490.

ON A PROBLEM OF ELIMINATION.

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I WRITE

$$P = (\alpha, \ldots \between x, y, z)^k, \quad Q = (\alpha', \ldots \between x, y, z)^k,$$

$$U = (a, \ldots \between x, y, z)^m, \quad V = (b, \ldots \between x, y, z)^n,$$

and I seek for the form of the relation between the coefficients $(\alpha, \ldots)$, $(\alpha', \ldots)$, $(a, \ldots)$, $(b, \ldots)$, in order that there may exist in the pencil

$$\lambda P + \mu Q = 0$$

a curve passing through *two* of the intersections of the curves $U = 0$, $V = 0$.

The ratio $\lambda : \mu$ may be determined so as that the curve $\lambda P + \mu Q = 0$ shall pass through one of the intersections of the curves $U = 0$, $V = 0$; or, what is the same thing, so as that the three curves shall have a common point; the condition for this is

$$\text{Reslt. } (\lambda P + \mu Q, \; U, \; V) = 0,$$

a condition of the form

$$(\lambda\alpha + \mu\alpha', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km} = 0\,;$$

or, what is the same thing,

$$(\alpha, \ldots, \alpha', \ldots)^{mn} (a, \ldots)^{kn} (b, \ldots)^{km} (\lambda, \mu)^{mn} = 0,$$

which, for shortness, may be written

$$(A, \ldots \between \lambda, \mu)^{mn} = 0.$$

Suppose this equation has equal roots, then we have

$$\text{Disct. Reslt. } (\lambda P + \mu Q,\ U,\ V) = 0,$$

the discriminant being taken in regard to λ, μ. This is of the form

$$(A, \ldots)^{2\,(mn-1)} = 0\,;$$

that is

$$(\alpha, \ldots, \alpha', \ldots)^{2mn\,(mn-1)}\,(a, \ldots)^{2kn\,(mn-1)}\,(b, \ldots)^{2km\,(mn-1)} = 0.$$

It is moreover clear that the nilfactum is a combinant of the functions P, Q; and the form of the equation is therefore

$$\left(\left\|\begin{matrix}\alpha, & \beta, \ldots \\ \alpha', & \beta', \ldots\end{matrix}\right\|\right)^{mn\,(mn-1)} (a, \ldots)^{2kn\,(mn-1)}\,(b, \ldots)^{2km\,(mn-1)} = 0.$$

Now the equation in question will be satisfied, 1°. if the curves $U=0$, $V=0$ touch each other; let the condition for this be $\nabla = 0$. 2°. If there exists a curve $\lambda P + \mu Q = 0$ passing through two of the intersections of the curves $U=0$, $V=0$; let the condition be $\Omega = 0$. There is reason to think that the equation contains the factor Ω^2, and that the form thereof is $\Omega^2\nabla = 0$.

Assuming that this is so, and observing that ∇, the osculant or discriminant of the functions U, V, is of the form

$$\nabla = (a, \ldots)^{n\,(n+2m-3)}\,(b, \ldots)^{m\,(m+2n-3)},$$

we have

$$\Omega^2 = \left(\left\|\begin{matrix}\alpha, & \beta, \ldots \\ \alpha', & \beta',\end{matrix}\right\|\right)^{mn\,(mn-1)} (a, \ldots)^{kn\,(n-1)\,(2m-1) + (k-1)\,n\,(n+2m-3)} \times$$
$$(b, \ldots)^{km\,(m-1)\,(2n-1) + (k-1)\,m\,(m+2n-3)},$$

and consequently

$$\Omega = \left(\left\|\begin{matrix}\alpha, & \beta, \ldots \\ \alpha', & \beta',\end{matrix}\right\|\right)^{\frac{1}{2}mn\,(mn-1)}$$
$$(a, \ldots)^{\frac{1}{2}n\,(n-1)\,k\,(2m-1) + \frac{1}{2}\,(k-1)\,n\,(n+2m-3)} \times$$
$$(b, \ldots)^{\frac{1}{2}m\,(m-1)\,k\,(2n-1) + \frac{1}{2}\,(k-1)\,m\,(m+2n-3)},$$

which is the solution of the proposed question. Suppose for instance $n = 1$, then

$$\Omega = \left(\left\|\begin{matrix}\alpha, & \beta, \ldots \\ \alpha', & \beta',\end{matrix}\right\|\right)^{\frac{1}{2}m\,(m-1)} (a, \ldots)^{(k-1)\,(m-1)}\,(b, \ldots)^{\frac{1}{2}m\,(m-1)\,k + \frac{1}{2}\,(k-1)\,(m-1)}.$$

If moreover $k = 1$, then

$$\Omega = \left(\left\|\begin{matrix}\alpha, & \beta, \ldots \\ \alpha', & \beta',\end{matrix}\right\|\right)^{\frac{1}{2}m\,(m-1)} (b, \ldots)^{\frac{1}{2}m\,(m-1)}\,;$$

this is right, for writing $P=\alpha x+\beta y+\gamma z$, $Q=\alpha' x+\beta' y+\gamma' z$, $V=bx+b'y+b''z$, then if two of the intersections of the curve $U=0$ with the line $V=0$ lie in a line with the point $P=0$, $Q=0$, then the point in question, that is the point $(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)$, must lie in the line $V=0$; and the condition reduces itself to

$$\{(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta \between b,\ b',\ b'')\}^{\frac{1}{2}m(m-1)}=0,$$

where the index $\frac{1}{2}m(m-1)$ is accounted for as denoting the number of pairs of points out of the m intersections of the curve $U=0$ with the line $V=0$.

If in general $k=1$, then writing as before $P=\alpha x+\beta y+\gamma z$, $Q=\alpha' x+\beta' y+\gamma' z$, we have

$$\Omega=(\beta\gamma'-\beta'\gamma,\ldots)^{\frac{1}{2}mn(mn-1)}(a,\ldots)^{\frac{1}{2}n(n-1)(2m-1)}(b,\ldots)^{\frac{1}{2}m(m-1)(2n-1)},$$

where $\Omega=0$ is the condition in order that the point $(\beta\gamma'-\beta'\gamma,\ldots)$ may lie *in lineâ* with two of the intersections of the curves $U=0$, $V=0$. Or writing $(X,\ Y,\ Z)$ for the coordinates of the given point, the condition is

$$\Omega=(a,\ldots)^{\frac{1}{2}n(n-1)(2m-1)}(b,\ldots)^{\frac{1}{2}m(m-1)(2n-1)}(X,\ Y,\ Z)^{\frac{1}{2}mn(mn-1)}=0.$$

I have found that if

$$U=(a,\ldots\between x,\ y,\ z)^m,\quad V=(b,\ldots\between x,\ y,\ z)^n,$$
$$W=(c,\ldots\between x,\ y,\ z)^p,\quad T=(d,\ldots\between x,\ y,\ z)^q,$$

the condition in order that the point $(X,\ Y,\ Z)$ may lie *in lineâ* with one of the intersections of the curves $U=0$, $V=0$, and one of the intersections of the curves $W=0$, $T=0$, is

$$\Omega=(a,\ldots)^{npq}(b,\ldots)^{mpq}(c,\ldots)^{mnq}(d,\ldots)^{mnp}(X,\ Y,\ Z)^{mnpq}=0.$$

Supposing that the curves $W=0$, $T=0$ become identical with the curves $U=0$, $V=0$ respectively, this becomes

$$\Omega=(a,\ldots)^{n^2\cdot 2m}(b,\ldots)^{m^2\cdot 2n}(X,\ Y,\ Z)^{mn\cdot mn}=0,$$

and the variation from the correct form given above is what might have been expected.