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ON EVOLUTES AND PARALLEL CURVES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XI. (1871), pp. 183—200.]

IN abstract geometry we have a conic called the Absolute; lines which are harmonics of each other in regard to the absolute, or, what is the same thing, which are such that each contains the pole of the other in regard to the absolute, are said to be at right angles. Similarly, points which are harmonics of each other in regard to the absolute, or, what is the same thing, which are such that each lies in the polar of the other, are said to be quadrantal.

A conic having double contact with the absolute is said to be a circle; the intersection of the two common tangents is the centre of the circle; the line joining the two points of contact, or chord of contact, is the axis of the circle.

Taking as a definition of equidistance that the points of a circle are equidistant from the centre, we arrive at the notion of distance generally, and we can thence pass down to that of equal circles; but the notion of equal circles may be established descriptively in a more simple manner:

Any two circles have an axis of symmetry, viz. this is the line joining their centres; and they have a centre of homology, viz. this is the intersection of their axes. They intersect in four points, lying in pairs on two lines through the centre of homology: they have also four tangents meeting in pairs in two points on the axis of symmetry. Now if the two lines through the centre of homology are harmonically related to the two axes, or, what is the same thing, if the two points on the axis of symmetry are harmonically related to the two centres, then the circles are *equal*.

Circles which are equal to the same circle are equal to each other, and the entire series of circles which are equal to a given circle, are said to be a system of circles of constant magnitude.

Starting from these general considerations, I pass to the question of evolutes and parallel curves: it will be understood that everything—lines at right angles, circles, poles, polars, reciprocal curves, &c.—refers to the absolute.

At any point of a curve we have a normal, viz. this is a line at right angles to the tangent; or, what is the same thing, it is the line joining the point with the pole of the tangent. The locus of the pole of the tangent is the reciprocal curve, and for any point of a given curve, the pole of the tangent at that point is the corresponding point of the reciprocal curve. Hence, also the normal is the line from the point to the corresponding point of the reciprocal curve. And the curve and its reciprocal have at corresponding points the same normal.

The envelope of the normals is the evolute; any curve having with the given curve the same normals (and therefore the same evolute) is a parallel curve; in other words, the parallel curve is any orthogonal trajectory of the normals of the given curve.

The parallel curve is also the envelope of a circle of constant radius having its centre on the given curve; or, again, it is the envelope of a circle of constant radius touching the given curve.

The theory in the above form is directly applicable to spherical, or rather conical, geometry; but in ordinary plane geometry the absolute degenerates into a point-pair, the two circular points at infinity, or say the points I, J; and this is a case that requires to be separately treated. The theory in the general case, the absolute a conic, is the more symmetrical and elegant, and it might appear advantageous to commence with this; but upon the whole I prefer the opposite course, and will commence with the case of plane geometry, the absolute a point-pair.

The subject connects itself with that of foci: I call to mind that a common tangent of the curve and the absolute is a focal tangent, and the intersection of two focal tangents a focus. In the case where the absolute is a point-pair, the focal tangents are the tangents from I to the curve, and the tangents from J to the curve, or say these are the I-tangents and the J-tangents; a focus is the intersection of an I-tangent and a J-tangent; the line IJ, when it touches the curve, and (when the curve passes through I and J or either of them) the tangents at I or J to the curve are usually not reckoned as focal tangents; and other singular tangents, for instance a double or stationary tangent through I or J, are also excluded from the focal tangents; and the number of foci is of course reckoned accordingly, viz. it is the product of the number of the I-tangents into that of the J-tangents. So when the absolute is a conic; if this is touched by the curve, the common tangent at the point of contact is not reckoned as a focal tangent; and we may also exclude any singular tangents which touch the absolute; and the number of foci is reckoned accordingly, viz. it is equal to the number of pairs of focal tangents.

Let the Plückerian numbers for the given curve be $(m, n, \delta, \kappa, \tau, \iota)$, viz. m the order, n the class, δ the number of nodes, κ of cusps, τ of bitangents, ι of inflexions; and suppose moreover that D is the deficiency, and α the statitude; viz.

$$\alpha = 3m + \iota, \quad = 3n + \kappa;$$

$$2D = (m-1)(m-2) - 2\delta - 2\kappa, \quad = (n-1)(n-2) - 2\tau - 2\iota, \quad = -2m - 2n + 2 + \alpha$$
$$= n - 2m + 2 + \kappa, \quad = m - 2n + 2 + \iota,$$

And let the corresponding numbers for the evolute be

$$(m'', n'', \delta'', \kappa'', \tau'', \iota''; D'', \alpha'').$$

These are most readily obtained, as in Clebsch's paper, "Ueber die Singularitäten algebraischer Curven," *Crelle*, t. LXIV. (1864), pp. 98—100, viz. it being assumed that we have

$$n'' = m + n, \ \iota'' = 0,$$

then by reason that the evolute has a (1, 1) correspondence with the original curve, the two curves have the same deficiency, or writing this relation under the form

$$m'' - 2n'' + \iota'' = m - 2n + \iota,$$

we have $m'' = 3m + \iota, = \alpha$; and the Plückerian relations then give the values of κ'', δ'', τ''.

In regard to these equations $n'' = m + n$, $\iota'' = 0$, I remark that if we have two curves of the orders m, m', and on these points P, Q having an (α, α') correspondence, the line PQ envelopes a curve of the class $m\alpha' + m'\alpha$, and the number of inflexions is in general $= 0$. Now in the present case, taking P on the given curve and Q the point of intersection with IJ of the normal (or harmonic of the tangent), the orders of the curves are $(m, 1)$, and the correspondence is $(n, 1)$; whence as stated $m'' = m + n$, $\iota'' = 0$.

The formulæ thus are

$$\begin{aligned} m'' &= \alpha, \\ n'' &= m + n, \\ \iota'' &= 0, \\ \kappa'' &= -3m - 3n + 3\alpha, \\ \alpha'' &= 3\alpha, \\ D'' &= D, \end{aligned}$$

in which formulæ it is assumed that the curve has no special relations to the points I, J; or, what is the same thing, that the line IJ intersects the curve in m points distinct from each other, and from the points I, J.

It is to be added (see Salmon's *Higher Plane Curves*, [Ed. 2], (1852), pp. 109 et seq.) that m of the κ'' cusps arise from the intersections of the curve with IJ, these cusps being situate on the line IJ, and each of them the harmonic of one of the intersections in question, and the cuspidal tangent being for each of them the line IJ. The intersections of the evolute by the line IJ are these m cusps each 3 times, and besides ι points arising from the ι inflexions of the curve; viz. at any inflexion the two consecutive normals intersect in a point on the line IJ, being in fact the harmonic of the intersection of IJ with the tangent at the inflexion. It was in this manner that Salmon obtained the number $3m + \iota$ of the points at infinity of the evolute, that is the expression $m'' = 3m + \iota$ $(= \alpha)$ for the order of the evolute.

The remaining $-4m - 3n + 3\alpha$ cusps arise from the points on the curve where there is a circle of 4-pointic intersection, or contact of the third order, and in this manner the number of them was found, Salmon, [Ed. 2], p. 113, in the particular case of a curve without nodes or cusps, and generally in Zeuthen's *Nyt Bidrag* &c., p. 91; the number of the points in question, in the foregoing form $-4m-3n+3\alpha$, is also obtained in my Memoir, "On the curves which satisfy given conditions," *Phil. Trans.* (1868), pp. 75—143, see p. 97, [406].

It is further to be noticed that the $m+n$ tangents to the evolute from either of the points I, J are made up of the line IJ counting m times (in respect that it is a tangent at each of the above-mentioned m cusps) and of the n tangents from the points in question to the original curve. Or taking the two points I, J conjointly, say the $2m+2n$ common tangents of the absolute and the evolute are made up of the line IJ (or axis of the absolute) counting m times, and of the $2n$ focal tangents of the original curve. The focal tangents of the original curve and of the evolute are thus the same $2n$ lines; and the two curves have the same foci.

The above are the ordinary values of m'', n'', ι'', κ'', but if the given curve touch the line IJ, then the evolute has at the point of contact an inflexion, the stationary tangent being the line IJ; and if the given curve pass through one or other of the points I, J, the evolute has in this case an inflexion on the tangent at the point in question, this tangent being the stationary tangent of the evolute: but observe that the inflexion is not at the point I or J in question: and for each inflexion there is a diminution $=1$ in the class, 3 in the order, and 5 in the number of cusps. Suppose that the point I is a f_2-tuple point on the given curve; then the evolute has f_1 inflexions; and similarly if the point J is a f_2-tuple point on the given curve, then the evolute has f_2 inflexions. Hence writing $f_1+f_2=f$, we have thus f inflexions; and if moreover the number of contacts with the line IJ be $=g$, then we have on this account g inflexions; or in all $f+g$ inflexions, and the formulæ become

$$\begin{aligned} m'' &= \alpha - 3f - 3g, \\ n'' &= m+n - f - g, \\ \iota'' &= f + g, \\ \kappa'' &= -3m-3n+3\alpha-5f-5g. \end{aligned}$$

It is to be noticed here that the number of the intersections of the given curve with the line IJ (other than the points I, J and the points of contact) is $=m-f_1-f_2-2g$, that is $m-f-2g$: each of these gives as before a cusp on the evolute, the cuspidal tangent being IJ; we have besides on the line IJ (in respect of the g contacts) g inflexions, the stationary tangent being the line IJ; and each of the ι inflexions gives for the evolute a point on the line IJ; hence the whole number of intersections with the line IJ is $3(m-f-2g)+3g+\iota$, $=3m+\iota-3f-3g$, which is thus the order of the evolute.

The tangents from the point I or J to the evolute are the line IJ counting $m-f-2g$ times in respect of the cusps on this line and $2g$ times in respect to the inflexions, that is $m-f$ times; the tangents at the point in question to the given curve each twice as touching the evolute at an inflexion, $2f_1$ or $2f_2$: and the remaining

$n-2f_1-g$, or $n-2f_2-g$ tangents from the point in question to the given curve; the whole number is thus $(m-f)+2f_1+(n-2f_1-g)$ or $(m-f)+2f_2+(n-2f_2-g)$, $=m+n-f-g$, the class of the evolute. The two values of n'' give

$$2n''=2m+(n-2f_1-g)+(n-2f_2-g),$$

viz. twice the class of the evolute = twice the order of the curve + the number of the focal I-tangents + that of the focal J-tangents; but this is not true for all relations whatever of the curve to the absolute.

The tangents from I to the given curve (excluding the line IJ and the tangents at I) are $n-2f_1-g$ tangents; and similarly the tangents from I to the evolute (excluding the line IJ and the stationary tangents through I) are the same $n-2f_1-g$ tangents; say the curve and the evolute have the same $n-2f_1-g$ I-tangents. Similarly they have the same $n-2f_2-g$ J-tangents; or together the same $2(n-f_1-f_2-g)$, $=2(n-f-g)$ focal tangents. And the curve and evolute have the same $(n-2f_1-g)(n-2f_2-g)$ foci.

The foregoing specialities f and g refer, g to the ordinary contacts of the line IJ with the curve, viz. the curve is supposed to have with the curve at an ordinary or non-singular point thereof a contact or 2-pointic intersection, and f, that is f_1 or f_2, to the multiple points having f_1 or f_2 ordinary branches, none of them touching the line IJ. Thus the formulæ do not apply to the cases of IJ passing through a node or a cusp of the given curve, or touching it at an inflexion; nor to the cases where at I or J the curve touches IJ, or where there is at I or J an ordinary double point with one of its branches touching IJ, or where there is at I or J a cusp, where the cuspidal tangent is or is not IJ.

It is easy to see that in the case of a multiple point of any kind whether situate on IJ or at I or J, each branch of the curve produces its own separate effect on the singularities of the evolute: thus if we have on IJ a double point neither branch touching IJ, then the separate effect of each branch is *nil*, therefore the effect of the double point is also *nil*: but if one branch touch the line IJ, then the whole effect is the same as if we had this branch only; viz. we have here the case $g=1$. And so if there is at I or J a double point with one branch touching the line IJ, then the effect of this branch is as if we had this branch only (a case not yet investigated) but the other branch is the case $f=1$. And so if we have at I or J a double point with two ordinary branches touching each other (tacnode or, if the two branches have a contact higher than the first order, oscnode), then if the branches do not touch the line IJ the case is $f=2$, but if they do, then the effect is twice that of an ordinary branch touching IJ. In support of these conclusions, observe that such multiple points, with ordinary branches, present themselves in the case of two or more curves which intersect or touch each other in any manner; and that the evolute of a system of two or more curves is simply the system of the evolutes of the several curves.

It follows as regards the relations of the given curve to the points I, J, and the effect thereby produced on the evolute, we only need to consider the case of a single branch; viz. the cases are

the given curve intersects the line IJ at a point other than I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ;

and the given curve passes through the point I or J, and belonging thereto there is a branch ordinary or singular,

not touching IJ,

touching IJ.

I have succeeded in determining the effect, not for a singular branch of any kind whatever, but for branches of the form $y=x^k$, $y^{k-1}=x^k$; viz. $k=2$, each of these is an ordinary branch, $k=3$, the first $y=x^3$ is an inflexional branch and the second $y=x^{\frac{3}{2}}$ a cuspidal branch; and so $k>3$ the two branches are respectively inflexional and cuspidal of a higher order. I do this very simply by consideration of the curve $x^{k-1}z=y^k$.

The curve in question $x^{k-1}z=y^k$, is a unicursal curve, and it has a reciprocal of the same form $X^{k-1}Z=Y^k$, hence

$$m=n=k;\quad 0=n-2m+2+\kappa,$$

whence

$$\iota=\kappa=k-2,\quad \tau=\delta=\tfrac{1}{2}(k-2)(k-3);$$

viz. the point $x=0$, $y=0$ is a cusp equivalent to $k-2$ cusps and $\frac{1}{2}(k-2)(k-3)$ nodes; and the point $z=0$, $y=0$ is an inflexion equivalent to $k-2$ inflexions and $\frac{1}{2}(k-2)(k-3)$ bitangents.

The equation $U=x^{k-1}z-y^k=0$ of the curve is satisfied by writing therein $x:y:z=1:\theta:\theta^k$; and these values give

$$d_xU : d_yU : d_zU=(k-1)\,x^{k-2}z : -ky^{k-1} : x^{k-1},\ =(k-1)\,\theta^k : -k\theta^{k-1} : 1.$$

Taking the coordinates of I, J to be (α, β, γ) and $(\alpha', \beta', \gamma')$ respectively, and X, Y, Z as current coordinates, the equation of the normal at the point (x, y, z) of the curve $U=0$ is readily found to be

$$(\alpha' d_xU+\beta' d_yU+\gamma' d_zU)\begin{vmatrix} X, & Y, & Z \\ \alpha, & \beta, & \gamma \\ x, & y, & z \end{vmatrix} + (\alpha\, d_xU+\beta\, d_yU+\gamma\, d_zU)\begin{vmatrix} X, & Y, & Z \\ \alpha', & \beta', & \gamma' \\ x, & y, & z \end{vmatrix}=0.$$

Hence for the curve in question the equation of the normal is

$$\{(k-1)\,\alpha'\theta^k - k\beta'\theta^{k-1} + \gamma'\}\,\{(\beta X - \alpha Y)\,\theta^k + (\alpha Z - \gamma X)\,\theta + (\gamma Y - \beta Z)\}$$
$$+\{(k-1)\,\alpha\theta^k - k\beta\theta^{k-1} + \gamma\}\,.\,\{(\beta' X - \alpha' Y)\,\theta^k + (\alpha' Z - \gamma' X)\,\theta + (\gamma' Y - \beta' Z)\} = 0,$$

or, expanding and reducing, this equation is

$$\begin{aligned}
&\theta^{2k}\,.\,(k-1)\ \ \{(\alpha\beta' + \alpha'\beta)\,X - 2\alpha\alpha'\,Y\}\\
&+\theta^{2k-1}\,.\quad -k\,\{2\beta\beta' X - (\alpha\beta' + \alpha'\beta)\,Y\}\\
&+\theta^{k+1}\,.\,(k-1)\,\{2\,\alpha\alpha'\,Z - (\alpha\gamma' + \alpha'\gamma)\,X\}\\
&+\theta^{k}\,.\,\{(k+1)\,(\beta\gamma' + \beta'\gamma)\,X + (k-2)\,(\gamma\alpha' + \gamma'\alpha)\,Y - (2k-1)\,(\alpha\beta' + \alpha'\beta)\,Z\}\\
&+\theta^{k-1}\,.\quad -k\,\{(\beta\gamma' + \beta'\gamma)\,Y - 2\beta\beta' Z\}\\
&+\theta\quad .\qquad \{(\gamma\alpha' + \gamma'\alpha)\,Z - 2\gamma\gamma' X\}\\
&+\qquad\qquad \{2\gamma\gamma' Y - (\beta\gamma' + \beta'\gamma)\,Z\} = 0,
\end{aligned}$$

where k is a positive integer not less than 2; hence except in the case $k = 2$, all the terms θ^{2k}, $\theta^{2k-1}, \ldots \theta$, θ^0, have different indices, and the coefficients $k-1$, k, &c. none of them vanish; if however $k = 2$, then the terms θ^{2k-1}, θ^{k+1} coalesce into a single term, as do also the terms θ^{k-1} and θ; moreover the coefficient $k-2$ is $= 0$.

The evolute is the envelope of the line represented by the foregoing equation, considering therein θ as an arbitrary parameter; viz. the equation is obtained by equating to zero the discriminant of the foregoing equation in θ. Hence in general the class of the evolute is $= 2k$, and its order is $= 2(2k-1)$; results which agree with the formulæ for n'', m'', since in the present case $m + n$, $= k + k$, $= 2k$, $\alpha = 3n + \kappa$, $= 3k + (k-2)$, $= 4k - 2$. And moreover there are not any inflexions, $\iota'' = 0$ as before.

The equation may however contain a factor in θ independent of (X, Y, Z), and throwing out this factor, say its order is $= s$, the expression for the class is $2k - s$, $= m + n - s$, and that for the order is $4k - 2 - 2s$, $= \alpha - 2s$. Moreover, in the original equation or in the equation thus reduced, it may happen that the equation will on writing therein $\Omega = 0$ (Ω a linear function of X, Y, Z) acquire a factor of the order ω, independent of (X, Y, Z); the line $\Omega = 0$ is in this case a stationary tangent, $= \omega - 1$ inflexions; and the discriminant contains the factor $\Omega^{\omega-1}$, which may be thrown out; that is we have here $n'' = 2k - s$, $\iota'' = \omega - 1$, $m'' = 4k - 2 - 2s - (\omega - 1)$; agreeing with the relation $m'' - 2n'' + 2 + \iota'' = 0$ which holds good in virtue of the evolute being a unicursal curve. It is in this manner that the values of m'', n'', ι'' are obtained in the several cases to be considered, viz.:

A_k Inflexion situate on IJ, which is not a tangent.
B_k Inflexion situate on IJ, which is a tangent.
C_k Cusp situate on IJ, which is not a tangent.
D_k Cusp situate on IJ, which is a tangent.
P_k Inflexion at J, IJ not a tangent.
Q_k Inflexion at J, IJ a tangent.
R_k Cusp at J, IJ not a tangent.
S_k Cusp at J, IJ a tangent.

The results are respectively as follows:

	A_k	B_k	C_k	D_k	P_k	Q_k	R_k	S_k
$m'' = \alpha \quad -$	0	$3k-3$	$k-2$	$k+1$	k	$3k-2$	$2k-2$	$2k$
$n'' = m+n \quad -$	0	$k-1$	0	1	1	k	$k-1$	k
$\iota'' = 0 \quad +$	0	$k-1$	$k-2$	$k-1$	$k-2$	$k-2$	0	0
$\kappa'' = -3m-3n+3\alpha \, -$	0	$5k-5$	$2k-4$	$2k+1$	$2k-1$	$5k-4$	$3k-3$	$3k$

	A_2C_2	B_2D_2	A_3	B_3	C_3	D_3	$\bar{P}_2\bar{R}_2$	Q_2S_2	P_3	Q_3	R_3	S_3
$m'' = \alpha \quad -$	0	3	0	6	0	4	3	4	3	7	4	6
$n'' = m+n \, -$	0	1	0	2	0	1	1	2	1	3	2	3
$\iota'' = 0 \quad +$	0	1	0	2	0	2	1	0	1	1	0	0
$\kappa'' = -3m-3n+3\alpha \, -$	0	5	0	10	0	7	5	6	5	11	6	9

read for instance in B_k, $m'' = \alpha - (3k-3)$, $n'' = m+n-(k-1)$, $\iota'' = 0+(k-1)$, and $\kappa'' = -3m-3n+3\alpha+(5k-5)$; and so in other cases.

A_2C_2 (that is indifferently A_2 or C_2) is when there is on IJ an ordinary point, IJ not a tangent; and so B_2D_2 when there is on IJ an ordinary point, IJ a tangent. Similarly P_2R_2 when there is at J an ordinary point, IJ not a tangent; only instead thereof I have written $\bar{P}_2\bar{R}_2$ to indicate that (for a reason which will appear) the numbers are *not* deducible from those for P_2 or R_2 by writing therein $k=2$; and Q_2S_2 is when there is at J an ordinary point, IJ a tangent.

Case A_k. We have to take the line IJ passing through the inflexion; the condition for this is $\beta\gamma' - \beta'\gamma = 0$: there is no speciality, or we have $n'' = 2k$, $m'' = 4k-2$, $\iota'' = 0$; whence also $\kappa'' = 0$; the value of κ'' being in every case deduced from those of m'', n'', ι'' by the formula

$$3m'' + \iota'' = 3n'' + \kappa''.$$

Case B_k. I write $\gamma = \gamma' = 0$, the equation of the normal is

$$\begin{array}{ll|ll}
\theta^{2k} & .\,(k-1)\,\{(\alpha\beta' + \alpha'\beta)\,X - 2\alpha\alpha' Y\} & \theta^{k+1} & (X,\ Y) \\
+\,\theta^{2k-1} & .-k\,\{2\beta\beta' X - (\alpha\beta' + \alpha'\beta)\,Y\} & +\,\theta^{k} & (X,\ Y) \\
+\,\theta^{k+1} & .\,(k-1)\,2\alpha\alpha' Z & +\,\theta^{2} & Z \\
+\,\theta^{k} & .-(2k-1)\,(\alpha\beta' + \alpha'\beta)\,Z & +\,\theta & Z \\
+\,\theta^{k-1} & .\ \ k\,.\ \ \ 2\beta\beta' Z \qquad\qquad = 0, & + & Z = 0,
\end{array}$$

where throwing out the factor θ^{k-1}, the form is as shown on the right hand. Writing $Z=0$, we have a factor θ^k, whence $\iota''=k-1$, and then $n''=k+1$, $m''=2k-(k-1)=k+1$, agreeing with the table. The process holds good for $k=2$.

Case C_k. I write $\beta=\beta'=0$; this brings as well the inflexion as the cusp upon the line IJ; but it has been seen (Case A_k) that there is not any reduction on account of this position of the inflexion, hence the whole effect will be due to the cusp. The equation is

$$\begin{array}{lll|l}
\theta^{2k} & .(k-1)\{-2\alpha\alpha' Y\} & & \theta^{2k}\ Y \\
+\ \theta^{k+1} & .(k-1)\{2\alpha\alpha' Z-(\alpha\gamma'+\alpha'\gamma)X\} & & +\ \theta^{k+1}(Z,\ X) \\
+\ \theta^{k} & .(k-2)(\gamma\alpha'+\gamma'\alpha)Y & & +\ \theta^{k}(k-2)\,Y \\
+\ \theta & . & \{(\gamma\alpha'+\gamma'\alpha)Z-2\gamma\gamma' X\} & +\ \theta\ (Z, X) \\
+ & & 2\gamma\gamma' Y=0, & +\quad Y=0,
\end{array}$$

so that here $n''=2k$. On writing $Y=0$, there is a factor $\left(1-\dfrac{\theta}{\infty}\right)^{k-1}$ thrown out (indicated by the reduction of the order from $2k$ to $k+1$), whence

$$\iota''=k-2,\quad m''=2(2k-1)-(k-2),\quad =3k.$$

The process holds good for $k=2$.

Case D_k. We may write $\alpha=\alpha'=0$; the equation is

$$\begin{array}{ll|l}
\theta^{2k-1} & .-k\,.\,2\beta\beta' X & \theta^{2k-1}\ X \\
+\ \theta^{k} & .\quad (k+1)(\beta\gamma'+\beta'\gamma)X & +\ \theta^{k}\ X \\
+\ \theta^{k-1} & .-k\,[(\beta\gamma'+\beta'\gamma)Y-2\beta\beta' Z] & +\ \theta^{k-1}(Y,\ Z) \\
+\ \theta & .-2\gamma\gamma' X & +\ \theta\quad X \\
 & +\,2\gamma\gamma' Y-(\beta\gamma'+\beta'\gamma)Z=0, & +\quad (Y,\ Z)=0,
\end{array}$$

so that $n''=2k-1$. Writing $X=0$, we have the factor $\left(1-\dfrac{\theta}{\infty}\right)^{k}$ (indicated by the reduction of order from $2k-1$ to $k-1$), whence $\iota''=k-1$, and then $m''=(4k-2)-2-(k-1)$, $=3k-3$, agreeing with the table. The process holds good for $k=2$.

Case P_k. We have $\beta'=\gamma'=0$; the equation is

$$\begin{array}{lll|l}
\theta^{2k} & .(k-1)\{\alpha'\beta X-2\alpha\alpha' Y\} & & \theta^{2k-1}(X,\ Y) \\
+\ \theta^{2k-1} & .-k & \{-\alpha'\beta Y\} & +\ \theta^{2k-2}\ Y \\
+\ \theta^{k+1} & .(k-1)\{-\alpha'\gamma X\} & & +\ \theta^{k}\ X \\
+\ \theta^{k} & .(k-2)\gamma\alpha' Y-(2k-1)\alpha'\beta Z & & +\ \theta^{k-1}(k-2)\,Y+Z \\
+\ \theta & . & \gamma\alpha' Z=0, & +\quad Z=0,
\end{array}$$

so that here $n''=2k-1$. Writing $Z=0$, we have the factor θ^{k-1}, whence $\iota''=k-2$, $m''=4k-4-(k-2)=3k-2$, agreeing with the table. If, however, $k=2$, then on writing $Z=0$ the equation (instead of the factor θ^{k-1}) acquires the factor $\theta^k\,(=\theta^2)$; so that here $n''=3$, $\iota''=1$, $m''=3$, agreeing with the column P_2R_2 of the table.

Case Q_k. We have $\gamma = 0$, $\beta' = \gamma' = 0$, viz. $\beta' = 0$ in the formulæ of B_k. The equation is

$$\begin{array}{ll|ll} & \theta^{2k} \,.\, (\;k-1)\,\alpha'\beta Y & & \theta^k \;\; X \\ + & \theta^{2k-1} \,.\, -k\,(-\,\alpha'\beta Y) & + & \theta^{k-1} Y \\ + & \theta^{k+1} \,.\, (\;k-1)\,2\alpha\alpha' Z & + & \theta \;\; Z \\ + & \theta^k \,.\, -(2k-1)\,\alpha'\beta Z = 0, & + & Z = 0, \end{array}$$

so that $n'' = k$. For $Z = 0$ there is the factor θ^{k-1}, hence $\iota'' = k-2$, $m'' = 2(k-1) - (k-2), = k$. The process holds good for $k = 2$.

Case R_k. We have $\beta = 0$, $\alpha' = 0$, $\beta' = 0$, viz. $\alpha' = 0$ in the formulæ of C_k. The equation is

$$\begin{array}{ll|ll} & \theta^{k+1} \,.\, (k-1)\,(-\,\alpha\gamma' X) & & \theta^{k+1} X \\ + & \theta^k \;\;\,.\, (k-2)\,(\alpha\gamma' Y) & + & \theta^k \;\; Y(k-2) \\ + & \theta \;\;\;\;\, . \;\;\;\;\; (\alpha\gamma' Z - 2\gamma\gamma' X) & + & \theta \;\; (Z,\, X) \\ + & \qquad\qquad 2\gamma\gamma' Y = 0, & + & Y = 0, \end{array}$$

so that $n'' = k+1$, $\iota'' = 0$, $m'' = 2k$. But observe that in the particular case $k = 2$, the form is $\theta^3 X + \theta(Z,\ X) + Y = 0$, the term $\theta^k Y$ disappearing on account of the factor $k-2$. Here on writing $X = 0$, there is a factor $\left(1 - \dfrac{\theta}{\infty}\right)^2$ (indicated by the reduction of order from 3 to 1), hence $\iota'' = 1$, $n'' = 3$, $m'' = 2\,.\,2 - 1 = 3$, agreeing with the column $\bar{P}_2\bar{R}_2$.

Case S_k. We have $\alpha = 0$, $\alpha' = \beta' = 0$, viz. $\beta' = 0$ in the formulæ of D_k. The equation is

$$\begin{array}{ll|ll} & \theta^k \;\;\,.\, (k+1)\,\beta\gamma' X & & \theta^k \;\; X \\ + & \theta^{k-1} . \;\;\;\; -k\beta\gamma' Y & + & \theta^{k-1} Y \\ + & \theta \;\;\;\;\, . \;\;\;\; -2\gamma\gamma' X & + & \theta \;\; X \\ + & \qquad\quad 2\gamma\gamma' Y - \beta\gamma' Z = 0, & + & Y + Z = 0, \end{array}$$

so that $n'' = k$, $\iota'' = 0$, $m'' = 2k-2$. The process applies to the case $k = 2$.

As to the formula for A_3, B_3, ... S_3, there is nothing special in these; they are simply deduced from those for A_k, B_k, ... S_k by writing therein $k = 3$. And we have thus the foregoing series of formulæ, which will apply to the greater part of the cases which ordinarily arise. For instance suppose there is at I or J a triple point = cusp + 2 nodes; there is here an ordinary branch and a cuspidal (ordinary cuspidal) branch and according as IJ touches neither branch, the ordinary branch, or the cuspidal branch, the corrections to m'', n'', ι'', κ'' are $\bar{R}_2 + R_3$, $S_2 + R_3$, $\bar{R}_2 + S_3$ respectively. Observe moreover that A_2C_2 is no speciality, B_2D_2 is the speciality $g = 1$, P_2R_2 the speciality $f = 1$.

There is a remarkable case in which the fundamental assumption of the (1, 1) correspondence of the evolute with the original curve ceases to be correct. In fact, in the case about to be considered of a parallel curve; the parallel to any given curve is *in general* a curve not breaking up into two distinct curves of the same order with such given curve, and when this is so (viz. when the parallel curve does not break up) each normal of the parallel curve is a normal at two distinct points thereof: the evolute of the parallel curve is thus the evolute of the given curve taken twice; and the parallel curve and its evolute have not a (1, 1) but (1, 2) correspondence. Hence, $(m, n, \delta, \kappa, \tau, \iota)$ the unaccented letters referring to the parallel curve, or say rather to a curve which has a (1, 2) correspondence with its evolute, and, as before, the twice accented letters to the evolute, it is *not true* that $m'' - 2n'' + \iota'' = m - 2n + \iota$; it will subsequently appear that the values of m'', n'' are correct, those of ι'', κ'' suffering a modification; viz. the formulæ are

$$\begin{aligned} m'' &= \alpha - 3f - 3g, \\ n'' &= m + n - f - g, \\ \iota'' &= f + g - \Theta, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f - 5g - \Theta, \end{aligned}$$

where, for the present, I leave Θ undetermined.

Coming now to the parallel curve, let the numbers in regard to it be m', n', δ', κ', τ', ι'; α', D'. Supposing in the first instance that the given curve does not stand in any special relation to I, J, the formulæ are

$$\begin{aligned} m' &= 2m + 2n, & \alpha &= 6n + 2\alpha, \\ n' &= 2n, & 2D' &= -4m + 2 + 2\alpha, = -2m + 2n + \alpha. \\ \iota' &= 2\alpha - 6m, \\ \kappa' &= 2\alpha. \end{aligned}$$

Considering the parallel curve as the envelope of a circle of constant radius having its centre on the given curve, it appears (e.g. by consideration of the case of the ellipse) that when the radius of the circle is $=0$, there is not any depression in the order of the parallel curve, but that the parallel curve reduces itself to the given curve twice, together with the system of tangents from the points I, J to the given curve: the order of the parallel curve is thus $m' = 2m + 2n$.

To find the class, consider the tangents from a given point to the parallel curve; about the point as centre describe a circle, radius k; then the tangents in question are respectively parallel to, and correspond each to each with, the common tangents of the circle and the given curve, and the number of these is $= 2n$, that is $n' = 2n$.

Each inflexion of the given curve gives rise to two inflexions of the parallel curve; and the inflexions of the parallel curve arise in this way only: that is $\iota' = 2\iota$, $= 2\alpha - 6m$. And the Plückerian relations then give $\kappa' = 2\alpha$; a value which may be investigated independently.

Attending now to the singularities f and g; the values of n', ι' are unaltered: to obtain m' we as before consider the particular case where the radius of the variable circle is $=0$: the parallel curve here breaks up into the original curve, together with the focal tangents from the points I, J; viz. we have

$$m' = 2n + (n - 2f_1 - g) + (n - 2f_2 - g), \quad = 2m + 2n - 2f - 2g;$$

and knowing m', n', ι' we have κ'.

The points on IJ of the original curve are I, J counting as f_1 and f_2 respectively; or together as f points: the points of contact counting as $2g$: and besides $m-f-2g$ points. As regards the parallel curve, we have the same points on IJ; but here I is $(n-g)$ tuple point, having in respect of each branch of the f_1-tuple point on the original curve a pair of branches touching each other, and in respect of each of the tangents from I to the given curve a single branch, together $2f_1 + (n - 2f_1 - g)$, $= n - g$ branches; and thus counting $n-g$ times: similarly J counts $n-g$ times. Hence also for the parallel curve $f_1' = f_2' = n - g$. In respect of each of the points g, we have a point where there are two branches touching each other and the line IJ; and thus counting 4 times, or together as $4g$: moreover, on account of the two branches at each of these points, $g' = 2g$. Lastly, each of the $m - f - 2g$ points is a node on the parallel curve; and as such counts twice; $m' = 2(n-g) + 4g + 2(m - f - 2g)$, $= 2m + 2n - 2f - 2g$ as above.

And we have thus the formulæ

$$\begin{aligned}
m' &= 2m + 2n - 2f - 2g, \\
n' &= 2n, \\
\iota' &= -6m + 2\alpha, = 2\iota, \\
\kappa' &= 2\alpha - 6f - 6g, = 6n + 2\kappa - 6f - 6g, \\
f' &= 2n - 2g, \\
g' &= 2g,
\end{aligned}$$

where observe that m' is $= 2n''$; that is, twice the class of the evolute (which relation however is not in all cases true for a curve with singularities); and further that $n' - f' - g'$ is $= 0$.

The case of a curve for which $n - f - g = 0$ is very interesting and remarkable. Recurring to the formulæ for the evolute, we have here $m'' = \kappa$, $n'' = m$, $\iota'' = n$, $\kappa'' = n - 3m + 3\kappa$. And for the parallel curve $m' = 2m$, $n' = 2n$, $\iota' = 2\iota$, $\kappa' = 2\kappa$, $f' = 2f$, $g' = 2g$; formulæ which lead to the assumption that the parallel curve here breaks up into two distinct curves, each such as the given curve.

Observe further that for a curve possessing the singularities f and g, but where $n - f - g$ is not $= 0$; then for the parallel curve we have as above $n' - f' - g' = 0$; or the parallel of the parallel curve should, according to the assumption, break up into two distinct curves such as the parallel curve; this is of course correct.

Consider the evolute of the parallel curve: since for the parallel curve $n' - f' - g' = 0$, the formulæ for the evolute thereof (viz. those containing the undetermined quantity Θ)

are $m'''=\kappa'$, $n'''=m'$, $\iota'''=n'-\Theta$, $\kappa'''=n'-3m'+3\kappa'-\Theta$, or substituting for m', n', κ' their values, and comparing with the formulæ in regard to the evolute, we have

$$\begin{aligned}
m''' &= \phantom{-6m+{}}2\alpha-6f-6g, && = 2m'',\\
n''' &= 2m+2n-2f-2g, && = 2n'',\\
\iota''' &= \phantom{-6m+{}}2n-\Theta, && = 2\iota''+2(n-f-g)-\Theta,\\
\kappa''' &= -6m-4n+6\alpha-12f-12g-\Theta, && = 2\kappa''+2(n-f-g)-\Theta,
\end{aligned}$$

where m'', n'', ι'', κ'' refer to the evolute. Hence by assuming $\Theta=2(n-f-g)$, the values of m''', n''', ι''', κ''' become $2m''$, $2n''$, $2\iota''$, $2\kappa''$, viz. the evolute of the parallel curve is the evolute of the original curve taken twice. Observe that in the foregoing value of Θ, the letters n, f, g refer not to the parallel curve, the evolute whereof is under consideration, but to the curve from which such parallel curve was derived; this value $\Theta=2(n-f-g)$ is not a value of Θ applicable to be substituted in the evolute-formulæ for the case of a curve which has with its evolute a (1, 2) correspondence.

Instead of the foregoing case of the f, viz. f- and g-singularities, we may, as regards the parallel curve, consider the original curve as having any I- and J-singularities whatever. Suppose in this case (excluding always the line IJ and the tangents at I or J) the number of tangents from I to the curve is $=n-I$, and the number of tangents from J to the curve is $=n-J$, then when the radius of the variable curve is $=0$, the parallel curve becomes the original curve twice together with the $(n-I)+(n-J)$, $=2n-I-J$ tangents; so that the order is $m'=2m+2n-I-J$[1]; we have, as before, $n'=2n$ and $\iota'=2\iota$, and these values give κ', so that the equations are

$$\begin{aligned}
m' &= \phantom{-6m+{}}2m+2n-I-J,\\
n' &= \phantom{-6m+{}}2n,\\
\iota' &= -6m+2\alpha, && = 2\iota,\\
\kappa' &= \phantom{-6m+{}}2\alpha-I-J, && = 6n+2\kappa-3I-3J.
\end{aligned}$$

Suppose $2n-I-J=0$; this implies $n-I=0$, $n-J=0$ since neither $n-I$ nor $n-J$ can be negative; viz. that there are no I- or J-tangents; and conversely, when this is the case $2n-I-J=0$: and we have then m', n', ι', $\kappa'=2m$, $2n$, 2ι, 2κ; viz. it is assumed, as before, that the parallel curve breaks up into two distinct curves such as the original curve; that is, the condition in order that the parallel curve should break up, is that the original curve has no focal tangents. Observe that the number of foci is $=(n-I)(n-J)$ which is $=0$ if only $n-I=0$ or $n-J=0$; but as regards real curves $I=J$, so that the equations $n-I=0$ and $n-J=0$ are one and the same equation, satisfied if $(n-I)(n-J)=0$; so that for a real curve without foci (real or imaginary) the parallel curve will break up. An instance given to me by Dr Salmon is the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}-c^{\frac{2}{3}}=0$ or $(x^2+y^2-c^2)^3+27c^2x^2y^2=0$, here $m=6$, $n=4$,

[1] That the order of the evolute is not (in every case of a curve with singularities) one-half this, or $=m+n-\frac{1}{2}(I+J)$, is at once seen by remarking that there is no reason why $I+J$ should be even.

$\delta=4$, $\kappa=6$, $\iota=0$, $\tau=3$: the points I, J are each of them a cusp, the tangents being the line IJ; the number of tangents from a cusp is $n-3$, $=1$, but for the cusp I or J, this tangent is the line IJ itself, so that we have $I=J=4$.

Theory when the Absolute is a conic.

When the Absolute is a conic the formulæ for the evolute are essentially the same as those in the former case, but the formulæ for the parallel curve are modified essentially and in a very remarkable manner. I observe that corresponding to a passage of the given curve through I or J we have a contact with the Absolute, so that in the present case f will properly denote the number of contacts of the given curve with the Absolute, and attending to this singularity only, viz. considering a given curve $(m, n, \delta, \kappa, \iota, \tau; \alpha, D)$ having f contacts with the Absolute, the formulæ for the evolute are

$$\begin{aligned} m'' &= \alpha - 3f, \\ n'' &= m + n - f, \\ \iota'' &= f, \\ \kappa'' &= -3m - 3n + 3\alpha - 5f. \end{aligned}$$

In the case $f=0$, these at once follow from the two equations $n''=m+n$, and $\iota''=0$. The normal is the line joining a point of the given curve with the pole of the tangent; or, what is the same thing, it is the line joining the point of the given curve with the corresponding point of the reciprocal curve: the degrees of the two curves are m, n, and the correspondence is a (1, 1) correspondence. Hence, by the general theorem previously referred to, it follows that we have $n''=m+n$, and $\iota''=0$. Compare herewith the demonstration of the theorem in the case where the Absolute is a point-pair.

The formulæ for the parallel curve are

$$\begin{aligned} m' &= 2m + 2n - 2f, \\ n' &= 2m + 2n - 2f, \\ \iota' &= 2\alpha - 6f, \\ \kappa' &= 2\alpha - 6f, \\ f' &= 2m + 2n - 2f, \end{aligned}$$

(so that $m'=n'$, $\iota'=\kappa'$). The intersections of the curve and Absolute are in this case the points f each twice, and besides $2m-2f$ points; similarly the common tangents are the tangents at f each twice and besides $2n-2f$ tangents. Now I remark that the parallel curve, when the radius of the variable circle is $=0$, reduces itself to the original curve twice, together with the $2n-2f$ common tangents, and the $2m-2f$ common points; the order is thus $=2m+(2n-2f)$, and the class $=2n+(2m-2f)$: and these are the values in the general case where the radius of the variable circle is not $=0$.

But in the remarkable case where the curve and its evolute have a (1, 2) correspondence, then I correct the formulæ by adding $-\Theta$ to the expressions for ι', κ' respectively. We have for the evolute of the parallel curve

$$\begin{aligned} m''' &= 2m'', \\ n''' &= 2n'', \\ \iota''' &= 2\iota'' + (2m+2n-4f) - \Theta, \\ \kappa''' &= 2\kappa'' + (2m+2n-4f) - \Theta, \end{aligned}$$

viz. assuming $\Theta = 2m+2n-4f$, this means that the evolute is the evolute of the original curve taken twice.

A very interesting case is when $m=n=f$: observe that neither $m-f$ nor $n-f$ can be negative, so that the assumed relation $m+n-2f=0$ would imply these two relations. We have here for the parallel curve $m'=2m$, $n'=2n$, $\iota'=2\iota$, $\kappa'=2\kappa$; the parallel curve in fact breaking up into two curves such as the given curve. And in this case the formulæ for the evolute assume the very simple form $m''=\kappa$, $n''=f$, $\iota''=f$, $\kappa''=-2f+3\kappa$.

Whatever the original curve may be, we have for the parallel curve $m'=n'=f'$, so that the formulæ for the evolute of the parallel curve are of the foregoing form $m'''=\kappa'$, $n'''=f'$, $\iota'''=f'-\Theta$, $\kappa'''=-2f'+3\kappa'-\Theta$, which agree with the above values of m''', n''', ι''', κ'''. In the particular case $m=n=f$, we have $\Theta=0$, so that the evolute-formulæ, if originally written down without the terms in Θ, would still be $m'''=2m''$, $n'''=2n''$ $\iota'''=2\iota''$ $\kappa'''=2\kappa''$; viz. the evolute is here the original evolute taken twice; as already seen, the parallel curve consisted of two curves such as the original curve, and each of these has for its evolute the evolute of the original curve.