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ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON, AND THE (2, 2) CORRESPONDENCE OF POINTS ON A CONIC.

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THE present paper includes, as will at once be seen, much that is perfectly well known; but the separate theories required, it seemed to me, to be put together; and there are, particularly as regards the unsymmetrical case afterwards referred to, some results which I believe to be new.

The porism of the in-and-circumscribed polygon has its foundation in the theory of the symmetrical (2, 2) correspondence of points on a conic; viz. a (2, 2) correspondence is such that to any given position of either point there correspond two positions of the other point; and in a symmetrical (2, 2) correspondence either point indifferently may be considered as the first point and the other of them will then be the second point of the correspondence. Or, what is the same thing, if x, y are the parameters which serve to determine the two points, then x, y are connected by an equation of the form $(*\between x, 1)^2(y, 1)^2 = 0$, which is symmetrical in regard to the two parameters (x, y). In the case of such symmetrical relation it is easy to show that the line joining the two points envelopes a conic. For the relation may be expressed in the form $(*\between 1, x+y, xy)^2 = 0$; we may imagine the coordinates (P, Q, R) fixed in such manner that for the point (x) on the first conic we have $P : Q : R = 1 : x : x^2$, and for the point (y), $P : Q : R = 1 : y : y^2$; the equation of the line joining the two points is then

$$\begin{vmatrix} P, & Q, & R \\ 1, & x, & x^2 \\ 1, & y, & y^2 \end{vmatrix} = 0;$$

that is

$$Pxy - Q(x+y) + R = 0,$$

or representing this by

$$P\xi + Q\eta + R\zeta = 0,$$

we have $\xi : \eta : \zeta = xy : -x-y : 1$; and consequently (ξ, η, ζ) are connected by a quadric equation; that is, the envelope is a conic.

The relation $(*\between x, 1)^2 (y, 1)^2 = 0$, whether symmetrical or not, leads as will be presently shown to a differential equation of the form

$$\frac{dx}{\sqrt{(X)}} \pm \frac{dy}{\sqrt{(Y)}} = 0,$$

where X, Y are quartic functions of x, y respectively; viz. these are unlike or like functions of the two variables according as the integral equation is not or is symmetrical in regard to the two variables. In the former case, however, the functions X, Y are so related to each other, that the two can be by a linear transformation converted into like functions of the variables: for instance, if y be changed into $ay_1 + b \div cy_1 + d$, then the constants may be determined in suchwise that Y is the same function of y_1, that X is of x; the original integral equation being hereby converted into a symmetrical equation $(*\between x, 1)^2 (y_1, 1)^2 = 0$ between x and y_1, so that in one point of view the unsymmetrical case is not really more general than the symmetrical one. It is to be added that the integral equation contains really one more constant than the differential equation (this is most readily seen in the symmetrical case, the differential equation depends only on the ratio of five constants a, b, c, d, e, whereas the integral equation depends on the ratio of six constants), so that the integral equation is really the complete integral of the differential equation.

Attending now to the symmetrical case; if A and B are corresponding points, then the corresponding points of B are A and a new point C; those of C are B and a new point D, and so on; so that the points form a series $A, B, C, D, \ldots$; and the porismatic property is that, if for a given position of A this series closes at a certain term, for instance, if $D = A$, then it will always thus close, whatever be the position of A. And this follows at once from the consideration of the differential equation $\dfrac{dx}{\sqrt{(X)}} = \dfrac{dy}{\sqrt{(Y)}}$; viz. as this is at once integrable *per se* in the form

$$\Pi(y) - \Pi(x) = \Pi(k),$$

this equation must be a transformation of the original equation $(*\between x, 1)^2 (y, 1)^2 = 0$, and equally with it represent the relation between the parameters x, y of the two points A, B; the constant of integration k is of course completely determined in terms of the coefficients of the last-mentioned equation, assumed to be given.

Hence forming the equations for the correspondences, B, C; C, D; ... and assuming that the series closes F, A; we have

$$\Pi(z) - \Pi(y) = \Pi(k),$$
$$\vdots$$
$$\Pi(x) - \Pi(u) = \Pi(k);$$

where, however, the $\Pi(x)$ of the last equation must be regarded as differing from that of the first equation by a period, say Ω, of the integral; hence adding, we have

$$\Omega = n\Pi(k),$$

or

$$\Pi(k) = \frac{1}{n}\Omega,$$

which gives between the constants of the integral equation $(*\between x, 1)^2 (y, 1)^2 = 0$, a relation which must be satisfied when the series closes at the n^{th} term (viz. when the term after this coincides with the first term); and this relation is independent of x, that is, of the position of the point A.

The analysis in regard to the differential equation is as follows:

Consider the equation

$$\begin{aligned} U = \;& y^2(ax^2 + 2bx + c) \\ & + 2y(a'x^2 + 2b'x + c') \\ & + (a''x^2 + 2b''x + c'') = 0, \end{aligned}$$

say

$$U = (P, Q, R\between y, 1)^2 = (L, M, N\between x, 1)^2 = 0,$$

we have

$$dU = 0 = (Py + Q)\,dy + (Lx + M)\,dx.$$

But the equation $U = 0$ gives $(Py + Q)^2 = Q^2 - PR$, $(Lx + M)^2 = M^2 - NL$, and the differential equation therefore becomes

$$dy\,\sqrt{(Q^2 - PR)} \pm dx\,\sqrt{(M^2 - NL)} = 0,$$

viz. it is

$$\frac{dy}{\sqrt{\{(ay^2 + 2a'y + a'')(cy^2 + 2c'y + c'') - (bx^2 + 2b'y + b'')^2\}}}$$
$$\pm \frac{dx}{\sqrt{\{(ax^2 + 2bx + c)(a''x^2 + 2b''x + c'') - (a'x^2 + 2b'x + c')^2\}}} = 0.$$

Suppose the equation is

$$\begin{aligned} & y^2(ax^2 + 2hx + g) \\ & + 2y(hx^2 + 2bx + f) \\ & + (gx^2 + 2fx + c) = 0, \end{aligned}$$

then the differential equation is

$$\frac{dy}{\sqrt{\{(ay^2 + 2hy + g)(gy^2 + 2fy + c)^2 - (hy^2 + 2by + f)^2\}}}$$
$$\pm \frac{dx}{\sqrt{\{(ax^2 + 2hx + g)(gx^2 + 2fx + c) - (hx^2 + 2bx + f)^2\}}} = 0,$$

say

$$\frac{dy}{\sqrt{(Y)}} = \frac{\pm\,dx}{\sqrt{(X)}}.$$

Now starting from the differential equation

$$\frac{dx}{\sqrt{\{(a,\ b,\ c,\ d,\ e\between x,\ 1)^4\}}} = \pm \frac{dy}{\sqrt{\{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4\}}},$$

the integral equation is known to be

$$\left[\frac{\sqrt{\{(a,\ b,\ c,\ d,\ e\between x,\ 1)^4\}} - \sqrt{\{(a,\ b,\ c,\ d,\ e\between y,\ 1)^4\}}}{x-y}\right] = a\,(x+y)^2 + 4b\,(x+y) + 6\,\theta,$$

where θ is the constant of integration. Writing, for shortness, $X = (a,\ b,\ c,\ d,\ e\between x,\ 1)^4$, $Y = (a,\ b,\ c,\ d,\ e\between y,\ 1)^4$, this is

$$X + Y - 2\sqrt{(XY)} = a\,(x^2-y^2)^2 + 4b\,(x-y)(x^2-y^2) + 6\,\theta\,(x-y)^2;$$

or, what is the same thing,

$$\begin{aligned} a\,(x^4+y^4) &- 2\sqrt{(XY)} = a\,(x^2-y^2)^2 + 4b\,(x-y)(x^2-y^2) + 6\,\theta\,(x-y)^2,\\ + 4b\,&(x^3+y^3)\\ + 6c\,&(x^2+y^2)\\ + 4d\,&(x+y)\\ + 2e&, \end{aligned}$$

viz. this gives

$$\begin{aligned} \sqrt{(XY)} = {} & ax^2y^2\\ & + 2b\,(x^2y+xy^2)\\ & + 3c\,(x^2+y^2)\\ & + 3\theta\,(x-y)^2\\ & + 2d\,(x+y)\\ & +\ e, \end{aligned}$$

and, rationalising, the integral equation becomes

$$\begin{aligned} & - 6a\theta x^2y^2\\ & - 4adxy\,(x+y)\\ & - ae\,(x+y)^2\\ & + 4b^2x^2y^2\\ & + 12bcxy\,(x+y) - 12b\theta xy\,(x+y)\\ & - 8bdxy\\ & - 4be\,(x+y)\\ & + 9c^2\,(x+y)^2 - 18c\theta\,(x^2+y^2)\\ & - 12cd\,(x+y)\\ & + 9\theta^2\,(x-y)^2 - 12d\theta\,(x+y) - 6e\theta + 4d^2 = 0; \end{aligned}$$

or, as it may be written,

$$\begin{aligned} & x^2y^2(4b^2-6a\theta) \\ & +(x^2y+xy^2)(-4ad+12bc-12b\theta) \\ & +(x^2+y^2)\ (-ae+9c^2-18c\theta+9\theta^2) \\ & +\quad xy\ (-2ae-8bd+18c^2-18\theta^2) \\ & +(x+y)\ (-4be+12cd-12d\theta) \\ & +\quad 4d^2-6e\theta=0. \end{aligned}$$

Comparing this with the original integral equation $V=0$, and the form of differential equation deduced therefrom, we ought to have identically

$$\begin{aligned} & [(4b^2-6a\theta)x^2+(-2ad+6bc-6b\theta)x+(-ae+9c^2-18c\theta+9\theta^2)] \\ & \quad\times[(-ae+9c^2-18c\theta+9\theta^2)x^2+(-2be+6cd-6d\theta)x+(4d^2-6e\theta)] \\ & -[(-2ad+6bc-6b\theta)x^2+(-ae-4bd+9c^2-9\theta^2)x+(-2be+6cd-6d\theta)]^2 \\ & =\text{multiple of } X, \\ & =\{(-4ad^2-4b^2e+24bcd)+(6ae-24bd-54c^2)\theta+108c\theta^2-54\theta^3\}(a,\ b,\ c,\ d,\ e\between x,\ 1)^4, \end{aligned}$$

by comparing the coefficients of x^4.

I obtain this otherwise:

Write

$$V=\alpha U+6\beta H,$$

then, forming the Hessian of V, we have

$$\begin{aligned} \tilde{H}V &= (\alpha^2-3I\beta^2)H+(I\alpha\beta+9J\beta^2)U, \\ &= \frac{(\alpha^2-3I\beta^2)}{6\beta}(V-\alpha U)+(I\alpha\beta+9J\beta^2)U, \\ &= \frac{\alpha^2-3I\beta^2}{6\beta}V+\frac{1}{6\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U, \end{aligned}$$

that is

$$d_x{}^2Vd_y{}^2V-(d_xd_yV)^2-\frac{2(\alpha^2-3I\beta^2)}{\beta}(x^2d_x{}^2V+2xyd_xd_yV+y^2d_y{}^2V)=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^3)U,$$

or writing

$$K=-\frac{2(\alpha^2-3I\beta^2)}{\beta},$$

this is

$$(d_x{}^2V+Ky^2)(d_y{}^2V+Kx^2)-(d_xd_yV-Kxy)^2=\frac{24}{\beta}(-\alpha^3+9I\alpha\beta^2+54J\beta^2)U,$$

so that the components are

$$d_x{}^2V+Ky^2,\quad d_xd_yV-Kxy,\quad d_y{}^2V+Kx^2,$$

$V=\alpha U+6\beta H=$

$$\alpha(a,\ b,\ c,\ d,\ e\between x,\ 1)^4+6\beta(ac-b^2,\ 2ad-2bc,\ ae+2bd-3c^2,\ 2be-2cd,\ ce-d^2\between x,\ 1)^4,$$

viz. the components are

$$\begin{array}{lll}(\alpha a+6\beta(ac-b^2), & \alpha b+3\beta(ad-bc), & \alpha c+\beta(ae+2bd-3c^2)+\tfrac{1}{12}K \between x,\ 1)^2, \\ (\alpha b+3\beta(ad-bc), & \alpha c+\beta(ae+2bd-3c^2)+\tfrac{1}{24}K, & \alpha d+3\beta(be-cd) \between x,\ 1)^2, \\ (\alpha c+\ \beta(ae+2bd-3c^2)+\tfrac{1}{12}K, & \alpha d+3\beta(be-cd), & \alpha e+6\beta(ce-d^2) \between x,\ 1)^2,\end{array}$$

where as before

$$K=-\frac{2(\alpha^2-3I\beta^2)}{\beta}.$$

I assume

$$\beta=-\tfrac{2}{3},\quad \alpha=4c-6\theta,\quad K=3\{(4c-6\theta)^2-\tfrac{4}{3}I\}.$$

$$\begin{aligned}\alpha a+6\beta(ac-b^2) &= a(4c-6\theta)-4(ac-b^2)=4b^2-6a\theta,\\ \alpha b+3\beta(ad-bc) &= b(4c-6\theta)-2(ad-bc)=-2ad+6bc-6b\theta,\\ \alpha d+3\beta(be-cd) &= d(4c-6\theta)-2(be-cd)=-2be+6cd-6d\theta,\\ \alpha e+6\beta(ce-d^2) &= e(4c-6\theta)-4(ce-d^2)=\ 4d^2-6e\theta,\\ \alpha c+\ \beta(ae+2bd-3c^2)-\tfrac{1}{24}K &= c(4c-6\theta)-\tfrac{2}{3}(ae+2bd-3c^2)-\tfrac{1}{8}\{(4c-6\theta)^2-\tfrac{4}{3}I\}\\ &= -\tfrac{1}{2}ae-2bd+\tfrac{9}{2}c^2-\tfrac{9}{2}\theta^2,\end{aligned}$$

$$\begin{aligned}\alpha c+\beta(ae+2bd-3c^2)+\tfrac{1}{12}K & \\ &= c(4c-6\theta)-\tfrac{2}{3}(ae+2bd-3c^2)+\tfrac{1}{4}\{(4c-6\theta)^2-\tfrac{4}{3}I\}\\ &= -ae+9c^2-18c\theta+9\theta^2,\end{aligned}$$

agreeing with the former result.

I return to the general form

$$\begin{aligned} & y^2(a,\ \ b,\ \ c \between x,\ 1)^2\\ &+2y(a',\ \ b',\ \ c' \between x,\ 1)^2\\ &+\quad (a'',\ \ b'',\ \ c'' \between x,\ 1)^2=0,\end{aligned}$$

giving

$$\frac{dx}{\sqrt{[(a,\ b,\ c \between x,\ 1)^2(a'',\ b'',\ c'' \between x,\ 1)^2-\{(a',\ b',\ c' \between x,\ 1)^2\}^2]}}$$

$$=\frac{dy}{\sqrt{[(a,\ a',\ a'' \between y,\ 1)^2(c,\ c',\ c'' \between y,\ 1)^2-\{(b,\ b',\ b'' \between y,\ 1)^2\}^2]}}.$$

Operate a linear transformation on the x, say

$$x=\frac{\lambda x'+\mu}{\nu x'+\rho};$$

the new coefficients are

$$(a,\ b,\ c\ \between\lambda,\ \nu)^2,\quad (a\ ,\ b\ ,\ c\ \between\lambda,\ \nu\between\mu,\ \rho),\quad (a\ ,\ b\ ,\ c\ \between\mu,\ \rho)^2,$$

$$(a',\ b',\ c'\ \between\lambda,\ \nu)^2,\quad (a',\ b',\ c'\between\lambda,\ \nu\between\mu,\ \rho),\quad (a',\ b',\ c'\between\mu,\ \rho)^2,$$

$$(a'',\ b'',\ c''\between\lambda,\ \nu)^2,\quad (a'',\ b'',\ c''\between\lambda,\ \nu\between\mu,\ \rho),\quad (a'',\ b'',\ c''\between\mu,\ \rho)^2:$$

assume now

$$\begin{aligned}
a'\lambda + (b'-\theta)\,\nu - \quad a\mu - \quad b\rho &= 0,\\
(b'+\theta)\,\lambda + \quad c'\nu - \quad b\mu - \quad c\rho &= 0,\\
a''\lambda + \quad b''\nu - \quad a'\mu - (b'+\theta)\,\rho &= 0,\\
b''\lambda + \quad c''\nu - (b'-\theta)\,\mu - \quad c'\rho &= 0,
\end{aligned}$$

then it is easy to show that

$$\begin{aligned}
(a,\ b,\ c\between\lambda,\ \nu\between\mu,\ \rho) &= (a',\ b',\ c'\between\lambda,\ \nu)^2,\\
(a',\ b',\ c'\between\mu,\ \rho)^2 &= (a'',\ b'',\ c''\between\lambda,\ \nu\between\mu,\ \rho),\\
(a,\ b,\ c\between\mu,\ \rho)^2 &= (a'',\ b'',\ c''\between\lambda,\ \nu)^2\\
&[= (a',\ b',\ c'\between\lambda,\ \nu\between\mu,\ \rho) + \theta\,(\lambda\rho - \mu\nu)],
\end{aligned}$$

and the equations give

$$\begin{vmatrix}
a' , & b'-\theta, & a\ , & b\\
b'+\theta, & c' , & b\ , & c\\
a'' , & b'' , & a' , & b'+\theta\\
b'' , & c'' , & b'-\theta, & c'
\end{vmatrix} = 0,$$

that is

$$\begin{aligned}
&(a'c'-b'^2+\theta^2)^2 + (a''c''-b''^2)\,(ac-b^2)\\
&-(a'b''-a''b'+a''\theta)\,(bc'-b'c+c\theta)\\
&+(a'c''-b''b'+b''\theta)\,(bb'-a'c+b\theta)\\
&+(b'b''-a''c'+b''\theta)\,(ac'-b'b+b\theta)\\
&-(b'c''-b''c'+c''\theta)\,(ab'-a'b+a\theta) = 0,
\end{aligned}$$

which is

$$\begin{aligned}
&(a'c'-b'^2)^2 + (a''c''-b''^2)\,(ac-b^2) \quad -2\theta \begin{vmatrix} a\ , & b\ , & c\\ a', & b', & c'\\ a'', & b'', & c''\end{vmatrix}\\
&+a'^2\,(-cc'')\\
&+b'^2\,(-ac''-2bb''-a''c)\\
&+c'^2\,(-aa'')\\
&+2b'c'\,(ab''+a''b) \qquad\qquad +\theta^2\,(2\,(a'c'-b'^2) - ac'' + 2bb'' - a''c)\\
&+2c'a'\,(-bb'')\\
&+2a'b'\,(bc''+b''c) \qquad\qquad +\theta^4 = 0.
\end{aligned}$$

If the original matrix be symmetrical $= \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, this is

$$\begin{aligned}
&(fh-b^2)^2+(ag-h^2)(cg-f^2) && -2\theta \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \\
&+h^2(-cg) && \\
&+b^2(-ac-g^2-2fh) && \\
&+f^2(-ag) && \\
&+2bf(af+gh) && +\theta^2\left[2(fh-b^2)-ac-g^2+2hf\right] \\
&+2fh(-fh) && \\
&+2bh(fg+ch) && +\theta^4=0,
\end{aligned}$$

that is

$$(b-g)\{(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\}$$
$$-2\theta(abc-af^2-bg^2-ch^2+2fgh)+\theta^2(4fh-2b^2-ac-g^2)+\theta^4=0,$$

satisfied by

$$\theta+b-g=0,$$

viz. the equation in θ is

$$(\theta+b-g)\{\theta^3-(b-g)\theta^2+(4fh-ac-2bg-b^2)\theta+(b^2-ac)(b+g)+2(af^2+ch^2-2bfh)\}=0.$$