496.

TABLES OF THE BINARY CUBIC FORMS FOR THE NEGATIVE DETERMINANTS, ≡ 0 (MOD. 4), FROM −4 to −400; AND ≡ 1 (MOD. 4), FROM −3 TO −99; AND FOR FIVE IRREGULAR NEGATIVE DETERMINANTS.

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THE theory of binary cubic forms for determinants, as well positive as negative, has been studied by M. Arndt in the memoir "Versuch einer Theorie der homogenen Functionen des dritten Grades mit zwei Variabeln," *Grunert's Archiv*, t. XVII. (1851, pp. 1—54); and in the later memoir, "Tabellarische Berechnung der reducirten binären cubischen Formen und Klassification derselben für alle negativen Determinanten $(-D)$ von $D=3$ bis $D\equiv 2000$," *ditto*, t. XXXI. (1858), pp. 335—445, he has given a very valuable Table of the forms for a Negative Determinant. It has appeared to me suitable to arrange this Table in the manner made use of for Quadratic Forms in my memoir "Tables des formes quadratiques binaires pour les déterminants négatifs $D\equiv -1$ jusqu'à $D\equiv -100$, pour les déterminants positifs non carrés depuis $D\equiv 2$ jusqu'à $D\equiv 99$, et pour les treize déterminants négatifs du premier millier," *Crelle*, t. LX. (1862), pp. 357—372, [335]; and confining myself to the limits of the last-mentioned tables I deduce from that of M. Arndt the three Tables which follow.

To explain the àrrangement, I give in the first instance the following extract from M. Arndt's Table:

D.	*Reducirte Formen mit Charakteristik.*			*Klassen.*
3	(0, 1, 1, 0) (2, 1, 2)	(1, 0, − 1, − 1) (2, 1, 2)	(1, 1, 0, − 1) (2, 1, 2)	(0, 1, 1, 0), (1, 0, − 1, ± 1)
4	(0, 1, 0, − 1) (2, 0, 2)	(1, 0, − 1, 0) (2, 0, 2)		(0, − 1, 0, 1)

D.	*Reducirte Formen mit Charakteristik.*	*Klassen.*
7	(0, 1, 1, − 1) (2, 1, 4)	(0, − 1, − 1, 1)
8	(0, 1, 0, − 2) (2, 0, 4)	(0, − 1, 0, 2)
11	(0, 1, 1, − 2) (2, 1, 6)	(0, − 1, − 1, 2)
12	(0, 1, 0, − 3) (2, 0, 6)	(0, − 1, 0, 3)
15 ⋮	(0, 1, 1, − 3) (2, 1, 8)	(0, − 1, − 1, 3)
44 ⋮	(0, 1, 0, − 11) (1, − 1, − 2, 0) (2, 0, 22) (6, 2, 8)	(0, − 1, 0, 11), (0, − 2, ± 1, 1)
112 ⋮	(0, 1, 0, − 28) (0, 2, 2, − 2) (1, 2, 0, − 4) (2, 0, 56) (8, 4, 16) (8, 4, 16) (1, − 1, − 3, − 1) (8, 4, 16)	(0, − 1, 0, 28), (0, 2, 2, − 2), (1, ± 1, − 3, ± 1)
144	(0, 1, 0, − 36) (0, 2, 2, − 3) (2, 0, 72) (8, 4, 20)	(0, − 1, 0, 36), (0, − 2, − 2, 3)
156	(0, 1, 0, − 39) (1, − 1, − 3, 1) (2, 0, 78) (8, 2, 20)	(0, − 1, 0, 39), (1, ∓ 1, − 3, ± 1)
216	(0, 1, 0, − 54) (1, − 2, − 3, 0) (2, 0, − 3, 0) (2, 0, 108) (14, 6, 18) (12, 0, 18)	(0, − 1, 0, 54), (0, ∓ 3, 0, ± 2), (0, ∓ 3, 2, ± 1)

The first column contains the value of the determinant, the second column contains the *reduced* forms, omitting the *contrary* and *opposite* forms; viz. for the cubic form (a, b, c, d), the contrary form (equal, that is, properly equivalent to the given form) is $(-a, -b, -c, -d)$; and the opposite form (improperly equivalent to the given form) is $(a, -b, c, -d)$ or $(-a, b, -c, d)$; this second column contains also the *characteristic* of each cubic form, viz. the cubic form (a, b, c, d) has for its characteristic the quadratic form

$$\{2(b^2 - ac),\ bc - ad,\ 2(c^2 - bd)\},$$

(so that the cubic form and its characteristic have the same determinant

$$-D = (bc - ad)^2 - 4(b^2 - ac)(c^2 - bd),\ \equiv 1 \text{ or } 0 \pmod 4)),$$

and a cubic form which corresponds to a reduced characteristic is itself a reduced form. The third column contains for each determinant the entire series of *unequal* cubic forms (that is of the forms whereof no two are properly equivalent to each other), the representatives of the *classes* for this determinant. M. Arndt has included in his table the non-primitive classes (for example Det. = − 112, the form $(0, 2, 2, -2)$), for which the terms (a, b, c, d) have a common divisor μ, but as these are at once deducible from the classes which belong to the determinant $= -\frac{D}{\mu^4}$, it seems better to omit the non-primitive classes.

The two opposite forms included in a single expression by means of the sign $\pm$ have opposite characteristics which are for the most part unequal to each other, for instance

$$\text{Det.} -44\,;\ (0, -2, 1, 1) \text{ has the characteristic } (6, -2, 8),$$
$$(0, 2, 1, -1) \quad ,, \quad ,, \quad (6, 2, 8),$$

where $(6, -2, 8)$, $(6, 2, 8)$ are unequal forms, but this is not always the case, for instance

$$\text{Det.} -112\,;\ (1, -1, -3, -1) \text{ has the characteristic } (8, -4, 16),$$
$$(1, 1, -3, 1) \quad ,, \quad ,, \quad (8, 4, 16),$$

where $(8, -4, 16) = (8, 4, 16)$, since each is an ambiguous form. Instead of the two unequal forms $(1, -1, -3, -1)$, $(1, 1, -3, 1)$ which correspond to the opposite (though equal) characteristics $(8, -4, 16)$, $(8, 4, 16)$, M. Arndt might have given the two forms $(1, 2, 0, -4)$ and $(1, -1, -3, -1)$ corresponding to the *same* characteristic $(8, 4, 16)$; but then it would not have appeared at a glance that the two classes were opposite to each other; and I presume that it is for this reason that he has selected the two representative forms $(1, -1, -3, -1)$ and $(1, 1, -3, 1)$. It must not, however, be imagined that the opposite cubic forms which correspond to opposite characteristics, which are ambiguous (and therefore equal to each other), are always, as in the last preceding example, unequal: for example Det. -144, there is only the form $(0, -2, -2, 3)$ given as corresponding to the ambiguous characteristic $(8, 4, 20)$; the opposite form $(0, 2, -2, -3)$ corresponding to the opposite but equal characteristic $(8, -4, 20)$ is equal to $(0, -2, -2, 3)$, and so does not give rise to a distinct opposite class. In the new tables, the sign $\pm$ is only employed in regard to opposite ambiguous characteristics; for instance, Det. -4×28 there are given (not included in a single expression by means of the sign $\pm$) the two forms $(1, -1, -3, 1)$, $(1, 1, -3, 1)$ corresponding to the characteristic $2\,(2, \pm 1, 4)$.

I remark that, in a few instances M. Arndt, in passing from the second to the third column, has modified the expression for a cubic form in such manner that the characteristic has ceased to be a reduced form; for instance, Det. -216, he has given in the third column the two forms $(0, \mp 3, 2, \pm 1)$ belonging to the characteristic $(18, \mp 6, 14)$; it would have been better, it appears to me, to preserve the expression of the second column $(1, -2, -3, 0)$, and adopt the two representative forms $(1, \mp 2, -3, 0)$; I have accordingly made this change.

I divide M. Arndt's table into two tables; the first of them corresponding to the determinants $\equiv 0 \pmod{4}$, the second to the determinants $\equiv 1 \pmod{4}$. In the first table I take for the characteristic the form

$$\{b^2 - ac,\ \tfrac{1}{2}(bc - ad),\ bd - c^2\},$$

which belongs to the determinant $-\frac{1}{4}D$, and I arrange the cubic classes according to their *order*; viz. we have the properly primitive order (*pp*) when the terms $(a, 3b, 3c, d)$ have no common divisor; and the improperly primitive order (*ip*) when the terms $(a, 3b, 3c, d)$ have no common divisor other than 3, or what is the same thing when a and d being each of them divisible by 3, the terms (a, b, c, d) have no common divisor. But, moreover, the characteristic $\{b^2 - ac, \frac{1}{2}(bc - ad), bd - c^2\}$, may be of the properly primitive order *pp*; or of the improperly primitive order *ip*; or it may be of a derived order $\mu(A', B', C'), = \mu . pp$ or $\mu . ip$, according as (A', B', C') considered as a form belonging to the determinant $B'^2 - A'C', = -\frac{1}{4\mu^2}D$, is of the properly primitive or the improperly primitive order. And in these different cases, the cubic class is said to be of the order *pp* on *pp*, *pp* on *ip*, *pp* on $\mu . pp$, *pp* on $\mu . ip$, *ip* on *pp*, &c., as the case may be.

For the determinants $\equiv 1$ (mod. 4), I retain the characteristic

$$\{2(b^2 - ac), bc - ad, 2(c^2 - bd)\},$$

and this being so, the division into orders is the same as in the former case; only as the characteristic, when primitive, is of necessity improperly primitive, the orders *pp* on *pp* and *ip* on *pp* no longer exist.

To every characteristic I annex in the tables the symbol of its composition; viz. 1 denotes the principal form, c a form which by its duplication, d a form which by its triplication, &c., produces the principal form, σ denotes the most simple form of order *ip*, σc, σd, &c., the forms obtained by combining σ with the forms c, d, &c., of the order *pp*. Similarly to a characteristic $\mu(A', B', C')$ I annex the symbol of composition of the form (A', B', C'), $\left(\text{considered as belonging to the determinant } B'^2 - A'C', = -\frac{D}{4\mu^2}\right)$ multiplying this symbol by the number μ; for instance, $\mu . 1$ denotes that (A', B', C') is the principal form, and similarly in other cases.

I have given a third table for the determinants

$$-4 \times 243, -4 \times 307, -4 \times 339, -4 \times 459, -4 \times 675,$$

where -243, -307, -339, -459, -675 are those of the thirteen irregular negative determinants in the first thousand for which the number of classes is divisible by 3. The number $-4 \times 675, = -2700$, is beyond the limits of M. Arndt's Table, but the calculation (at least for the order *pp* on *pp*) presents no difficulty.

I remark that, according to M. Arndt (*Grunert*, t. XVII. p. 19), the number of cubic forms corresponding to a given characteristic (A, B, C) is equal to the number of proper transformations of $(A, -B, C)$, Det. D, into $(\frac{1}{2}A^2, B^2 - \frac{1}{2}AC, \frac{1}{2}C^2)$, Det. DB^2, so that when there is no such transformation, there exists no cubic form corresponding to the characteristic (A, B, C). This includes, I believe, the theorem in a letter of mine to M. Hermite, *Quarterly Mathematical Journal*, t. I. (1857), p. 85, [162], viz. that for a *pp* form (A, B, C) of negative determinant, there is either no corresponding cubic form, or else a single corresponding cubic form, according as (A, B, C) does not, or does, produce by its triplication the principal form; but the particular theorem, in the cases to which it applies, is the more convenient one: it shows at once that for a regular negative determinant the number of cubic forms corresponding to a properly primitive characteristic (or, what is the same thing, number of cubic classes of the order (*pp* or *ip*) on *pp*) is 1 or 3, according as the number of quadratic classes is not, or is, divisible by 3.

The inspection of the tables gives rise to other remarks, but at present I abstain from pursuing the subject further; I will only notice that in some instances, for example Det. − 224, the classes which correspond to characteristics of the principal genus are partly of the order *pp* on *pp* and partly of the order *ip* on *pp*.

Table I. *of the binary cubic forms, the determinants of which are the negative numbers* ≡ 0 (*mod.* 4) *from* − 4 *to* − 400.

Det. 4 ×	Classes	Order	on	Charact.	Compn.
1	0, − 1, 0, 1	*pp*	*pp*	1, 0, 1	1
2	0, − 1, 0, 2	*pp*	*pp*	1, 0, 2	1
3	0, − 1, 0, 3	*ip*	*pp*	1, 0, 3	1
4	0, − 1, 0, 4	*pp*	*pp*	1, 0, 4	1
	1, − 1, − 1, 1	*pp*	2*pp*	2 (1, 0, 1)	2 . 1
5	0, − 1, 0, 5	*pp*	*pp*	1, 0, 5	1
6	0, − 1, 0, 6	*ip*	*pp*	1, 0, 6	1
7	0, − 1, 0, 7	*pp*	*pp*	1, 0, 7	1
	1, 0, − 2, 2 1, 0, − 2, − 2	*pp*	2*ip*	2 (2, ± 1, 4)	2σ
8	0, − 1, 0, 8	*pp*	*pp*	1, 0, 8	1
	0, − 2, 0, 1	*pp*	2*pp*	2 (1, 0, 2)	2 . 1
9	0, − 1, 0, 9	*ip*	*pp*	1, 0, 9	1
10	0, − 1, 0, 10	*pp*	*pp*	1, 0, 10	1
11	0, − 1, 0, 11	*pp*	*pp*	1, 0, 11	1
	0, − 2, − 1, 1			3, 1, 4	*d*
	0, − 2, 1, 1			3, − 1, 4	d^2
12	0, − 1, 0, 12	*ip*	*pp*	1, 0, 12	1
13	0, − 1, 0, 13	*pp*	*pp*	1, 0, 13	1
14	0, − 1, 0, 14	*pp*	*pp*	1, 0, 14	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
15	0, −1, 0, 15	ip	pp	1, 0, 15	1
	1, −2, 0, 2	pp	ip	4, ±1, 4	σc
	1, 2, 0, −2				
16	0, −1, 0, 16	pp	pp	1, 0, 16	1
17	0, −1, 0, 17	pp	pp	1, 0, 17	1
18	0, −1, 0, 18	ip	pp	1, 0, 18	1
	1, 1, −2, −2	pp	3pp	3 (1, 0, 2)	3 . 1
	1, −1, −2, 2				
19	0, −1, 0, 19			1, 0, 19	1
	0, 2, 1, −2	pp	pp	4, 1, 5	d
	0, −2, 1, 2			4, −1, 5	d^2
20	0, −1, 0, 20	pp	pp	1, 0, 20	1
	0, −2, 2, 1	pp	2pp	2 (2, 1, 3)	2c
21	0, −1, 0, 21	ip	pp	1, 0, 21	1
22	0, −1, 0, 22	pp	pp	1, 0, 22	1
23	0, −1, 0, 23			1, 0, 23	1
	1, −1, −2, 4	pp	pp	3, −1, 8	d^2
	1, 1, −2, −4			3, 1, 8	d
24	0, −1, 0, 24	ip	pp	1, 0, 24	1
	0, −2, 0, 3	ip	2pp	2 (2, 0, 3)	2c
25	0, −1, 0, 25	pp	pp	1, 0, 25	1
	1, −2, −1, 2	pp	5pp	5 (1, 0, 1)	5 . 1
	1, 2, −1, 2				
26	0, −1, 0, 26			1, 0, 26	1
	1, 0, −3, 2	pp	pp	3, −1, 9	g^2
	1, 0, −3, −2			3, 1, 9	g^4
27	0, −1, 0, 27			1, 0, 27	1
	0, −2, 1, 3	ip	pp	4, −1, 7	d^2
	0, 2, 1, −3			4, 1, 7	d
	0, −3, 0, 1	pp	3pp	3 (1, 0, 3)	3 . 1
28	0, −1, 0, 28	pp	pp	1, 0, 28	1
	1, 1, −3, 1	pp	2ip	2 (2, ±1, 4)	2σ
	1, −1, −3, −1				

Det. 4 ×	Classes	Order	on	Charact.	Compn.
29	0, – 1, 0, 29			1, 0, 29	1
	2, 1, – 2, – 2	*pp*	*pp*	5, 1, 6	g^2
	2, – 1, – 2, 2			5, – 1, 6	g^4
30	0, – 1, 0, 30	*ip*	*pp*	1, 0, 30	1
31	0, – 1, 0, 31			1, 0, 31	1
	2, 1, – 2, – 3	*pp*	*pp*	5, 2, 7	*d*
	2, – 1, – 2, 3			3, – 2, 7	d^2
32	0, – 1, 0, 32	*pp*	*pp*	1, 0, 32	1
33	0, – 1, 0, 33	*ip*	*pp*	1, 0, 33	1
34	0, – 1, 0, 34	*pp*	*pp*	1, 0, 34	1
35	0, – 1, 0, 35			1, 0, 35	1
	0, 2, 1, – 4	*pp*	*pp*	4, 1, 9	g^2
	0, – 2, 1, 4			4, – 1, 9	g^4
36	0, – 1, 0, 36	*ip*	*pp*	1, 0, 36	1
	0, – 2, – 2, 3	*ip*	2*pp*	2 (2, 1, 5)	2*c*
37	0, – 1, 0, 37	*pp*	*pp*	1, 0, 37	1
38	0, – 1, 0, 38			1, 0, 38	1
	2, – 2, – 1, 3	*ip*	*pp*	6, 2, 7	g^2
	2, 2, – 1, – 3			6, – 2, 7	g^4
39	0, – 1, 0, 39	*ip*	*pp*	1, 0, 39	1
	1, 1, – 3, – 1	*pp*	*ip*	4, – 1, 10	σe
	1, – 1, – 3, 1			4, 1, 10	σe^3
40	0, – 1, 0, 40	*pp*	*pp*	1, 0, 40	1
	0, – 2, 0, 5	*pp*	2*pp*	2 (2, 0, 5)	2 . 1
41	0, – 1, 0, 41	*pp*	*pp*	1, 0, 41	1
42	0, – 1, 0, 42	*ip*	*pp*	1, 0, 42	1
43	0, – 1, 0, 43			1, 0, 43	1
	0, – 2, 1, 5	*pp*	*pp*	4, – 1, 11	d^2
	0, 2, 1, – 5			4, 1, 11	*d*
44	0, – 1, 0, 44	*pp*	*pp*	1, 0, 44	1
	1, 2, – 1, – 4				
	1, – 2, – 1, 4	*pp*	2*ip*	2 (2, ± 1, 6)	2σ

Det. 4 ×	Classes	Order	on	Charact.	Compn.
45	0, − 1, 0, 45	ip	pp	1, 0, 45	1
	2, 0, − 3, 3	pp	3pp	3 (2, ± 1, 3)	3c
	2, 0, − 3, − 3				
46	0, − 1, 0, 46	pp	pp	1, 0, 46	1
47	0, − 1, 0, 47	pp	pp	1, 0, 47	1
	1, − 2, − 2, 2	pp	ip	6, 1, 8	σf
	1, 2, − 2, − 2			6, − 1, 8	$σf^4$
48	0, − 1, 0, 48	ip	pp	1, 0, 48	1
	1, − 1, − 3, 3	pp	4pp	4 (1, 0, 3)	4 . 1
	1, 1, − 3, − 3				
49	0, − 1, 0, 49	pp	pp	1, 0, 49	1
50	0, − 1, 0, 50	pp	pp	1, 0, 50	1
	2, 0, − 3, 2			6, − 2, 9	g^4
	2, 0, − 3, − 2			6, 2, 9	g^2
51	0, − 1, 0, 51	ip	pp	1, 0, 51	1
	0, − 2, 1, 6			4, − 1, 13	g^4
	0, 2, 1, − 6			4, 1, 13	g^2
52	0, − 1, 0, 52	pp	pp	1, 0, 52	1
	0, − 2, − 2, 5	pp	2pp	2 (2, 1, 7)	2c
53	0, − 1, 0, 53	pp	pp	1, 0, 53	1
	1, − 3, 0, 2			6, − 1, 9	g^2
	1, 3, 0, − 2			6, 1, 9	g^4
54	0, − 1, 0, 54	ip	pp	1, 0, 54	1
	1, 2, − 3, 0	pp		7, − 3, 9	g^4
	1, − 2, − 3, 0	pp		7, 3, 9	g^2
	0, − 3, 0, 2	pp	3pp	3 (2, 0, 3)	3c
	0, 3, 0, − 2				
55	0, − 1, 0, 55	pp	pp	1, 0, 55	1
	1, − 1, − 3, 5	pp	ip	4, − 1, 14	$σe^3$
	1, 1, − 3, − 5			4, 1, 14	σe
56	0, − 1, 0, 56	pp	pp	1, 0, 56	1
	0, − 2, 0, 7	pp	2pp	2 (2, 0, 7)	$2e^2$
57	0, − 1, 0, 57	ip	pp	1, 0, 57	1
58	0, − 1, 0, 58	pp	pp	1, 0, 58	1

Det. 4×	Classes	Order	on	Charact.	Compn.
59	0, − 1, 0, 59			1, 0, 59	1
	0, − 2, 1, 7	*pp*	*pp*	4, − 1, 15	j^6
	0, 2, 1, − 7			4, 1, 15	j^3
60	0, − 1, 0, 60	*ip*	*pp*	1, 0, 60	1
	1, 0, − 4, 4	*pp*	2*ip*	2 (2, ± 1, 8)	2σ
	1, 0, − 4, − 4				
61	0, − 1, 0, 61			1, 0, 61	1
	1, − 2, − 1, 6	*pp*	*pp*	5, − 2, 13	g^2
	1, 2, − 1, − 6			5, 2, 13	g^4
62	0, − 1, 0, 62	*pp*	*pp*	1, 0, 62	1
63	0, − 1, 0, 63	*ip*	*pp*	1, 0, 63	1
	1, 0, − 4, 2	*pp*	*ip*	4, − 1, 16	σe
	1, 0, − 4, − 2			4, 1, 16	σe^3
64	0, − 1, 0, 64	*pp*	*pp*	1, 0, 64	1
	1, 0, − 4, 0	*pp*	4*pp*	4 (1, 0, 4)	4 . 1
65	0, − 1, 0, 65	*pp*	*pp*	1, 0, 65	1
66	0, − 1, 0, 66	*ip*	*pp*	1, 0, 66	1
67	0, − 1, 0, 67			1, 0, 67	1
	0, − 2, 1, 8	*pp*	*pp*	4, − 1, 17	d^2
	0, 2, 1, − 8			4, 1, 17	d
68	0, − 1, 0, 68	*pp*	*pp*	1, 0, 68	1
	0, − 2, − 2, 7	*pp*	2*pp*	2 (2, 1, 9)	2e^2
69	0, − 1, 0, 69	*ip*	*pp*	1, 0, 69	1
70	0, − 1, 0, 70	*pp*	*pp*	1, 0, 70	1
71	0, − 1, 0, 71	*pp*	*pp*	1, 0, 71	1
	1, − 3, 1, 3	*pp*	*ip*	8, − 3, 10	σh^3
	1, 3, 1, − 3			8, 3, 10	σh^4
72	0, − 1, 0, 72	*ip*	*pp*	1, 0, 72	1
	0, − 2, 0, 9	*ip*	2*pp*	2 (2, 0, 9)	2c
	1, − 2, − 2, 4	*pp*	6*pp*	6 (1, 0, 2)	6 . 1
	1, 2, − 2, − 4				
	2, − 3, 0, 3	*pp*	3*pp*	3 (3, ± 1, 3)	3c
	2, 3, 0, − 3				
73	0, − 1, 0, 73	*pp*	*pp*	1, 0, 73	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
74	0, − 1, 0, 74	pp	pp	1, 0, 74	1
75	0, − 1, 0, 75	ip	pp	1, 0, 75	1
	0, − 2, 1, 9			4, − 1, 19	g^2
	0, 2, 1, − 9			4, 1, 19	g^4
76	0, − 1, 0, 76	pp	pp	1, 0, 76	1
	0, − 4, 1, 1			5, − 2, 16	g^2
	0, 4, 1, − 1			5, 2, 16	g^4
77	0, − 1, 0, 77	pp	pp	1, 0, 77	1
78	0, − 1, 0, 78	ip	pp	1, 0, 78	1
79	0, − 1, 0, 79	pp	pp	1, 0, 79	1
	2, − 2, − 2, 3	pp	ip	8, − 1, 10	σf
	2, 2, − 2, − 3			8, 1, 10	σf^4
80	0, − 1, 0, 80	pp	pp	1, 0, 80	1
81	0, − 1, 0, 81	pp	pp	1, 0, 81	1
	0, − 3, 2, 2	ip		9, − 3, 10	g^2
	0, 3, 2, − 2	ip		9, 3, 10	g^4
82	0, − 1, 0, 82	pp	pp	1, 0, 82	1
83	0, − 1, 0, 83	pp	pp	1, 0, 83	1
	0, − 2, 1, 10			4, − 1, 21	j^6
	0, 2, 1, − 10			4, 1, 21	j^3
84	0, − 1, 0, 84	ip	pp	1, 0, 84	1
	0, − 2, − 2, 9	ip	2pp	2 (2, 1, 11)	2σ
85	0, − 1, 0, 85	pp	pp	1, 0, 85	1
86	0, − 1, 0, 86	pp	pp	1, 0, 86	1
87	0, − 1, 0, 87	ip	pp	1, 0, 87	1
	1, − 2, − 3, 2	pp		7, − 2, 13	g^4
	1, 2, − 3, − 2	pp		7, 2, 13	g^2
88	0, − 1, 0, 88	pp	pp	1, 0, 88	1
	0, − 2, 0, 11	pp	2pp	2 (2, 0, 11)	2σ
89	0, − 1, 0, 89	pp	pp	1, 0, 89	1
	1, − 1, − 4, 2			5, − 1, 18	m^8
	1, 1, − 4, − 2			5, 1, 18	m^4
90	0, − 1, 0, 90	ip	pp	1, 0, 90	1

Det. 4 ×	Classes	Order	on	Charact.	Compn.
91	0, − 1, 0, 91	pp	pp	1, 0, 91	1
	0, − 2, 1, 11			4, − 1, 23	g^4
	0, 2, 1, − 11			4, 1, 23	g^2
92	0, − 1, 0, 92	pp	pp	1, 0, 92	1
	2, − 3, 0, 4			9, − 4, 12	g^4
	2, 3, 0, − 4			9, 4, 12	g^2
93	0, − 1, 0, 93	ip	pp	1, 0, 93	1
94	0, − 1, 0, 94	ip	pp	1, 0, 94	1
95	0, − 1, 0, 95	pp	pp	1, 0, 95	1
	1, − 2, − 2, 6	pp	ip	6, − 1, 16	σi^7
	1, 2, − 2, − 6			6, 1, 16	σi
96	0, − 1, 0, 96	ip	pp	1, 0, 96	1
97	0, − 1, 0, 97	pp	pp	1, 0, 97	1
98	0, − 1, 0, 98	pp	pp	1, 0, 98	1
99	0, − 1, 0, 99	ip	pp	1, 0, 99	1
	0, − 2, 1, 12			4, − 1, 25	g^2
	0, 2, 1, − 12			4, 1, 25	g^4
100	0, − 1, 0, 100	pp	pp	1, 0, 100	1
	0, − 2, − 2, 11	pp	2pp	2 (2, 1, 13)	2 . 1
	1, − 1, − 4, 4	pp	5pp	5 (1, 0, 4)	5 . 1
	1, 1, − 4, − 4				
	1, − 3, − 1, 3	pp	10pp	10 (1, 0, 1)	10 . 1
	1, 3, − 1, − 3				

Table II. *of the binary cubic forms the determinants of which are the positive numbers* ≡ 1 (mod. 4) *from* − 3 *to* − 99.

Det. 4 ×	Classes	Order	on	Charact.	Compn.
3	0, 1, 1, 0	ip	ip	2, ± 1, 2	σ
	1, 0, − 1, 1	pp			
	1, 0, − 1, − 1	pp			
7	0, − 1, − 1, 1	pp	ip	2, 1, 4	σ
11	0, − 1, − 1, 2	pp	ip	2, 1, 6	σ
15	0, − 1, − 1, 3	ip	ip	2, 1, 8	σ
19	0, − 1, − 1, 4	pp	ip	2, 1, 10	σ

Det. 4×	Classes	Order	on	Charact.	Compn.
23	0, − 1, − 1, 5			2, 1, 12	σ
	1, − 1, − 1, 2	pp	ip	4, − 1, 6	σd
	1, 1, − 1, − 2			4, 1, 6	$σd^2$
27	0, − 1, − 1, 6	ip	ip	2, 1, 14	σ
	1, 1, − 2, − 1				
	1, − 1, − 2, 1	pp	3ip	3 (2, ± 1, 2)	3σ
	2, − 1, − 1, 2				
31	0, − 1, − 1, 7			2, 1, 16	σ
	1, 0, − 2, 1	pp	ip	4, − 1, 8	σd
	1, 0, − 2, − 1			4, 1, 8	$σd^2$
35	0, − 1, − 1, 8	pp	ip	2, 1, 18	σ
39	0, − 1, − 1, 9	ip	ip	2, 1, 20	σ
43	0, − 1, − 1, 10	pp	ip	2, 1, 22	σ
47	0, − 1, − 1, 11	pp	ip	2, 1, 24	σ
51	0, − 1, − 1, 12	ip	ip	2, 1, 26	σ
55	0, − 1, − 1, 13	pp	ip	2, 1, 28	σ
59	0, − 1, − 1, 14			2, 1, 30	σ
	1, − 1, − 2, 1	pp	ip	6, 1, 10	σj
	1, 1, − 2, − 1			6, − 1, 10	$σj^2$
63	0, − 1, − 1, 15	ip	ip	2, 1, 32	σ
67	0, − 1, − 1, 16	pp	ip	2, 1, 34	σ
71	0, − 1, − 1, 17	pp	ip	2, 1, 36	σ
75	0, − 1, − 1, 18	ip	ip	2, 1, 38	σ
79	0, − 1, − 1, 19	pp	ip	2, 1, 40	σ
83	0, − 1, − 1, 20			2, 1, 42	σ
	1, − 1, − 2, 3	pp	ip	6, − 1, 14	$σj^2$
	1, 1, − 2, − 3			6, 1, 14	σj
87	0, − 1, − 1, 21	ip		2, 1, 44	σ
	1, − 2, 0, 3	pp	ip	8, − 3, 12	$σg^2$
	1, 2, 0, − 3	pp		8, 3, 12	$σg^4$
91	0, − 1, − 1, 22	pp	ip	2, 1, 46	σ
95	0, − 1, − 1, 23	pp	ip	2, 1, 48	σ
99	0, − 1, − 1, 24	ip	ip	2, 1, 50	σ
	1, 0, − 3, 3				
	1, 0, − 3, − 3	pp	3ip	3 (2, ± 1, 6)	3σ

Table III. *of the binary cubic forms the determinants of which are the negative numbers* – 972, – 1228, 1336, – 1836 *et* – 2700. (– 4 × 675 = – 2700 is beyond Arndt's Tables.)

Det. 4 ×	Classes	Order	on	Charact.	Compn.
243	0, – 1, 0, 243	*ip*		1, 0, 243	1
	1, – 1, – 6, 0	*pp*		7, 3, 36	d
	1, 1, – 6, 0	*pp*		7, – 3, 36	d^2
	0, 2, 1, – 30	*ip*		4, 1, 61	d_1
	2, 3, – 2, – 5	*pp*	*pp*	13, – 2, 19	dd_1
	0, 3, 2, – 8	*pp*		9, 3, 28	d^2d_1
	0, – 2, 1, 30	*ip*		4, – 1, 61	d_1^2
	0, – 3, 2, 8	*pp*		9, – 3, 28	dd_1^2
	2, – 3, – 2, 5	*pp*		13, 2, 19	$d^2d_1^2$
307	0, – 1, 0, 307			1, 0, 307	1
	1, 1, – 6, – 8			7, 1, 44	d
	1, – 1, – 6, 8			7, – 1, 44	d^2
	0, 2, 1, – 38			4, 1, 77	d_1
	1, – 3, – 2, 8	*pp*	*pp*	11, – 1, 28	dd_1
	4, 1, – 4, – 3			17, 4, 19	d^2d_1
	0, – 2, 1, 38			4, – 1, 77	d_1^2
	4, – 1, – 4, 3			17, – 4, 19	dd_1^2
	1, 3, – 2, 8			11, 1, 28	$d^2d_1^2$
339	0, – 1, 0, 339	*ip*		1, 0, 339	1
	1, 0, – 7, – 4	*pp*		7, 2, 49	d
	1, 0, – 7, 4	*pp*		7, – 2, 49	d^2
	0, 2, 1, – 42	*ip*		4, 1, 85	d_1
	3, 0, – 5, – 4	*pp*	*pp*	15, 6, 25	dd_1
	2, – 3, – 2, 8	*pp*		13, – 5, 28	d^2d_1
	0, – 2, 1, 42	*ip*		4, – 1, 85	d_1^2
	2, 3, – 2, – 8	*pp*		13, 5, 28	dd_1^2
	3, 0, – 5, 4	*pp*		15, – 6, 25	$d^2d_1^2$
459	0, – 1, 0, 459	*ip*		1, 0, 459	1
	0, 3, 2, – 16	*pp*		9, 3, 52	d
	0, – 3, 2, 16	*pp*		9, – 3, 52	d^2
	0, 2, 1, – 57	*ip*		4, 1, 115	d_1
	1, 4, – 3, – 4	*pp*	*pp*	19, – 4, 25	dd_1
	2, – 1, 6, 0	*pp*		13, 3, 36	d^2d_1
	0, – 2, 1, 57	*ip*		4, – 1, 115	d_1^2
	2, 1, – 6, 0	*pp*		13, – 3, 36	dd_1^2
	1, – 4, – 3, 4	*pp*		19, 4, 25	$d^2d_1^2$
	0, – 3, 0, 17	*pp*	3*pp*	3 (3, 0, 17)	$3g^3$

Det. 4 ×	Classes	Order	on	Charact.	Compn.
675	0, − 1, 0, 675	ip	pp	1, 0, 675	1
	0, 3, 2, − 24			9, 3, 76	d
	0, − 3, 2, 24			9, − 3, 76	d^2
	0, 2, 1, − 168			4, 1, 169	d_1
	0, 5, − 4, − 3			25, − 10, 31	dd_1
	3, − 1, − 6, 0			19, 3, 36	d^2d_1
	0, − 1, 1, 168			4, − 1, 169	d_1^2
	3, 1, − 6, 0			19, − 3, 36	dd_1^2
	0, − 5, − 4, 3			25, 10, 31	$d^2d_1^2$

N.B. For this last determinant -4×675, there may possibly be other cubic classes based on a non-primitive characteristic; I have not ascertained whether such forms do or do not exist.