416.

ON THE THEORY OF RECIPROCAL SURFACES.

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620. IN further developing the theory of reciprocal surfaces it has been found necessary to take account of other singularities, some of which are as yet only imperfectly understood. It will be convenient to give the following complete list of the quantities which present themselves:

n, order of the surface.

a, order of the tangent cone drawn from any point to the surface.

δ, number of nodal edges of the cone.

κ, number of its cuspidal edges.

ρ, class of nodal torse.

σ, class of cuspidal torse.

b, order of nodal curve.

k, number of its apparent double points.

f, number of its actual double points.

t, number of its triple points.

j, number of its pinch-points.

q, its class.

c, order of cuspidal curve.

h, number of its apparent double points.

θ, number of its points of an unexplained singularity.

χ, number of its close-points.

ω, number of its off-points.

r, its class.

β, number of intersections of nodal and cuspidal curves, stationary points on cuspidal curve.

γ, number of intersections, stationary points on nodal curve.

i, number of intersections, not stationary points on either curve.

C, number of cnicnodes of surface.

B, number of binodes.

And corresponding reciprocally to these:

n', class of surface.

a', class of section by arbitrary plane.

δ', number of double tangents of section.

κ', number of its inflexions.

ρ', order of node-couple curve.

σ', order of spinode curve.

b', class of node-couple torse.

k', number of its apparent double planes.

f', number of its actual double planes.

t', number of its triple planes.

j', number of its pinch-planes.

q', its order.

c', class of spinode torse.

h', number of its apparent double planes.

θ', number of its planes of a certain unexplained singularity.

χ', number of its close-planes.

ω', number of its off-planes.

r', its order.

β', number of common planes of node-couple and spinode torse, stationary planes of spinode torse.

γ', number of common planes, stationary planes of node-couple torse.

i', number of common planes, not stationary planes of either torse.

C', number of cnictropes of surface.

B', number of its bitropes.

In all 46 quantities.

621. In part explanation, observe that the definitions of ρ and σ agree with those given, Art. 609: the nodal torse is the torse enveloped by the tangent planes along the nodal curve; if the nodal curve meets the curve of contact a, then a tangent plane of the nodal torse passes through the arbitrary point, that is, ρ will be the number of these planes which pass through the arbitrary point, viz. the class of the torse. So also the cuspidal torse is the torse enveloped by the tangent planes along the cuspidal curve; and σ will be the number of these tangent planes which pass through the arbitrary point, viz. it will be the class of the torse. Again, as regards ρ' and σ': the node-couple torse is the envelope of the bitangent planes of the surface, and the node-couple curve is the locus of the points of contact of these planes; similarly, the spinode torse is the envelope of the *parabolic* planes of the surface, and the spinode curve is the locus of the points of contact of these planes; viz. it is the curve UH of intersection of the surface and its Hessian; the two curves are the reciprocals of the nodal and cuspidal torses respectively, and the definitions of ρ', σ' correspond to those of ρ and σ.

622. In regard to the nodal curve b, we consider k the number of its apparent double points (excluding actual double points); f the number of its actual double points (each of these is a point of contact of two sheets of the surface, and there is thus at the point a single tangent plane, viz. this is a plane f', and we thus have $f'=f$); t the number of its triple points; and j the number of its pinch-points—these last are not singular points of the nodal curve *per se*, but are singular in regard to the curve as nodal curve of the surface; viz. a pinch-point is a point at which the two tangent planes are coincident. The curve is considered as not having any stationary points other than the points γ, which lie also on the cuspidal curve; and the expression for the class consequently is $q = b^2 - b - 2k - 2f - 3\gamma - 6t$.

623. In regard to the cuspidal curve c we consider h the number of its apparent double points; and upon the curve, not singular points in regard to the curve *per se*, but only in regard to it as cuspidal curve of the surface, certain points in number θ, χ, ω respectively. The curve is considered as not having any actual double or other multiple points, and as not having any stationary points except the points β, which lie also on the nodal curve; and thus the expression for the class is $r = c^2 - c - 2h - 3\beta$.

624. The points γ are points where the cuspidal curve with the two sheets (or say rather half-sheets) belonging to it are intersected by another sheet of the surface; the curve of intersection with such other sheet belonging to the nodal curve of the surface has evidently a stationary (cuspidal) point at the point of intersection.

As to the points β, to facilitate the conception, imagine the cuspidal curve to be a semi-cubical parabola, and the nodal curve a right line (not in the plane of the curve) passing through the cusp; then intersecting the two curves by a series of parallel planes, any plane which is, say, above the cusp, meets the parabola in two real points and the line in one real point, and the section of the surface is a curve with two real cusps and a real node; as the plane approaches the cusp, these approach together, and, when the plane passes through the cusp, unite into a singular point in the nature of a triple point (= node + two cusps); and when the plane passes below the cusp, the two cusps of the section become imaginary, and the nodal line changes from crunodal to acnodal.

625. At a point i the nodal curve crosses the cuspidal curve, being on the side away from the two half-sheets of the surface acnodal, and on the side of the two half-sheets crunodal, viz. the two half-sheets intersect each other along this portion of the nodal curve. There is at the point a single tangent plane, which is a plane i'; and we thus have $i = i'$.

626. As already mentioned, a cnicnode C is a point where, instead of a tangent plane, we have a tangent quadricone; and at a binode B the quadricone degenerates into a pair of planes. A cnictrope C' is a plane touching the surface along a conic; in the case of a bitrope B', the conic degenerates into a flat conic or pair of points.

627. In the original formulæ for $a(n-2)$, $b(n-2)$, $c(n-2)$, we have to write $\kappa - B$ instead of κ, and the formulæ are further modified by reason of the singularities θ and ω. So in the original formulæ for $a(n-2)(n-3)$, $b(n-2)(n-3)$, $c(n-2)(n-3)$, we have instead of δ to write $\delta - C - 3\omega$; and to substitute new expressions for $[ab]$, $[ac]$, $[bc]$, viz. these are

$$[ab] = ab - 2\rho - j,$$

$$[ac] = ac - 3\sigma - \chi - \omega,$$

$$[bc] = bc - 3\beta - 2\gamma - i.$$

The whole series of equations thus is

(1) $a' = a.$

(2) $f' = f.$

(3) $i' = i.$

(4) $a = n\ \ (n-1) - 2b - 3c.$

(5) $\kappa' = 3n\,(n-2) - 6b - 8c.$

(6) $\delta' = \frac{1}{2}n\,(n-2)\,(n^2-9) - (n^2-n-6)\,(2b+3c) + 2b\,(b-1) + 6bc + \frac{9}{2}c\,(c-1).$

(7) $a\,(n-2) = \kappa - B + \rho + 2\sigma + 3\omega.$

(8) $b\,(n-2) = \rho + 2\beta + 3\gamma + 3t.$

(9) $c\,(n-2) = 2\sigma + 4\beta + \gamma + \theta + \omega.$

(10) $a\,(n-2)\,(n-3) = 2\,(\delta - C - 3\omega) + 3\,(ac - 3\sigma - \chi - 3\omega) + 2\,(ab - 2\rho - j\quad).$

(11) $b\,(n-2)\,(n-3) = 4k + (ab - 2\rho - j\quad) + 3\,(bc - 3\beta - 2\gamma - i).$

(12) $c\,(n-2)\,(n-3) = 6h + (ac - 3\sigma - \chi - 3\omega) + 2\,(bc - 3\beta - 2\gamma - i).$

(13) $q = b^2 - b - 2k - 2f - 3\gamma - 6t.$

(14) $r = c^2 - c - 2h - 3\beta.$

Also, reciprocal to these

(15) $a = n'(n'-1) - 2b' - 3c'$.

(16) $\kappa = 3n'(n'-2) - 6b' - 8c'$.

(17) $\delta = \frac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \frac{9}{2}c'(c'-1)$.

(18) $a'(n'-2) = \kappa' - B' + \rho' + 2\sigma' + 3\omega'$.

(19) $b'(n'-2) = \rho + 2\beta' + 3\gamma' + 3t'$.

(20) $c'(n'-2) = 2\sigma' + 4\beta' + \gamma' + \theta' + \omega'$.

(21) $a'(n'-2)(n'-3) = 2(\delta' - C' - 3\omega') + 3(a'c' - 3\sigma' - \chi' - 3\omega') + 2(a'b' - 2\rho' - j')$.

(22) $b'(n'-2)(n'-3) = 4k' + (a'b' - 2\rho' - j') + 3(b'c' - 3\beta' - 2\gamma' - i')$.

(23) $c'(n'-2)(n'-3) = 6h' + (a'c' - 3\sigma' - \chi' - 3\omega') + 2(b'c' - 3\beta' - 2\gamma' - i')$.

(24) $q' = b'^2 - b' - 2k' - 2f' - 3\gamma' - 6t'$.

(25) $r' = c'^2 - c' - 2h' - 3\beta'$,

together with one other independent relation, in all 26 relations between the 46 quantities.

628. The new relation may be presented under several different forms, equivalent to each other in virtue of the foregoing 25 relations; these are

(26) $2(n-1)(n-2)(n-3) - 12(n-3)(b+c) + 6q + 6r + 24t + 42\beta + 30\gamma - \frac{3}{2}\theta = \Sigma$;

(27) $26n - 12c - 4C - 10B + \beta - 7j - 8\chi + \frac{1}{2}\theta - 4\omega = \Sigma$,

in each of which two equations Σ is used to denote the same function of the accented letters that the left-hand side is of the unaccented letters.

(28)
$$\begin{aligned}\beta' + \tfrac{1}{2}\theta' = {} & 2r(n-2)(11n-24)\\ & +(-66n+184)b\\ & +(-93n+252)c\\ & +22(2\beta+3\gamma+3t)\\ & +27(4\beta+\gamma+\theta)\\ & +\beta+\tfrac{1}{2}\theta\\ & -24C-28B-27j-38\chi-73\omega\\ & +4C'+10B'+7j'+8\chi'-4\omega'.\end{aligned}$$

Or, reciprocally,

(29)
$$\begin{aligned}\beta + \tfrac{1}{2}\theta = {} & 2n'(n'-2)(11n'-24)\\ & +(-66n'+184)b'\\ & +(-93n'+252)c'\\ & +22(2\beta'+3\gamma+3t')\\ & +27(4\beta'+\gamma'+\theta')\\ & +\beta'+\tfrac{1}{2}\theta'\\ & -24C'-28B'-27j'-38\chi'-73\omega'\\ & +4C+10B+7j+8\chi-4\omega.\end{aligned}$$

The foregoing equation (26) in fact expresses that the surface and its reciprocal have the same deficiency; viz. the expression for the deficiency is

$$(30)\quad \text{Deficiency} = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + i - \tfrac{1}{8}\theta,$$
$$= \tfrac{1}{6}(n'-1)(n'-2)(n'-3) - \&c.$$

629. The equation (28) (due to Prof. Cayley) is the correct form of an expression for β', first obtained by him (with some errors in the numerical coefficients) from independent considerations, but which is best obtained by means of the equation (26); and (27) is a relation presenting itself in the investigation. In fact, considering a as standing for its value $n(n-1)-2b-3c$, we have from the first 25 equations

$$\begin{array}{r|l}
6 & a = \Sigma \\
+2 & 3n - c - \kappa = \Sigma \\
-2 & a(n-2) - \kappa + B - \rho - 2\sigma - 3\omega = \Sigma \\
-4 & b(n-2) - \rho - 2\beta - 3\gamma - 3t = \Sigma \\
-6 & c(n-2) - 2\sigma - 4\beta - \gamma - \theta - \omega = \Sigma \\
+2 & n + \kappa - \sigma - 2C - 4B - 2j - 3\chi - 3\omega = \Sigma \\
-3 & 2q - 2\rho + \beta + j = \Sigma \\
-2 & 3r + c - 5\sigma - \beta - 4\theta + \chi - \omega = \Sigma
\end{array}$$

and multiplying these equations by the numbers set opposite to them respectively, and adding, we find

$$-2n^3 + 12n^2 + 4n + b(12n-36) + c(12n-48)$$
$$-6q - 6r - 4C - 10B - 41\beta - 30\gamma - 24t - 7j - 8\chi + 2\theta - 4\omega = \Sigma,$$

and adding thereto (26) we have the equation (27); and from this (28), or by a like process, (29), is obtained without much difficulty. As to the 8 Σ-equations or symmetries, observe that the first, third, fourth, and fifth are in fact included among the original equations (for an expression which vanishes is in fact $=\Sigma$); we have from them moreover $3n - c = 3a' - \kappa'$, and thence $3n - c - \kappa = 3a' - \kappa - \kappa'$, which is $=\Sigma$, or we have thus the second equation; but the sixth, seventh, and eighth equations have yet to be obtained.

630. The equations (15), (16), (17) give

$$n' = a(a-1) - 2\delta - 3\kappa,$$
$$c' = 3a(a-2) - 6\delta - 8\kappa,$$
$$b' = \tfrac{1}{2}a(a-2)(a^2-9) - (a^2-a-6)(2\delta+3\kappa) + 2\delta(\delta-1) + 6\delta\kappa + \tfrac{9}{2}\kappa(\kappa-1);$$

from (7), (8), (9) we have

$$(a-b-c)(n-2) = \kappa - B - 6\beta - 4\gamma - 3t - \theta + 2\omega,$$
$$(a-2b-3c)(n-2)(n-3) = 2(\delta - C) - 8k - 18h - 6bc + 18\beta + 12\gamma + 6i - 6\omega,$$

and substituting these values for κ and δ, and for a its value $=n(n-1)-2b-3c$ we obtain the values of n', c', b'; viz. the value of n' is

$$\begin{aligned} n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 8b + 9c^2 + 15c \\ -8k - 18h + 18\beta + 12\gamma + 12i - 9t \\ -2C - 3B - 3\theta. \end{aligned}$$

Observe that the effect of a cnicnode C is to reduce the class by 2, and that of a binode B to reduce it by 3.

631. We have

$$(n-2)(n-3) = n^2 - n + (-4n+6) = a + 2b + 3c + (-4n+6),$$

and making this substitution in the equations (10), (11), (12), which contain $(n-2)(n-3)$, these become

$$\begin{aligned} a(-4n+6) &= 2(\delta - C) - a^2 - 4\rho - 9\sigma - 2j - 3\chi - 15\omega, \\ b(-4n+6) &= 4k - 2b^2 - 9\beta - 6\gamma - 3i - 2\rho - j, \\ c(-4n+6) &= 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma - \chi - 3\omega, \end{aligned}$$

(the foregoing equations (C) Salmon p. 586); and adding to each equation four times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2 - 2a &= 2(\delta - C) + 4(\kappa - B) - \sigma - 2j - 3\chi - 3\omega, \\ 2b^2 - 2b &= 4k - \beta + 6\gamma + 12t - 3i + 2\rho - j, \\ 3c^2 - 2c &= 6h + 10\beta + 4\theta - 2i + 5\sigma - \chi + \omega. \end{aligned}$$

Writing in the first of these $a^2 - 2a = n' + 2\delta + 3\kappa - a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n' - a &= -2C - 4B + \kappa - \sigma - 2j - 3\chi - 3\omega, \\ 2q + \beta + 3i + j &= 2\rho, \\ 3r + c + 2i + \chi &= 5\sigma + \beta + 4\theta + \omega, \end{aligned}$$

which give at once the last three of the 8 Σ-equations.

The reciprocal of the first of these is

$$\sigma' = a - n + \kappa' - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

or writing herein $a = n(n-1) - 2b - 3c$ and $\kappa' = 3n(n-2) - 6b - 8c$, this is

$$\sigma' = 4n(n-2) - 8b - 11c - 2j' - 3\chi' - 2C' - 4B' - 3\omega',$$

giving the order of the spinode curve; viz. for a surface of the order n without singularities this is $= 4n(n-2)$, the product of the orders of the surface and its Hessian.

632. Instead of obtaining the second and third equations as above, we may to the value of $b(-4n+6)$ add twice the value of $b(n-2)$; and to twice the value of $c(-4n+6)$ add three times the value of $c(n-2)$, thus obtaining equations free from ρ and σ respectively; these equations are

$$\begin{aligned} b(-2n+2) &= 4k - 2b^2 - 5\beta - 3i + 6t - j, \\ c(-5n+6) &= 12h - 6c^2 - 5\gamma - 4i - 2\chi + 3\theta - 3\omega, \end{aligned}$$

equations which, introducing therein the values of q and r, may also be written

$$\begin{aligned} b(2n-4) &= 2q + 5\beta + 6\gamma + 6t + 3i + j + 4f, \\ c(5n-12) + 3\theta &= 6r + 18\beta + 5\gamma + 4i + 2\chi + 3\omega. \end{aligned}$$

Considering as given, n the order of the surface; the nodal curve with its singularities b, k, f, t; the cuspidal curve and its singularities c, h; and the quantities β, γ, i which relate to the intersections of the nodal and cuspidal curves; the first of the two equations gives j, the number of pinch-points, being singularities of the nodal curve quoad the surface; and the second equation establishes a relation between θ, χ, ω, the numbers of singular points of the cuspidal curve quoad the surface.

In the case of a nodal curve only, if this be a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2 = 0$, and the first equation is

$$b(-2n+2) = 4k - 2b^2 + 6t - j;$$

or, assuming $t=0$, say $j = 2(n-1)b - 2b^2 + 4k$, which may be verified; and so in the case of a cuspidal curve only, when this is a complete intersection $P=0$, $Q=0$, the equation of the surface is $(A, B, C \between P, Q)^2 = 0$, where $AC - B^2 = MP + NQ$; and the second equation is

$$c(-5n+6) = 12h - 6c^2 - 2\chi + 3\theta - 3\omega,$$

or, say

$$2\chi + 3\omega = (5n-6)c - 6c^2 + 12h + 3\theta,$$

which may also be verified.

633. We may in the first instance out of the 46 quantities consider as given the 14 quantities

$$n; \quad b, k, f, t; c, h, \theta, \chi \quad ; \beta, \gamma, i; C, B,$$

then of the 26 relations, 17 determine the 17 quantities

$$\begin{array}{llll} a, \delta, \kappa, \rho, \sigma; j, q & ; r, \omega & ; & \\ n'; a', \delta', \kappa' & ; b', f' & ; c' & ; \quad i', \end{array}$$

and there remain the 9 equations

(18), (19), (20), (21), (22), (23), (24), (25), (28),

connecting the 15 quantities

$$\rho', \sigma'; k', t', j', q'; h', \theta', \chi', \omega', r'; \beta', \gamma'; C', B'.$$

Taking then further as given the 5 quantities j', χ', ω', C', B',

$$\begin{array}{llll} \text{equations (18) and (21) give} & \rho', \sigma', & & \\ \text{equation} & (19) & \text{gives} & 2\beta' + 3\gamma' + 3t', \\ \quad ,, & (20) & \quad ,, & 4\beta' + \gamma' + \theta', \\ \quad ,, & (28) & \quad ,, & \beta' + \tfrac{1}{2}\theta', \end{array}$$

so that taking also t' as given, these last three equations determine β', γ', θ'; and finally

$$\begin{array}{llll} \text{equation} & (22) & \text{gives} & k', \\ \quad ,, & (23) & \quad ,, & h', \\ \quad ,, & (24) & \quad ,, & q', \\ \quad ,, & (25) & \quad ,, & r', \end{array}$$

viz. taking as given in all 20 quantities, the remaining 26 will be determined.

634. In the case of the general surface of the order n, without singularities, we have as follows:

$n = n,$

$a = n(n-1),$

$\delta = \tfrac{1}{2}n(n-1)(n-2)(n-3),$

$\kappa = n(n-1)(n-2),$

$n' = n(n-1)^2,$

$a' = n(n-1),$

$\delta' = \tfrac{1}{2}n(n-2)(n^2-9),$

$\kappa' = 3n(n-2),$

$b' = \tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12),$

$k' = \tfrac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3+1068n^2-1214n+1464),$

$t' = \tfrac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960),$

$q' = n(n-2)(n-3)(n^2+2n-4),$

$\rho' = n(n-2)(n^3-n^2+n-12),$

$c' = 4n(n-1)(n-2),$

$h' = \tfrac{1}{2}n(n-2)(16n^4-64n^3+80n^2-108n+156),$

$r' = 2n(n-2)(3n-4),$

$\sigma' = 4n(n-2),$

$\beta' = 2n(n-2)(11n-24),$

$\gamma' = 4n(n-2)(n-3)(n^3-3n+16),$

the remaining quantities vanishing.

635. The question of singularities has been considered under a more general point of view by Zeuthen, in the memoir "Recherche des singularités qui ont rapport à une droite multiple d'une surface," *Math. Annalen*, t. IV. pp. 1—20, 1871. He attributes to the surface:

A number of singular points, viz. points at any one of which the tangents form a cone of the order μ, and class ν, with $y+\eta$ double lines, of which y are tangents to branches of the nodal curve through the point, and $z+\zeta$ stationary lines, whereof z are tangents to branches of the cuspidal curve through the point, and with u double planes and v stationary planes; moreover, these points have only the properties which are the most general in the case of a surface regarded as a locus of points; and Σ denotes a sum extending to all such points. {The foregoing general definition includes the cnicnodes ($\mu=\nu=2$, $y=\eta=z=\zeta=u=v=0$), and [also, but not properly] the binodes ($\mu=2$, $\eta=1$, $\nu=y=$ &c. $=0$), [it includes also the off-points ($\mu=\nu=3$, $z=v=1$, $y=\eta=(=0)$].}

And, further, a number of singular planes, viz. planes any one of which touches along a curve of the class μ' and order ν', with $y'+\eta'$ double tangents, of which y' are generating lines of the node-couple torse, $z'+\zeta'$ stationary tangents, of which z' are generating lines of the spinode torse, u' double points and v' cusps; it is, moreover, supposed that these planes have only the properties which are the most general in the case of a surface regarded as an envelope of its tangent planes; and Σ' denotes a sum extending to all such planes. {The definition includes the cnictropes ($\mu'=\nu'=2$, $y'=\eta'=z'=\zeta'=u'=v'=0$), and [also, but not properly] the bitropes ($\mu'=2$, $\eta'=1$, $\nu'=y'=$ &c. $=0$), [it includes also the off-planes ($\mu'=\nu'=3$, $z'=v'=1$, $y'=\eta'=\zeta'=0$)].}

636. This being so, and writing

$$x=\nu+2\eta+3\zeta,\quad x'=\nu'+2\eta'+3\zeta',$$

the equations (7), (8), (9), (10), (11), (12), contain in respect of the new singularities additional terms, viz. these are

$$\begin{aligned}
a\,(n-2) &= \ldots + \Sigma\,[x\,(\mu-2)-\eta-2\zeta],\\
b\,(n-2) &= \ldots + \Sigma\,[y\,(\mu-2)],\\
c\,(n-2) &= \ldots + \Sigma\,[z\,(\mu-2)],\\
a\,(n-2)\,(n-3) &= \ldots + \Sigma\,[x\,(-4\mu+7)+2\eta+4\zeta],\\
b\,(n-2)\,(n-3) &= \ldots + \Sigma\,[y\,(-4\mu+8)] - \Sigma'\,(4u'+3v'),\\
c\,(n-2)\,(n-3) &= \ldots + \Sigma\,[z\,(-4\mu+9)] - \Sigma'\,(2v'),
\end{aligned}$$

and there are of course the reciprocal terms in the reciprocal equations (18), (19), (20), (21), (22), (23). These formulæ are given without demonstration in the memoir just referred to: the principal object of the memoir, as shown by its title, is the consideration not of such singular points and planes, but of the multiple right lines of a surface; and in regard to these, the memoir should be consulted.