lines, but) of the cases of these tables; the numbers in each case satisfy the foregoing Plückerian relations. To fix the ideas, I attend to the second table. We require to know in each case, say the numbers in the (n, r, α) columns; these being known, the other three numbers will be determined.

20. First for the n column; for the Cases 2, 6, 8, the plane is not a tangent plane; the number of tangent planes which pass through a fixed point in the plane is still $=n$. For Cases 3, 7, 9, the plane is a tangent plane, it therefore counts 1 among the tangent planes which pass through a fixed point thereof; and the number of the remaining tangent planes is $=n-1$. And so for Cases 4 and 5, the plane counts for 2 among the tangent planes which pass through a fixed point thereof, and the number of the remaining tangent planes is $=n-2$.

21. Next as to the r and α columns.

Case (2). The plane passes through a generating line, and therefore besides cuts the torse in a curve of the order $r-1$; this generating line is a stationary tangent cutting the curve $r-1$ in the point of contact with the curve m, counting 3 times (in all $\alpha+1$ stationary tangents), and in $r-4$ other points.

Case (3). The plane cuts the torse in the generating line counting twice, and in a curve of the order $r-2$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m (so that the number of stationary tangents remains $=\alpha$), and besides cuts the curve in $r-4$ points as in Case 2.

Case (4). The plane cuts the torse in two generating lines, each counting twice, and in a curve of the order $r-4$; each of the generating lines is in regard to this curve an ordinary tangent at the point of contact with the curve m (number of stationary tangents remains $=\alpha$), and besides cuts the curve in $r-6$ other points.

Case (5). The plane cuts the torse in a generating line counting 3 times, and in a curve of the order $r-3$; the generating line is in regard to this curve an ordinary tangent at the point of contact with the curve m, and besides cuts it in $r-5$ points. The plane being in the present case a plane α, its intersections with the remaining planes α, give the $\alpha-1$ stationary tangents.

Case (6). The plane meets the torse in a generating line counting twice, and in a curve of the order $r-2$; the generating line is in regard to the curve a singular tangent meeting it in the point of contact with the curve m, counting 4 times, and besides meeting it in $r-6$ points. The generating line in respect of this four-pointic intersection counts as a stationary tangent twice; and the number of stationary tangents is $=\alpha+2$.

Case (7). The plane meets the torse in the generating line counting 3 times, and in a curve of the order $r-3$; the generating line is in regard to this curve a stationary tangent at the point of contact with the curve m, counting 3 times, and besides meeting it in $r-6$ points as in Case 6. The whole number of stationary tangents is thus $=\alpha+1$.

Case (8). The plane meets the torse in a generating line counting twice and in a curve of the order $r-2$. The generating line is in respect of the curve a stationary tangent at each of the points of contact with the curve m, viz. each of these points counts 3 times, and there are besides $r-8$ intersections. Moreover, the generating line counting as 2 stationary tangents, the whole number of stationary tangents is $=\alpha+2$.

Case (9). The plane meets the torse in a generating line counting 3 times, and in a curve of the order $r-3$. The generating line is in respect of the curve an ordinary tangent at one of the points of contact with the curve m (viz. the point at which the plane is a tangent plane of the torse), so that we have here two intersections; and it is a stationary tangent at the other of the points of contact with the curve m (viz. the point at which the plane is not a tangent plane of the torse), so that there are here 3 intersections; there are therefore $r-8$ other intersections. The generating line reckons once as a stationary tangent, and the whole number of stationary tangents is $=\alpha+1$.

22. The $r-6$ points in Cases 6 and 7 are the points of intersection of the stationary tangent v by other tangents of the curve m; and so the $r-8$ points in Cases 8 and 9 are the points of intersection of the double tangent ω by other tangents of the curve m; these numbers $r-6$ and $r-8$ will present themselves in the sequel.

We may as to the α-column sum up by saying, that in Cases 2, 7, 9 there is a generating line which reckons as a stationary tangent; in Case 6 a generating line, which, in respect of 4 consecutive intersections, reckons as two stationary tangents; and in Case 8 a generating line, which, in respect of two pairs of 3 consecutive intersections, reckons as two stationary tangents.

In the $(x+\omega)$ and $(m+v)$-columns, observe that in Cases 6, 7, 8, 9, we have in the first two ω, $v-1$, and in the last two $\omega-1$, v; viz. these numbers refer to the intersections of the plane with the tangent ω, v respectively; the actual numbers $\begin{pmatrix} x-2r+9 \\ +\omega \end{pmatrix}$ and $\begin{pmatrix} x-2r+10 \\ +\omega-1 \end{pmatrix}$, &c. are equal for the Cases 6 and 8, and also for the Cases 7 and 9. So in the $g+G$ column in Case 4, we have $G-1$ for G.

The Nodal Curve x; Intersections with the Curve m; and Singularities.

23. The intersections of the curve m with the nodal curve x, are points α, β, γ, H, v or ω.

24. At a point α, four consecutive points of the curve m lie in a plane; the point may be considered as the intersection of two consecutive tangents, viz. of the line through two consecutive points with that through the next two consecutive points; and it is thus a point on the curve x. We may imagine the points A, A' starting from α in opposite directions along the curve m, and moving in such manner that the tangents at these two points respectively continually intersect; we have thus a portion of the curve x, proceeding apparently in one direction only from the point α; and being, as regards the portion in question, an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from α, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve x thus meets the curve m in the several points α, the curve x, each time that it passes through such a point of intersection, changing its character from crunodal to acnodal. The two *half-sheets*[1] of the torse cross each other in the crunodal portion, extending

[1] In a curve (plane or twisted), the portions extending each way from a cusp (and considered without reference to a termination) are called half-branches; and so in a surface which has a cuspidal curve, or in particular, in a torse, the portions extending each way from the cuspidal curve (and considered without reference to a termination) are called half-sheets. In the case of higher singularities of a like nature, we may speak of a partial branch or partial sheet (as the case may be).

in one direction from the point α of the nodal curve; and the acnodal portion extends in the opposite direction from the same point.

25. A point β, or stationary point (cusp) on the curve m, is the intersection of four consecutive osculating planes; and it is thus a point of intersection of two consecutive tangents (viz. of the line of intersection of two consecutive osculating planes, and of that of the next two consecutive osculating planes), consequently a point on the curve x. We may imagine the points B, B' starting from β, and moving along the two half-branches of the curve m, in such manner that the tangents at the two points respectively continually intersect. We have thus a portion of the curve x, proceeding apparently in one direction only from the point β (and having at this point a common tangent with the curve m), and being as regards this portion thereof an intersection of two real sheets of the torse; that is, a crunodal curve. The curve x, however, really extends in the opposite direction from the point β, but it is as to this portion thereof an intersection of two imaginary sheets of the torse; that is, an acnodal curve. The nodal curve thus meets the curve m in each of the points β, the curve x, each time that it passes through such a point, changing its character from crunodal to acnodal. There are at β three partial sheets of the torse; viz. if we imagine at this point the half-tangent b in the sense of the two half-branches of the curve m, and the half-tangent b' in the opposite sense, then we have through b' and one of the half-branches a partial sheet, and through b' and the other half-branch a partial sheet; these two partial sheets touching along b', and intersecting in the crunodal portion of the curve x; and a third partial sheet through b and the two half-branches of the curve m.

26. At a point γ on the cuspidal curve m, we have, traversing the curve and the two half-sheets which meet along it, another sheet of the torse, meeting the two half-sheets respectively in two half-branches, which are a portion of the nodal curve x, and which unite together (as at a cusp), in the point γ, which is thus a cusp or stationary point on the curve x.

27. A point H is the intersection of two branches of the cuspidal curve m. There are for each branch two half-sheets; and we have thus at the point H four (say) quarter-branches of the curve x, touching each other at the point (viz. the common tangent is the intersection of the osculating planes belonging to the two branches of the curve m respectively). I find in a special manner that in regard to the curve x a point H is equivalent to six double points, plus two stationary points; it thus causes a reduction $2 \cdot 6 + 3 \cdot 2 = 18$ in the class of the curve.

28. The nodal curve x has in each of the double tangent planes G an actual double point. In fact the plane is an osculating plane of the curve m at two points thereof; that is, it contains two consecutive tangents R, R', and two other consecutive tangents S, S'; hence RS is a point on the nodal curve x; and not only so, but this is an actual double point, the two tangents being R and S; for since R is met by S and S', there is a consecutive point on the line R; that is, R is a tangent; and similarly since S is met by R and R' there is a consecutive point on the line S; that is, S is also a tangent.

29. At a point υ the tangent has with the curve x a 3-pointic intersection (whence also the tangent is a stationary tangent υ in regard to the curve x), the curves m and x have also a 3-pointic intersection at υ.

30. At each of the two points ω the tangent has with the curve x a 3-pointic intersection (viz. the tangent is in regard to the curve x more than a double tangent ω, instead of two 2-pointic intersections, or ordinary contacts, there are two 3-pointic intersections; I am unable to perceive this directly, but accept it on other grounds). But as at each of the points ω the intersection of the tangent with the curve m is 2-pointic, the intersection of the curves x and m is only 2-pointic.

31. The curves m and x meet in the points α each once, β each 3 times, γ each twice, υ each 3 times, 2ω each twice, and H each 8 times: we have thus

$$\alpha + 3\beta + 2\gamma + 3\upsilon + 4\omega + 8H$$

for the number of actual intersections of the two curves; and the number of apparent intersections is therefore

$$= mx - \alpha - 3\beta - 2\gamma - 3\upsilon - 4\omega - 8H,$$

a result which is required in the sequel.

32. Consider the cone (vertex an arbitrary point) through the curve x, or say simply the cone x.

Each line n (*quà* ordinary line of the system) is met by $r-4$ other lines, the $r-4$ points being situate on the curve x; the line n at each of these points *touches* the cone x, and it therefore besides intersects it in $x - 2r + 8$ points.

33. A line υ meets the curve x in the point υ counting 3 times, and in $r-6$ points each a stationary point of the curve; it consequently meets the cone x, in the point υ counting 3 times, in each of the $r-6$ points counting twice, and besides in $x - 2r + 9$ points.

34. A line ω meets the curve x in the two points ω each counting 3 times, and in $r-8$ points each an actual double point of the curve; it consequently meets the cone x in the two points ω each counting 3 times, in the $r-8$ points each counting twice, and besides in $x - 2r + 10$ points.

35. Each of the $r-8$ points in which the double tangent ω meets another tangent of m is an actual double point of the curve x; and each of the $r-6$ points in which a stationary tangent υ meets another tangent of m is a stationary point or cusp on the curve x. We have thus as singularities of the curve x, the k apparent double points, G actual double points, $\omega(r-8)$ ditto, t triple points, H 4-branch cuspidal points, γ stationary or cuspidal points, $\upsilon(r-6)$ ditto. In regard to the effect upon the class of the curve x, each of the points t is equivalent to 3 double points and produces a reduction $=6$; each of the points H is equivalent to 6 double points $+2$ cusps, and produces a reduction $2 \cdot 6 + 3 \cdot 2 = 18$, as already mentioned. We have thus the relation

$$q = x(x-1) - 2k - 2G - 3\upsilon(r-6) - 2\omega(r-8) - 3\gamma - 6t - 18H,$$

which is one of the Salmon-Cremona equations.

36. The stationary points of the curve x are the points γ, the $v(r-6)$ points, and the points H, each counting as 2 stationary points; that is, we have

$$2D_x = q - 2x + 2 + \gamma + v(r-6) + 2H,$$

used above for finding the value of the deficiency D_x.

The remaining Salmon-Cremona Equations.

37. The formulæ for $m(r-2)$, $x(r-2)$ and $x(r-2)(r-3)$ correspond to those for $c(n-2)$, $b(n-2)$ and $b(n-2)(n-3)$ in the case of a general surface (*Solid Geometry*, 2nd Ed., Nos. 522 and 525 [4th Ed., Nos. 610, 613]), being obtained as follows; considering as before the cone, vertex an arbitrary point, which passes through the curve m, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve m is $= m(r-2)$; and, similarly, considering the cone, vertex the same arbitrary point, which passes through the curve x, the total number of lines hereof which have with the torse a 3-pointic intersection at the curve x is $= x(r-2)$; viz. these numbers $m(r-2)$ and $x(r-2)$ are the numbers of the intersections of the two curves respectively with the surface of the order $(r-2)$, which is the second polar of the arbitrary point in regard to the torse. And so also in the cone, vertex the arbitrary point, which passes through the curve x, the total number of lines which touch the torse at a point not on the curve x is $= x(r-2)(r-3)$; viz. this is obtained by Salmon, the number of intersections of the curve x with a certain surface of the order $(r-2)(r-3)$ (Salmon, Nos. 269 and 273, writing r for n), but in a preferable manner by Cremona thus; the cone through the curve x meets the torse in the curve x counting twice and in a residual curve of the order $x(r-2)$; the lines in question are the tangents from the vertex to this curve, which is a curve meeting each line of the cone $r-2$ times; we may on the surface of the cone metrically, as in the plane, construct the polar of the vertex in regard to the curve of the order $x(r-2)$; viz. we have thus on the cone a curve meeting each generating line $(r-3)$ times; and this polar curve meets the curve of the order $x(r-2)$ in $x(r-2)(r-3)$ points (this would be clearly the case if only the curve of the order $x(r-2)$ were the complete intersection of the cone by a surface of the order $r-2$, for then the polar curve would be the intersection of the cone by the polar surface of the order $r-3$; but in the case in hand, where this is not so, some additional considerations would be required in order to sustain the result), and we have thus the number $x(r-2)(r-3)$ of the lines in question.

38. I consider the second polar surface, order $r-2$, which belongs to a given arbitrary point. Any point on the torse, such that the line joining it with the arbitrary point cuts the torse 3-pointically at the point on the torse, is a point on the second polar $r-2$. Such points are, as will be shewn, the points n, $n(r-4)$, β, γ, v, 2ω, $v(r-6)$, $\omega(r-8)$, t, H.

39. We may consider through any such point and the arbitrary point either a particular section of the torse or any section whatever. If the line joining the two points has at the point in question a 3-pointic intersection with the section, then it has a 3-pointic intersection with the torse, viz. the point possesses the required

property. For the points n and $n(r-8)$, the plane may be taken to be the osculating plane of the curve m. This meets the torse in the line n twice, and in a curve of the order $r-2$ touching this line at the point n and meeting it in the $r-4$ points. Hence, considering the complete section made up of the line twice and the curve of the order $r-2$, the line from the arbitrary point to the point n or to any one of the $r-4$ points meets the section 3-pointically; viz. the lines to the points n and to the points $n(r-4)$ meet the torse 3-pointically.

40. For the points β; we may consider any section through the arbitrary point and β; in effect any section through the point β. A plane section through a point β has at this point an invisible triple point, that is a point not in appearance differing from an ordinary point of the curve, but which by considering a consecutive position of the plane of section is seen to be equivalent to a double point and two cusps; viz. the node is a point of intersection of the plane with the curve x, the cusps are the two intersections with the curve m, in the neighbourhood of the point β. The line from β to the arbitrary point has thus with the torse a 3-pointic intersection at β.

41. Similarly for the points γ; we take any section through the arbitrary point and γ; in effect any section through the point γ. The section through a point γ has at this point a triple point, at which an ordinary branch passes through a cusp, and thus equivalent to a cusp $+2$ nodes; in fact, for a consecutive position of the cutting plane, the section has actually a cusp and two nodes; the cusp at the intersection of the plane with the curve m, the nodes at the two intersections of the plane with the curve x in the neighbourhood of the cusp γ. The line through the point γ has thus a 3-pointic intersection with the torse.

42. For a point υ I consider the section through the tangent υ; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 4-pointic intersection at the point υ, and besides meeting it in $r-6$ points. Hence in the plane, a line through υ, or through one of the $r-6$ points has at such point a 3-pointic intersection with the curve. And thus the lines through the points υ and $\upsilon(r-6)$ respectively have a 3-pointic intersection with the torse.

43. Similarly for a point ω, I consider the section through a tangent ω; this is made up of the tangent twice, and of a curve of the order $r-2$ having with the tangent a 3-pointic intersection at each of the points ω, and besides meeting it in $r-8$ points. Hence in the plane a line through either of the points ω or through one of the $r-8$ points has with the curve a 3-pointic intersection, and thus the lines through the points 2ω and $\omega(r-8)$ respectively have a 3-pointic intersection with the torse.

44. For a point H I consider any section through the arbitrary point and H; in effect any section through H. There is at H a singularity $=6$ nodes $+2$ cusps. But a line through H in the plane of the section cuts the section in 4 points only; that is, the line from the arbitrary point to H has with the curve a 4-pointic intersection at H; and *à fortiori* it may be regarded as a line of 3-pointic intersection. The lines to the points H have thus a (4-pointic, that is, more than) 3-pointic intersection with the torse.

45. For a point t I consider any section through the arbitrary point and t; in effect any section through t; there is at t a triple point, and the line through t has thus a 3-pointic intersection at t. Hence a line through t has 3-pointic intersections with the torse.

46. The several points above referred to lie on the curve m or on the curve x, or on both of these curves; and each curve has at these points respectively a simple or multiple intersection with the polar surface $r-2$, as shown in the table.

	Intersection of second polar $r-2$	
	with curve m	curve x
Points n	2	0
" $n(r-4)$	0	1
" β	4	2
" γ	1	3
" v	4	4
" 2ω	2	3
" $v(r-6)$	0	3
" $\omega(r-8)$	0	2
" H	4	12
" t	0	3

where the figures 1, 2, &c. denote a simple intersection, 2-pointic intersection, &c. of the curve and surface; 0 denotes of course that there is not any intersection, viz. that the curve does not pass through the point referred to.

47. Several of the foregoing numbers are obtained without difficulty; thus we see that the points n, $n(r-4)$ are ordinary points on the second polar $r-2$, the surface at each of the points n *touching* the curve m, but at each of the points $n(r-4)$ simply cutting the curve x. So also the points β, γ, v, 2ω, $v(r-6)$, $\omega(r-8)$ are ordinary points on the second polar; at a point β the two half-branches of the curve m touch the surface in a special manner so as to give a 4-pointic intersection; whereas the curve x simply touches the surface. At γ the curve m cuts the surface, but the two half-branches of x *touch* the surface. At v, each of the curves m, x has a 4-pointic intersection with the surface; at each of the two points ω, the curve m touches the surface, but the curve x has with it a 3-pointic intersection, and at $\omega(r-8)$ it simply touches the surface.

48. The point H is in the nature of a biplanar point on the polar surface; this appears, or is at least indicated, by the circumstance that the line to the arbitrary point has with the torse (not a 3-pointic but) a 4-pointic intersection; the two branches of the curve m each simply cut the two coincident sheets of the polar surface, giving 2×2, $=4$ intersections; but for the curve x, the four partial branches each touch the

two coincident sheets; for a single sheet the number of intersections would be 6, but for the two coincident sheets it is twice this, or $=12$. Finally, a point t is an ordinary point on the second polar, each of the three branches of x simply cuts the surface, or the number of intersections is $=3$.

49. The last table gives at once

$$m(r-2) = \quad 2n+4\beta+\quad\gamma+4v+2.2\omega+4H,$$

$$x\ (r-2) = (r-4)+2\beta+3\gamma+4v+3.2\omega+3v(r-6)+2\omega(r-8)+12H+3t,$$

which are the true theoretical forms of the equations for $m(r-2)$ and $x(r-2)$, in which these were obtained by Cremona.

50. The $x(r-2)(r-3)$ points are those points in which the Cremona $x(r-2)$ curve is met 2-pointically by the line from the arbitrary point (I recall that taking the arbitrary point as the vertex of a cone through the curve x, this cone, say the cone x, meets the torse in the curve x twice, and in the $x(r-2)$ curve in question); viz. these points are either points of contact of tangents from the vertex to the $x(r-2)$ curve; or they are double points, or else cusps of the $x(r-2)$ curve; in which several cases respectively they count 1, 2 or 3 times, among the $x(r-2)(r-3)$ points.

51. The points of contact are the $n(x-2r+8)$ points of intersection of the lines n with the cone x. We have in fact a plane n through the vertex of the cone, and in this plane two consecutive lines of the system; hence at each of the $x-2r+8$ points the generating line of the cone meets the two consecutive lines of the system; that is, there is with the curve $x(r-2)$ a 2-pointic intersection, not arising out of any singularity of the curve, and consequently a contact of this curve with the generating line of the cone.

52. The actual double points of the curve $x(r-2)$ are first the $2k$ apparently coincident points of the curve x, and secondly the $\omega(x-2r+10)$ points on the lines ω. For first if we consider through the vertex a line meeting the curve x in two points, say A, B, this meets the torse in these points each twice and in $r-4$ other points. Now imagine a line from the vertex to the point P in the vicinity of A, this meets the torse in the point P twice and in $r-2$ points, which are points on the $x(r-2)$ curve; hence as P travels through A, 2 of the $r-2$ points come together at B, and again separate, that is B is an actual double point on the $x(r-2)$ curve; and similarly A is an actual double point on the curve; and we have thus the $2k$ double points. Secondly, since the line ω is a nodal line on the torse, a generating line of the cone, in the neighbourhood of and considered as travelling through one of the $x-2r+10$ points, meets the torse in two points which come to coincide and then again separate; that is each of the $x-2r+10$ points is an actual double point on the curve $x(r-2)$; and the whole number of these is $=\omega(x-2r+10)$.

53. The stationary points of the curve $x(r-2)$ are first the points on the curve m which apparently coincide with the curve x; viz. the number of these, as was seen, is $=mx-\alpha-3\beta-2\gamma-3v-4\omega-8H$; secondly, the $v(x-2r+9)$ points on the lines v;

thirdly, the points H each counting as 4 cusps. For first consider a generating line meeting the curve x in B and the curve x in A; if we imagine on the curve x a point Q which approaches and ultimately coincides with B, the generating line through Q meets the torse in the neighbourhood of its cuspidal edge in two points which come ultimately to coincide with the point A, and we thus see that A is a stationary point on the $x(r-2)$ curve.

54. Secondly, observing that the line v is a cuspidal line on the torse, and considering in like manner a generating line of the x cone, which approaches and comes ultimately to coincide with one of the $x-2r+9$ points, we see that this is a stationary point on the $x(r-2)$ curve. And thirdly, any line through a point H meets the torse in this point counting 4 times, and in $r-4$ other points. Hence considering the generating line of the x cone, which travelling along any one of the four partial branches of the x curve comes ultimately to coincide with H, 2 of the $r-2$ points on such generating line come to coincide at the point H; and we have thus the point H as a singular point on the $x(r-2)$ curve; viz. it reckons as a stationary point once in respect of each of the four partial branches of the curve x (it must be assumed that this is so, but a further proof is required), that is as 4 cusps on the $x(r-2)$ curve.

55. By what precedes we have

$$\begin{aligned} x(r-2)(r-3) = {} & n(x-2r+8) \\ & + 2\{2k+\omega(x-2r+10)\} \\ & + 3\{(mx-\alpha-3\beta-2\gamma-3v-4\varpi-8H)+v(x-2r+9)+4H\}, \end{aligned}$$

which is the true theoretical form in which the equation for $x(r-2)(r-3)$ was obtained by Cremona.