412.

A MEMOIR ON CUBIC SURFACES.

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THE present Memoir is based upon, and is in a measure supplementary to that by Professor Schläfli, "On the Distribution of Surfaces of the Third Order into Species, in reference to the presence or absence of Singular Points, and the reality of their Lines," *Phil. Trans.* vol. CLIII. (1863), pp. 193—241. But the object of the Memoir is different. I disregard altogether the ultimate division depending on the reality of the lines, attending only to the division into (twenty-two, or as I prefer to reckon it) twenty-three cases depending on the nature of the singularities. And I attend to the question very much on account of the light to be obtained in reference to the theory of Reciprocal Surfaces. The memoir referred to furnishes in fact a store of materials for this purpose, inasmuch as it gives (partially or completely developed) the equations in plane-coördinates of the several cases of cubic surfaces, or, what is the same thing, the equations in point-coordinates of the several surfaces (orders 12 to 3) reciprocal to these respectively. I found by examination of the several cases, that an extension was required of Dr Salmon's theory of Reciprocal Surfaces in order to make it applicable to the present subject; and the preceding "Memoir on the Theory of Reciprocal Surfaces," [411], was written in connexion with these investigations on Cubic Surfaces. The latter part of the Memoir is divided into sections headed thus:—"Section $\text{I} = 12$, equation $(X, Y, Z, W)^3 = 0$" &c. referring to the several cases of the cubic surface; but the paragraphs are numbered continuously throughout the Memoir.

Article Nos. 1 to 13. *The twenty-three Cases of Cubic Surfaces—Explanations and Table of Singularities.*

1. I designate as follows the twenty-three cases of cubic surfaces, adding to each of them its equation:

$$\text{I} = 12, \qquad (X, Y, Z, W)^3 = 0,$$

$$\text{II} = 12 - C_2, \qquad W(a, b, c, f, g, h \between X, Y, Z)^2 + 2kXYZ = 0,$$

$$\text{III} = 12 - B_3, \qquad 2W(X + Y + Z)(lX + mY + nZ) + 2kXYZ = 0$$

IV $= 12 - 2C_2$, $WXZ + Y^2(\gamma Z + \delta W) + (a, b, c, d)(X, Y)^3 = 0$,
V $= 12 - B_4$, $WXZ + (X + Z)(Y^2 - aX^2 - bZ^2) = 0$,
VI $= 12 - B_3 - C_2$, $WXZ + Y^2Z + (a, b, c, d)(X, Y)^3 = 0$,
VII $= 12 - B_5$, $WXZ + Y^2Z + YX^2 - Z^3 = 0$,
VIII $= 12 - 3C_2$, $Y^3 + Y^2(X + Z + W) + 4aXZW = 0$,
IX $= 12 - 2B_3$, $WXZ + (a, b, c, d)(X, Y)^3 = 0$,
X $= 12 - B_4 - C_2$, $WXZ + (X + Z)(Y^2 - X^2) = 0$,
XI $= 12 - B_6$, $WXZ + Y^2Z + X^3 - Z^3 = 0$,
XII $= 12 - U_6$, $W(X + Y + Z)^2 + XYZ = 0$,
XIII $= 12 - B_3 - 2C_2$, $WXZ + Y^2(X + Y + Z) = 0$,
XIV $= 12 - B_5 - C_2$, $WXZ + Y^2Z + YX^2 = 0$,
XV $= 12 - U_7$, $WX^2 + XZ^2 + Y^2Z = 0$,
XVI $= 12 - 4C_2$, $W(XY + XZ + YZ) + XYZ = 0$,
XVII $= 12 - 2B_3 - C_2$, $WXZ + XY^2 + Y^3 = 0$,
XVIII $= 12 - B_4 - 2C_2$, $WXZ + (X + Z)Y^2 = 0$,
XIX $= 12 - B_6 - C_2$, $WXZ + Y^2Z + X^3 = 0$,
XX $= 12 - U_8$, $WX^2 + XZ^2 + Y^3 = 0$,
XXI $= 12 - 3B_3$, $WXZ + Y^3 = 0$,
XXII $= 3$, $S(1, 1)$, $WX^2 + ZY^2 = 0$,
XXIII $= 3$, $S(\overline{1, 1})$, $X(WX + YZ) + Y^3 = 0$;

2. Where C_2 denotes a conic-node diminishing the class by 2; B_3, B_4, B_5, B_6 a biplanar node diminishing (as the case may be) the class by 3, 4, 5, or 6; and U_6, U_7, U_8 a uniplanar node diminishing (as the case may be) the class by 6, 7, or 8. The affixed explanation, which I shall usually retain in connexion with the Roman number, shows therefore in each case what the class is, and also the singularities which cause the reduction: thus XIII $= 12 - B_3 - 2C_2$ indicates that there is a biplanar node, B_3, diminishing the class by 3, and two conic-nodes, C_2, each diminishing the class by 2; and thus that the class is $12 - 3 - 2 . 2, = 5$. As regards the cases XXII and XXIII, these are surfaces having a nodal right line, and are consequently scrolls, each of the class 3, viz. XXII is the scroll $S(1, 1)$ having a simple directrix right line distinct from the nodal line, and XXIII is the scroll $S(\overline{1, 1})$ having a simple directrix right line coincident with the nodal line: see as to this my "Second Memoir on Skew Surfaces, otherwise Scrolls," *Phil. Trans.* vol. CLIV. (1864), pp. 559—577, [340].

3. The nature of the points C_2, B_3, B_4, B_5, B_6, U_6, U_7, U_8 requires to be explained.

$C (= C_2)$ is a conic-node, where, instead of the tangent plane, we have a proper quadric cone.

$B (= B_3, B_4, B_5$ or $B_6)$ is a biplanar-node, where the quadric cone becomes a plane-pair (two distinct planes): the two planes are called the biplanes, and their line of intersection is the edge:

In B_3, the edge is not a line on the surface—in the other cases it is; this implies that the surface is touched along the edge by a plane, viz. in B_4, B_5 the edge is torsal, in B_6 it is oscular:

In B_4, the tangent plane is distinct from each of the biplanes:

In B_5, the tangent plane coincides with one of the biplanes; we have thus an ordinary biplane, and a torsal biplane:

In B_6, the tangent plane coinciding with one of the biplanes becomes oscular; we have thus an ordinary biplane, and an oscular biplane.

$U\ (=U_6,\ U_7$ or $U_8)$ is a uniplanar-node, where the quadric cone becomes a coincident plane-pair; say, the plane is the uniplane. It is to be observed that there is not in this case any edge. The uniplane meets the cubic surface in three lines, or say "rays," passing through the uniplanar-node, viz.

In U_6, the rays are three distinct lines:

In U_7, two of them coincide:

In U_8, they all three coincide.

4. To connect these singular points with the theory of the preceding Memoir, it is to be observed that they are respectively equivalent to a certain number of the cnicnodes $C\,(=C_2)$ and binodes $B\,(=B_3)$, viz. we have

$$\begin{aligned} C_2 &= C, \\ B_3 &= \quad\ B, \\ B_4 &= 2C, \\ B_5 &= C + B, \\ &\begin{cases} B_6 = 3C, \\ U_6 = 3C, \end{cases} \\ U_7 &= 2C + B, \\ U_8 &= C + 2B. \end{aligned}$$

5. I take the opportunity of remarking that although the expressions cnicnode and binode properly refer to the simple singularities C and B, yet as $C_2 = C$, C_2 is properly spoken of as a cnicnode, and we may (using the term binode as an abbreviation for biplanar-node) speak of any of the singularities B_3, B_4, B_5, B_6 as a binode. Thus the surface $\mathrm{X} = 12 - B_4 - C_2$ has a binode B_4 and a cnicnode C_2; although theoretically the binode B_4 is equivalent to two cnicnodes, and the surface belongs to those with three cnicnodes, or for which $C = 3$. I use also the expression unode for shortness, instead of uniplanar-node, to denote any of the singularities U_6, U_7, U_8.

6. The foregoing equations (substantially the same as Schläfli's) are *Canonical forms;* the reduction of the equation of any case of surface to the above form is not always obvious. It would appear that each equation is from its simplicity in the form best adapted to the separate discussion of the surface to which it belongs; there is the disadvantage that the equations do not always (when from the geometrical connexion of the surfaces they ought to do so) lead the one to the other; for instance, $\mathrm{V} = 12 - B_4$ includes $\mathrm{VII} = 12 - B_5$, but we cannot from the equation $WXZ + (X+Z)(Y^2 - aX^2 - bZ^2) = 0$ of the former pass to the equation $WXZ + Y^2Z + YX^2 - X^3 = 0$ of the latter. This would be a serious imperfection if the object were to form a theory of the quaternary function $(X, Y, Z, W)^3$; but the equations are in the present Memoir used only as means to an end, the establishment of the geometrical theory of the surfaces to which they respectively belong, and the imperfection is not material.

7. I have used the capital letters (X, Y, Z, W) in place of Schläfli's (x, y, z, w), reserving these in place of his (p, q, r, s) for plane-coordinates of the cubic surfaces, or (what is the same thing) point-coordinates of the reciprocal surfaces; but I have in several cases interchanged the coordinates (X, Y, Z, W) so that they do not *in this order* correspond to Schläfli's (x, y, z, w): this has been done so as to obtain a greater uniformity in the representation of the surfaces. To explain this, let A, B, C, D be the vertices of the tetrahedron formed by the coordinate planes $A = YZW$, $B = ZWX$, $C = WXY$, $D = XYZ$; the coordinate planes have been chosen so that determinate vertices of the tetrahedron shall correspond to determinate singularities of the surface.

8. Consider first the surfaces which have no nodes B or U. It is clear that the nodes C_2 might have been taken at any vertices whatever of the tetrahedron; they are taken thus: there is always a node C_2 at D; when there is a second node C_2, this is at C, the third one is at A, and the fourth at B.

9. Consider next the surfaces which have a binode B_3, B_4, B_5, or B_6; this is taken to be at D, and the biplanes to be $X = 0$, $Z = 0$ (1) (the edge being therefore DB), viz. in B_5 or B_6, where the distinction arises, $X = 0$ is the ordinary biplane, $Z = 0$ the torsal or (as the case may be) oscular biplane. If there is a second node, this of necessity lies in an ordinary biplane; it may be and is taken to be in the biplane $X = 0$, at C. I suppose for a moment that this is a node C_2. It is only when the binode is B_3 or B_4 that there can be a third node, for it is only in these cases that there is a second ordinary biplane $Z = 0$; but in these cases respectively the third node, a C_2, may be and is taken to be in the biplane $Z = 0$, at A.

10. The only case of two binodes is when each is a B_3. Here the first is as above at D, its biplanes being $X = 0$, $Z = 0$; and the second is as above in the biplane $X = 0$, at C; the biplanes thereof are then $X = 0$ (which is thus a biplane common to the two binodes, or say a common biplane), and a remaining biplane which may be and is taken to be $W = 0$. If there is a third node, this may be either C_2 or B_3, but it will in either case lie in the biplane $Z = 0$ of the first binode, and also in the biplane $W = 0$ of the second binode, that is, in the line BA; and it may be and is taken to be at A; if a binode, then its biplanes are of necessity $Z = 0$, $W = 0$; and the plane $Y = 0$ will be the plane through the three binodes D, C, A.

11. If there is a unode, then this may be and is taken to be at D, and its uniplane may be taken to be $X = 0$; in the surface $\text{XII} = 12 - U_6$ the uniplane is, however, taken to be $X + Y + Z = 0$. There is never, besides the unode, any other node.

12. The result is that the nodes, in the order of their speciality, are in the equations taken to be at D, C, A, B respectively; and that (except in the case $\text{III} = 12 - B_3$) the biplanes of the first binode are $X = 0$, $Z = 0$ (for a binode B_5 or B_6, $X = 0$ being the ordinary biplane, $Z = 0$ the special biplane), those of the second binode $X = 0$, $W = 0$, those of the third binode $Z = 0$, $W = 0$, and that (except in the case $\text{XII} = 12 - U_6$) the uniplane is $X = 0$. For example, in the surface $\text{XVII} = 12 - 2B_3 - C_2$, as represented by its equation $WXZ + Y^2Z + X^3 = 0$, we have a B_3 at D, the biplanes being $X = 0$, $Z = 0$, a B_3 at C, the biplanes being $X = 0$, $W = 0$ (therefore $X = 0$ the common biplane), and a C_2 at A.

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(1) In the case, however, of a single B_3, $\text{III} = 12 - B_3$, the biplanes are taken to be $X + Y + Z = 0$, $lX + mY + nZ = 0$.

13. It will be convenient (anticipating the results of the investigations contained in the present Memoir) to give at once the following Table of Singularities; the several symbols have of course the significations explained in the former Memoir.

	I $=12$.	II $=12-C_2$.	III $=12-B_3$.	IV $=12-2C_2$. V $=12-B_4$.	VI $=12-B_3-C_2$. VII $=12-B_5$.	VIII $=12-3C_2$. X $=12-B_4-C_2$. XI $=12-B_6$. XII $=12-U_6$.	IX $=12-2B_3$.	XIII $=12-B_3-2C_2$. XIV $=12-B_5-C_2$. XV $=12-U_7$.	XVI $=12-4C_2$. XVIII $=12-B_4-2C_2$. XIX $=12-B_6-C_2$.	XVII $=12-2B_3-C_2$. XX $=12-U_8$.	XXI $=12-3B_3$.	XXII $=S(1, 1)$. XXIII $=S(\overline{1, 1})$.	
n	3	3	3	3	3	3	3	3	3	3	3	3	n
a	6	6	6	6	6	6	6	6	6	6	6	4	a
δ	0	1	0	2	1	3	0	2	4	1	0	0	δ
κ	6	6	7	6	7	6	8	7	6	8	9	3	κ
b	0	0	0	0	0	0	0	0	0	0	0	1	b
k	0	0	0	0	0	0	0	0	0	0	0	0	k
t	0	0	0	0	0	0	0	0	0	0	0	0	t
q	0	0	0	0	0	0	0	0	0	0	0	0	q
ρ	0	0	0	0	0	0	0	0	0	0	0	1	ρ
j	0	0	0	0	0	0	0	0	0	0	0	2	j
c	0	0	0	0	0	0	0	0	0	0	0	0	c
h	0	0	0	0	0	0	0	0	0	0	0	0	h
r	0	0	0	0	0	0	0	0	0	0	0	0	r
σ	0	0	0	0	0	0	0	0	0	0	0	0	σ
θ	0	0	0	0	0	0	0	0	0	0	0	0	θ
χ	0	0	0	0	0	0	0	0	0	0	0	0	χ
β	0	0	0	0	0	0	0	0	0	0	0	0	β
γ	0	0	0	0	0	0	0	0	0	0	0	0	γ
i	0	0	0	0	0	0	0	0	0	0	0	0	i
C	0	1	0	2	1	3	0	2	4	1	0	0	C
B	0	0	1	0	1	0	2	1	0	2	3	0	B
	I	II	III	IV V	VI VII	VIII X XI XII	IX	XIII XIV XV	XVI XVIII XIX	XVII XX	XXI	XXII XXIII	
n'	12	10	9	8	7	6	6	5	4	4	3	3	n'
a'	6	6	6	6	6	6	6	6	6	6	6	4	a'
δ'	0	0	0	0	0	0	0	0	0	0	0	0	δ'
κ'	9	9	9	9	9	9	9	9	9	9	9	3	κ'
b'	27	15	9	7	3	3	0	1	3	0	0	1	b'
k'	216	60	18	12	3	0	0	0	0	0	0	0	k'
t'	45	15	6	3	0	1	0	0	1	0	0	0	t'
q'	0	0	0	0	0	0	0	0	0	0	0	0	q'
ρ'	27	15	9	7	3	3	0	1	3	0	0	1	ρ'
j'	0	0	0	1	0	3	0	1	6	0	0	2	j'
c'	24	18	16	12	10	6	8	4	0	2	0	0	c'
h'	180	96	72	38	24	6	12	2	0	0	0	0	h'
r'	30	24	42	17	24	9	32	5	0	2	0	0	r'
σ'	12	12	12	10	9	6	8	4	0	2	0	0	σ'
θ'	0	0	16	0	8	0	16	0	0	0	0	0	θ'
χ'	0	0	0	0	1	0	0	2	0	2	0	0	χ'
β'	54	30	18	13	6	3	0	1	0	0	0	0	β'
γ'	0	0	0	0	0	0	0	0	0	0	0	0	γ'
i'	0	0	0	0	0	0	0	0	0	0	0	0	i'
C'	0	0	0	0	0	0	0	0	0	0	0	0	C'
B'	0	0	0	0	0	0	1	0	0	1	3	0	B

Article Nos. 14 to 19. *Explanation in regard to the Determination of the Number of certain Singularities.*

14. In the several cases I to XXI, we have a cubic surface ($n=3$), with singular points C and B but without singular lines. The section by an arbitrary plane is thus a curve, order $n=3$, that is, a cubic curve, without nodes or cusps, and therefore of the class $a'=6$, having $\delta'=0$ double tangents and $\kappa'=9$ inflexions. The tangent cone with an arbitrary point as vertex is a cone of the order $a=6$, having in the case $\text{I}=12$, $\delta=0$ nodal lines and $\kappa=6$ cuspidal lines, but with (in the several other cases) C nodal lines and B cuspidal lines (or rather singular lines tantamount to C double lines and B cuspidal lines): the class of the cone, or order of the reciprocal surface, is thus $n'=6\,.\,5-2(0+C)-3(6+B)=12-2B-3C$.

15. In the general case $\text{I}=12$, there are on the cubic surface 27 lines, lying by 3's in 45 planes; these 27 lines constitute the node-couple curve of the order $\rho'=27$, and the node-couple torse consists of the pencils of planes through these lines respectively, being thus of the class $\rho'=b'=27$; the 45 planes are triple tangent planes of the node-couple torse, which has thus $t'=45$ triple tangent planes. But in the other cases it is only certain of the 27 lines, say the "facultative lines" (as will be explained), which constitute the node-couple curve of the order ρ': the pencils of planes through these lines constitute the node-couple torse of the class $b'=\rho'$; the t' planes, each containing three facultative lines, are the triple tangent planes of the node-couple torse. Or if (as is somewhat more convenient) we refer the numbers b', t' to the reciprocal surface, then the lines, reciprocals of the facultative lines, constitute the nodal curve of the order b'; and the points t', each containing three of these lines, are the triple points of the nodal curve. Inasmuch as the nodal curve consists of right lines, the number k' of its apparent double points is given by the formula $2k'=b'^2-b'-6t'$; and comparing with the formula $q'=b'^2-b'-2k'-3\gamma'-6t'$, we have $q'+3\gamma'=0$, that is, $q'=0$ (q' the class of the nodal curve), and also $\gamma'=0$.

16. In the general case $\text{I}=12$, the spinode curve is the complete intersection of the cubic surface by the Hessian surface of the order 4, and it is thus of the order $\sigma'=12$; but in the other cases the complete intersection consists of the spinode curve together with certain right lines not belonging to the curve, and the spinode curve is of an order σ' less than 12: this will be further explained, and the reduction accounted for (see *post*, Nos. 24 *et seq.*).

17. Again, in the general case $\text{I}=12$, each of the 27 lines is a double tangent of the spinode curve, and the tangent planes of the surface at the points of contact are common tangent planes of the spinode torse and the node-couple torse, stationary planes of the spinode torse; or we have $\beta'=2\rho'=54$. In the other cases, however, instead of the 27 lines we must take only the facultative lines, each of which is or is not a double or a single tangent of the spinode curve; and the tangent planes of the surface at the points of contact are the common tangent planes as above—that is, the number of contacts gives β' not in general $=2\rho'$.

18. There are not, except as above, any common tangent planes of the two torses, that is, not only $\gamma'=0$ as already mentioned, but also $i'=0$. I do not at present account *à priori* for the values $\theta'=16$, 8, and 16, which present themselves in the Table. The cubic surface cannot have a plane of conic contact, and we have thus in every case $C'=0$; but the value of B' is not in every case $=0$.

19. In what precedes we see how a discussion of the equation of the cubic surface should in the several cases respectively lead to the values b', t', ρ', σ', β', j', χ', B', and how in the reciprocal surface the nodal curve of the order b' is known by means of the facultative lines of the original cubic surface. The cuspidal curve c' might also be obtained as the reciprocal of the spinode-torse; but this would in general be a laborious process, and it is the less necessary, inasmuch as the equation of the reciprocal surface is in each case obtained in a form putting in evidence the cuspidal curve.

Article Nos. 20 to 23. *The Lines and Planes of a Cubic Surface; Facultative Lines; Explanation of Diagrams.*

20. In the general surface $\mathrm{I}=12$, we have 27 lines and 45 triple-tangent planes, or say simply, planes: through each line pass 5 planes, in each plane lie 3 lines. For the surfaces II to XXI (the present considerations do not of course apply to the Scrolls) several of the lines come to coincide with each other, and several of the planes also come to coincide with each other; but the number of the lines is always reckoned as 27, and that of the planes as 45. If we attend to the distinct lines and the distinct planes, each line has a multiplicity, and the sum of these is $=27$; and so each plane has a multiplicity, and the sum of these is $=45$. Again, attending to a particular line in a particular plane, the line has a frequency 1, 2, or 3, that is, it represents 1, 2, or 3 of the 3 lines in the plane (this is in fact the distinction of a scrolar, torsal, or oscular line); and similarly, the plane has a frequency 1, 2, 3, 4, or 5, according to the number which it represents of the 5 planes through the line. It requires only a little consideration to perceive that the multiplicity of the plane into its frequency in regard to the line is equal to the multiplicity of the line into its frequency in regard to the plane. Observe, further, that if M be the multiplicity of the plane, then, considering it in regard to the lines contained therein, we get the products (M, M, M), $(2M, M)$, or $3M$, according as the three lines are or are not distinct, but that the sum of the products is always $=3M$, and that in regard to all the planes the total sum is 3×45, $=135$. And so if M' be the multiplicity of the line, then, considering it in regard to the planes which pass through it, we get the products (M', M', M', M', M'), $(2M', M', M', M')$, $\ldots(5M')$, as the case may be, but that the sum of the products is $=5M'$, and that in regard to all the lines the sum is 5×27, $=135$, as before.

21. The mode of coincidence of the lines and planes, and the several distinct lines and planes which are situate in or pass through the several distinct planes and lines respectively, are shown in the annexed diagrams I to XXI[1]: the multiplicity

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[1] See the commencements of the several sections.

of each line appears by the upper marginal line, and that of each plane by the left-hand marginal column (thus in diagram I, $27 \times 1 = 27$ and $45 \times 1 = 45$, 1 is the multiplicity of each line, and it is also the multiplicity of each plane); the frequencies of a line and plane in regard to each other appear by the dots in the square opposite to the line and plane in question, these being read, for the frequency of the line vertically, and for the frequency of the plane horizontally; thus $^{\cdot}\vdots$ indicates that the frequency of the line is $=3$, and the frequency of the plane is $=2$. There should be and are in every line of the diagram 3 dots, and in every column of the diagram 5 dots (a symbol $^{\cdot}\vdots$ being read as just explained, 2 dots in the line, 3 dots in the column).

22. For the surface $\mathrm{I} = 12$, there is of course no distinction between the lines, but these form only a single class, and the like for the planes; but for the other surfaces the lines and planes form separate classes, as shown in the diagrams by the lower marginal explanation of the lines, and the right-hand marginal explanation of the planes. I use here and elsewhere "ray" to denote a line passing through a single node; "axis" to denote a line joining two nodes; "edge" (as above) to denote the edge of a binode; any other line is a "mere line." An axis is always torsal or oscular; when it is torsal, the plane touching along the axis contains a third line which is the "transversal" of such axis; but a transversal may be a mere line, a ray, or an axis; in the case $\mathrm{XVI} = 12 - 4C_2$, each transversal is a transversal in regard to two axes.

23. In the general case $\mathrm{I} = 12$, each of the 27 lines is, as already mentioned, part of the node-couple curve; and the node-couple curve is made up of the 27 lines, and is thus a curve of the order 27. In fact each plane through a line meets the cubic surface in this line, and in a conic; the line and conic meet in two points, and the plane (that is in any plane) through the line is thus a double tangent plane touching the surface at the two points in question; the locus of the points of contact, that is the line itself, is thus part of the node-couple curve. But in the other cases, II to XXI, certain of the lines do not belong to the node-couple curve (this will be examined in detail in the several cases respectively); but I wish to show here how in a general way a line passing through a node, say a nodal ray, is not part of the node-couple curve. To fix the ideas, consider the surface $\mathrm{II} = 12 - C_2$; there are here through C_2 six lines, or say rays: attending to any one of these, a plane through the ray meets the surface in the ray itself and in a conic; the ray and the conic meet as before in two points, one of them being the point C_2: the plane touches the surface at the other point, *but it does not touch the surface at* C_2. (I am not sure, and I leave it an open question, whether we ought to say that at a node C_2 there is *no* tangent plane, or to say that only the tangent planes of the nodal cone are tangent planes of the surface; but, at any rate, an arbitrary plane through C_2 is *not* a tangent plane.) The plane through the ray is only a single tangent plane, not a double tangent plane; and the ray is not part of the node-couple curve. We say that a line of the surface is or is not "facultative" according as it does or does not form part of the node-couple curve.

Article Nos. 24 to 26. *Axis; the different kinds thereof.*

24. A line joining two nodes is an axis; such a line is always a line, and it is a torsal or oscular line, of the surface. But some further distinctions are requisite; using the expressions in their strict sense, cnicnode $= C$, binode $= B$, an axis is a CC-axis joining two cnicnodes, or it is a CB-axis joining a cnicnode and a binode, or it is a BB-axis joining two binodes. A CC-axis is torsal, the transversal being a mere line, not a ray through either of the cnicnodes; a CB-axis is torsal, the transversal being a ray of the binode; a BB-axis is oscular. The distinction is of course carried through as regards the higher biplanar nodes B_4, B_5, B_6, and the uniplanar nodes U_6, U_7, U_8: thus $(B_3 = B)$ the edge of a binode B_3 is not an axis at all, but $(B_4 = 2C)$ the edge of a binode B_4 is a CC-axis; $(B_5 = B + C)$ the edge of a binode B_5 is a CB-axis; $(B_6 = 3C)$ the edge of a binode B_6 is a thrice-taken CC-axis; $(U_6 = 3C)$ each of the rays is regarded as a CC-axis; $(U_7 = B + 2C)$ the double ray is regarded as a twice-taken CB-axis, and the single ray as a CC-axis; $(U_8 = 2B + C)$ the ray is regarded as a BB-axis + a twice-taken CB-axis.

25. It has been mentioned that the intersection of the surface with the Hessian consists of the spinode curve, together with certain right lines; these lines are in fact the axes—viz. the examination of the several cases shows that in the complete intersection each CC-axis presents itself twice, each CB-axis 3 times, and each BB-axis 4 times. We thus see that a CC-axis, or rather the torsal plane along such axis, is the pinch-plane or singularity $j' = 1$; the CB-axis, or rather the torsal plane along such axis, the close-plane or singularity $\chi' = 1$; and the BB-axis, or oscular plane along such axis, the bitrope or singularity $B' = 1$; for a cubic surface with singular lines the expression of σ' being in fact $\sigma' = 12 - 2j' - 3\chi' - 4B'$. There are, however, some cases requiring explanation; thus for the case $\text{VIII} = 12 - B_5$, where the edge is by what precedes a CB-axis, the complete intersection is made up of the edge 4 times and of an octic curve; the consideration of the reciprocal surface shows, however, that the edge taken once is really part of the spinode curve (viz. that this curve is made up of the edge taken once and of the octic curve, its order being thus $\sigma' = 9$); and the interpretation then of course is that the intersection is made up of the edge taken 3 times (as for a CB-axis it should be) and of the spinode curve.

26. I remark in further explanation, that in the several sections, in showing how the complete intersection of the cubic surface with the Hessian is made up, I have not referred to the axes in the above precise significations; thus $\text{XIV} = 12 - B_5 - C_2$, the binode B_5 is $C + B$, and the edge is thus a CB-axis, while the axis B_5C_2 is a CB-axis + a CC-axis $(\chi' = 1 + 1, = 2, j' = 1)$. The complete intersection should therefore consist of the spinode curve, + edge (as a CB-axis) 3 times + axis (as a CB-axis + a CC-axis) $2 + 3, = 5$ times: it is in the section stated (in perfect consistency herewith, but without the full explanation) that the intersection is made up of the axis 5 times, the edge 4 times, and a cubic curve—which cubic curve together with the edge once constitutes the spinode curve; and so in other cases: this explanation will, I think, remove all difficulty.

Article Nos. 27 to 32. *On the Determination of the Reciprocal Equation.*

27. Consider in general the cubic surface $(*\between X, Y, Z, W)^3=0$, and in connexion therewith the equation $Xx+Yy+Zz+Ww=0$, which regarding therein X, Y, Z, W as current coordinates, and x, y, z, w as constants, is the equation of a plane. If from the two equations we eliminate one of the coordinates, for instance W, we obtain

$$(*\between Xw,\ Yw,\ Zw,\ -(Xx+Yy+Zz))^3=0,$$

which, (X, Y, Z) being current coordinates, is obviously the equation of the cone, vertex $(X=0, Y=0, Z=0)$, which stands on the section of the cubic surface by the plane. Equating to zero the discriminant of this function in regard to (X, Y, Z), we express that the cone has a nodal line; that is, that the section has a node, or, what is the same thing, that the plane $xX+yY+zZ+wW=0$ is a tangent plane of the cubic surface; and we thus by the process in fact obtain the equation of the cubic surface in the reciprocal or plane coordinates (x, y, z, w). Consider in the same equation x, y, z, w as current coordinates, (X, Y, Z) as given parameters, the equation represents a system of three planes, viz. these are the planes $xX+yY+zZ+wW'=0$, where W' has the three values given by the equation $(*\between X, Y, Z, W')^3=0$, or, what is the same thing, X, Y, Z, W' are the coordinates of any one of the three points of intersection of the cubic surface by the line $\frac{x}{X}=\frac{y}{Y}=\frac{z}{Z}$; (X, Y, Z, W') belongs to a point on the surface, and

$$xX+yY+zZ+wW'=0$$

is the polar plane of this point in regard to a quadric surface $X^2+Y^2+Z^2+W^2=0$; the equation

$$(*\between Xw,\ Yw,\ Zw,\ -(Xx+Yy+Zz))^3=0$$

is thus the equation of a system of 3 planes, the polar planes of three points of the cubic surface (which three points lie on an arbitrary line through the point $x=0$, $y=0$, $z=0$). In equating to zero the discriminant in regard to (X, Y, Z), we find the envelope of the system of three planes, or say of a plane, the polar plane of an arbitrary point on the cubic surface,—or we have the equation of the reciprocal surface, being, as is known, the same thing as the equation of the cubic surface in the reciprocal or plane coordinates (x, y, z, w). In what precedes we have the explanation of an ordinary process of finding the equation of the reciprocal surface, this equation being thereby given by equating to zero the discriminant of a function $(*\between X, Y, Z)^3$, that is, of a ternary cubic function.

28. The process, as last explained, is a special one, viz. the position of a point on the surface is determined by means of certain two parameters, the ratios $X : Y : Z$ which fix the position of the line joining this point with the point $(x=0, y=0, z=0)$. More generally we may consider the position of the point as determined by means of any two parameters; the equation of the polar plane then contains the two parameters, and by taking the envelope in regard to the two parameters considered as variable, we have the equation of the reciprocal surface.

29. But let the parameters, say θ, ϕ, be regarded as varying successively; if ϕ alone vary, we have on the surface a curve Θ, the equation whereof contains the parameter θ, and when θ varies this curve sweeps over the surface. The envelope in regard to ϕ of the polar plane of a point of the surface is a torse, the reciprocal of the curve Θ, and the envelope of the torse is the reciprocal surface. In particular the curve Θ may be the plane section by any plane through a fixed line, say, by the plane $P - \theta Q = 0$; the section is a cubic curve, the reciprocal is a sextic cone having its vertex in a fixed line (the reciprocal of the line $P = 0$, $Q = 0$), and the reciprocal surface is thus obtained as the envelope of this cone; assuming that the equation of the sextic cone has been obtained, this is an equation of a certain order in the parameter θ; or writing $\theta = P : Q$, we obtain the equation of the reciprocal surface by equating to zero the discriminant of a *binary* function of (P, Q).

30. With a variation, this process is a convenient one for obtaining the reciprocal of a cubic surface: we take the fixed line to be one of the lines on the cubic surface; the curve Θ is then a conic, its reciprocal is a quadricone, and the envelope of this quadricone is the required reciprocal surface. This is really what Schläfli does (but the process is not explained) in the several instances in which he obtains the equation of the reciprocal surface by means of a binary function. I remark that it would be very instructive, for each case of surface, to take the variable plane successively through the several kinds of lines on the particular surface; the equation of the reciprocal surface would thus be obtained under different forms, putting in evidence the relation to the reciprocal surface of the fixed line made use of. But this is an investigation which I do not enter upon: I adopt in each case Schläfli's process, without explanation, and merely write down the ternary or (as the case may be) binary function by means of which the equation of the reciprocal surface is obtained.

31. It is to be mentioned that there is a reciprocal process of obtaining the equation of the reciprocal surface; we may imagine, touching the cubic surface along any curve, a series of planes; that is, a torse circumscribed about the surface, and the equation whereof contains a variable parameter θ; the reciprocal figure is a curve, the equations whereof contain the parameter θ; the locus of this curve is the reciprocal surface; that is, the equation of the reciprocal surface is obtained by eliminating θ from the equations of the curve. In particular let the torse be the circumscribed cone having its vertex at any point of a fixed line; the reciprocal figure is then a plane curve, the plane of which passes through the line which is the reciprocal of the fixed line; it is moreover clear that if the position of the vertex on the fixed line be determined by the parameter θ *linearly* (for instance if the vertex be given as the intersection of the fixed line by a plane $P - \theta Q = 0$), then the equation of the plane of the curve will be of the form $P' = \theta Q'$, containing the parameter θ linearly; the other equation of the plane curve will contain θ rationally, and the elimination will be at once effected by substituting in this other equation for θ its value, $= P' \div Q'$. And observe moreover that if the fixed line be a line on the cubic surface, then the cone is a quadricone having for its reciprocal a conic; the reciprocal surface is thus given as the locus of a variable conic, the plane of which always passes through a fixed line; there are thus on the reciprocal surface

series of such conics. It would be very instructive and interesting to carry out the investigation in detail.

32. The equation of the reciprocal surface is found by equating to zero the discriminant of a ternary or a binary function[1], viz. this is a ternary cubic, or a binary quartic, cubic, or quadric. The equation as given in the form disct. = 0, contains a factor which for the adopted forms of equations is always a power or product of powers of w, z, x[2] known *à priori*, and which is thrown out without difficulty, the equation being thereby reduced to the proper order. There is the singular advantage that the process puts in evidence the cuspidal curve of the resulting reciprocal surface, viz. for a ternary cubic, the form obtained is $S^3 - T^2 = 0$, and for a binary quartic it is the equivalent form $I^3 - 27J^2 = 0$; but for the factor thrown out as just mentioned, we should have simply $(S = 0,\ T = 0)$, or, as the case may be, $(I = 0,\ J = 0)$ for equations of the cuspidal curve; the existence of the factor occasions however a modification, viz. the intersection of the two surfaces is not an indecomposable curve, and the cuspidal curve is in most cases, not the complete intersection, but a partial intersection of the two surfaces. In several cases it thus happens that the cuspidal curve is obtained as a curve $\left\| \begin{matrix} P, & Q, & R \\ P', & Q', & R' \end{matrix} \right\| = 0$, without or with further speciality. Similarly when the equation of the reciprocal surface is obtained by means of a binary cubic; if the coefficients hereof (functions of course of the coordinates x, y, z, w) be A, B, C, D, then the surface is

$$(AD - BC)^2 - 4(AC - B^2)(BD - C^2) = 0,$$

having the cuspidal curve $\left\| \begin{matrix} A, & B, & C \\ B, & C, & D \end{matrix} \right\| = 0$, subject however to modification in the case of a thrown out factor.

Article Nos. 33 and 34. *Explanation as to the Sections of the Memoir.*

33. As regards the following Sections I to XXIII, it is to be observed that for the general surface I = 12, I do not attempt to form the equation of the reciprocal surface, and in some of the other cases, II = 12 − C_2 &c., the equation of the reciprocal surface is either not obtained in a completely developed form, or it is too complicated to allow of its being dealt with, for instance so as to put in evidence the nodal curve of the surface. Portions of the theory given in the latter sections are consequently omitted in the earlier ones, and in particular in the Section I there is given only the diagram of the 27 lines and the 45 planes (with however developments as to notation and otherwise which have no place in the subsequent sections), and with the analytical expressions for the several lines and planes, although from the

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[1] In some easy cases, for instance XVI = 12 − 4C_2, the equation of the reciprocal surface is obtained otherwise by a direct elimination.

[2] The factor is in general a power or product of powers of the linear functions which, equated to zero, give the equations of the planes reciprocal to the several nodes of the surface.

want of the equation of the reciprocal surface these analytical expressions have no present application. And so in some of the next following sections, no application is made of the analytical expressions of the lines and planes.

34. I call to mind that if a line be given as the intersection of the two planes

$$AX + BY + CZ + DW = 0, \quad A'X + B'Y + C'Z + D'W = 0,$$

then the six coordinates of the line are

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h \\ = AD' - A'D, & BD' - B'D, & CD' - C'D, & BC' - B'C, & CA' - C'A, & AB' - A'B, \end{array}$$

and that in terms of its six coordinates the line is given as the common intersection of the four planes

$$\left(\begin{array}{cccc} . & h, & -g, & a \\ -h, & . & f, & b \\ g, & -f, & . & c \\ -a, & -b, & -c, & . \end{array}\right)(X, Y, Z, W) = 0,$$

and that (reciprocating as usual in regard to $X^2 + Y^2 + Z^2 + W^2 = 0$) the coordinates of the reciprocal line are (f, g, h, a, b, c); that is, this is the common intersection of the four planes

$$\left(\begin{array}{cccc} . & c, & -b, & f \\ -c, & . & a, & g \\ b, & -a, & . & h \\ -f, & -g, & -h, & . \end{array}\right)(x, y, z, w) = 0.$$

It is in some cases more convenient to consider a line as determined as the intersection of two planes rather than by means of its six coordinates; thus, for instance, to speak of the line $X = 0$, $Y = 0$ rather than of the line (0, 0, 0, 1, 0, 0); and in some of the sections I have preferred not to give the expressions of the six coordinates of the several lines.

Article Nos. 35 to 46. § I = 12, Equation $(X, Y, Z, W)^3 = 0$.

35. There is in the system of the 27 lines and the 45 planes a complicated and many-sided symmetry which precludes the existence of any unique notation: the notation can only be obtained by starting from some arrangement which is not unique, but one of a system of several like arrangements. The notation employed in my original paper "On the Simple Tangent Planes of Surfaces of the Third Order," *Camb. and Dub. Math. Journ.* vol. IV. 1849, pp. 118—132, [76], and which is shown in the right hand and lower margins of the diagram, starts from such an arrangement; but

it is so complicated that it can hardly be considered as at all putting in evidence the relations of the lines and planes; that of Dr Hart (Salmon, "On the Triple Tangent Planes of a Surface of the Third Order," same volume, pp. 252—260), depending on an arrangement of the 27 lines according to a cube of 3 each way, is a singularly elegant one, and will be presently reproduced.

36. But the most convenient one is Schläfli's, starting from a double-sixer; viz. we can (and that in 36 different ways) select out of the 27 lines two systems each of six lines, such that no two lines of the same system intersect, but that each line of the one system intersects all but the corresponding line of the other system; or, say, if the lines are

1, 2, 3, 4, 5, 6
1′, 2′, 3′, 4′, 5′, 6′,

then these have the thirty intersections

	1′	2′	3′	4′	5′	6′
1		.	.	.	.	.
2	.		.	.	.	.
3	.	.		.	.	.
4	.	.	.		.	.
5	.	.	.	.		.
6	.	.	.	.	.	

Any two lines such as 1, 2′ lie in a plane which may be called 12′; similarly the lines 1′, 2 lie in a plane which may be called 1′2; these two planes meet in a line 12; and any three lines such as 12, 34, 56 meet in pairs, lying in a plane 12.34.56. We have thus the entire system of the 27 lines and 45 planes, as in effect completely explained by what has been stated, but which is exhibited in full in the diagram.

37. The diagram of the lines and planes is

Lines.

I=12

27×1=27

$\overline{27}$ $\overline{27}$

Planes.

45×1=45

	56	46	45	36	35	34	26	25	24	23	16	15	14	13	12	6′	5′	4′	3′	2′	1′	6	5	4	3	2	1	
12′															•					•							•	w
13′														•					•								•	ζ
14′													•					•									•	z
15′												•					•										•	z
16′											•					•											•	$\bar{\text{z}}$
21′															•						•					•		ξ
23′										•									•							•		θ
24′									•									•								•		$\bar{\text{f}}$
25′								•									•									•		$\bar{\text{p}}_1$
26′							•									•										•		p
31′														•							•				•			$\bar{\theta}$
32′										•										•					•			η
34′						•												•							•			g
35′					•												•								•			q
36′				•												•									•			q_1
41′													•								•			•				f
42′									•											•				•				y
43′						•													•					•				$\bar{\text{h}}$
45′			•														•							•				m
46′		•														•								•				m_1
51′												•									•		•					p
52′								•												•			•					y
53′					•														•				•					$\bar{\text{r}}$
54′			•															•					•					n_1
56′	•															•							•					l
61′											•										•	•						p_1
62′							•													•		•						$\bar{\text{y}}$
63′				•															•			•						$\bar{\text{r}}$
64′		•																•				•						n
65′	•																•					•						l_1
12.34.56	•					•									•													x
12.35.46		•			•										•													x
12.36.45			•	•											•													x
13.24.56	•								•					•														h
13.25.46		•						•						•														r
13.26.45			•				•							•														$\bar{\text{r}}_1$
14.23.56	•									•			•															$\bar{\text{g}}$
14.25.36				•				•					•															$\bar{\text{n}}$
14.26.35					•		•						•															$\bar{\text{n}}_1$
15.23.46		•								•		•																$\bar{\text{q}}_1$
15.24.36				•					•			•																$\bar{\text{m}}_1$
15.26.34						•	•					•																$\bar{\text{l}}_1$
16.23.45			•							•	•																	$\bar{\text{q}}$
16.24.35					•				•		•																	$\bar{\text{m}}$
16.25.34						•		•			•																	$\bar{\text{l}}_1$
	a_5	a_7	a_9	a_8	a_6	a_4	b_8	b_6	b_4	b_2	c_8	c_6	c_4	c_3	a_1	c_9	c_7	c_5	c_2	b_1	a_3	b_9	b_7	b_5	b_3	a_2	c_1	

$\overline{45}$ $\overline{45}$

38. It has been mentioned that the number of double-sixers was $=36$, these are as follows:

$$\begin{array}{cccccclr} 1, & 2, & 3, & 4, & 5, & 6 & \text{Assumed primitive} & 1 \\ 1', & 2', & 3', & 4', & 5', & 6' & & \\ \hline 1, & 1', & 23, & 24, & 25, & 26 & \text{Like arrangements} & 15 \\ 2, & 2', & 13, & 14, & 15, & 16 & & \\ \hline 1, & 2, & 3, & 56, & 46, & 45 & \text{Like arrangements} & \underline{20} \\ 23, & 13, & 12, & 4, & 5, & 6 & & 36 \\ \hline \end{array}$$

where, if we take any column $\frac{1}{1'}$ of two lines, we have the complete number 216 of pairs of non-intersecting lines (each line meets 10 lines, there are therefore $27-1-10$, $=16$, which it does not meet, and the number of non-intersecting pairs is thus $\frac{1}{2}.27.16=216$).

39. We can out of the 45 planes select, and that in 120 ways, a trihedral-pair, that is, two triads of planes, such that the planes of the one triad, intersecting those of the other triad, give 9 of the 27 lines. Analytically if $X=0$, $Y=0$, $Z=0$ and $U=0$, $V=0$, $W=0$ are the equations of the six planes, then the equation of the cubic surface is $XYZ+kUVW=0$. See as to this *post*, No. 44.

The trihedral plane pairs are:

$$\begin{array}{llllr} 12', & 23', & 31' & & \\ 1'2, & 2'3, & 3'1 & \text{No. is} = & 20 \\ \hline 12', & 34', & 14.23.56 & & \\ 2'3, & 4'1, & 12.34.56 & = & 90 \\ \hline 14.25.36, & 35.16.24, & 26.34.15 & & \\ 14.35.26, & 25.16.34, & 36.24.15 & = & \underline{10} \\ \hline & & & & 120 \end{array}$$

The construction of the last set is most easily effected by the diagram

$$\underbrace{\begin{array}{ccc} 1\,2\,3 & \times & 4\,5\,6 \\ 3\,1\,2 & & 5\,6\,4 \\ 2\,3\,1 & & 6\,4\,5 \end{array}}_{\Vert}$$

$$\begin{array}{ccc} 14 & 25 & 36 \\ 35 & 16 & 24 \\ 26 & 34 & 15 \end{array}$$

It is immaterial how the two component triads 123 and 456 are arranged, we obtain always the same trihedral pair.

40. Dr Hart arranges the 27 lines, cubically, thus:

$$\begin{array}{ccc|ccc|ccc} A_1 & B_1 & C_1 & a_1 & b_1 & c_1 & \alpha_1 & \beta_1 & \gamma_1 \\ A_2 & B_2 & C_2 & a_2 & b_2 & c_2 & \alpha_2 & \beta_2 & \gamma_2 \\ A_3 & B_3 & C_3 & a_3 & b_3 & c_3 & \alpha_3 & \beta_3 & \gamma_3 \end{array}$$

where letters of the same alphabet denote lines in the same plane, if only the letters are the same or the suffixes the same; thus A_1, A_2, A_3 lie in a plane $A_1A_2A_3$; A_1, B_1, C_1 lie in a plane $A_1B_1C_1$. Letters of different alphabets denote lines which meet according to the Table

$a_1\ \ b_2\ \ c_3$ A_1 $\alpha_1\ \ \beta_2\ \ \gamma_3$	$b_1\ \ c_2\ \ a_3$ B_1 $\beta_1\ \ \gamma_2\ \ \alpha_3$	$c_1\ \ a_2\ \ b_3$ C_1 $\gamma_1\ \ \alpha_2\ \ \beta_3$
$c_2\ \ a_3\ \ b_1$ A_2 $\beta_3\ \ \gamma_1\ \ \alpha_2$	$a_2\ \ b_3\ \ c_1$ B_2 $\gamma_3\ \ \alpha_1\ \ \beta_2$	$b_2\ \ c_3\ \ a_1$ C_2 $\alpha_3\ \ \beta_1\ \ \gamma_2$
$b_3\ \ c_1\ \ a_2$ A_3 $\gamma_2\ \ \alpha_3\ \ \beta_1$	$c_3\ \ a_1\ \ b_2$ B_3 $\alpha_2\ \ \beta_3\ \ \gamma_1$	$a_3\ \ b_1\ \ c_2$ C_3 $\beta_2\ \ \gamma_3\ \ \alpha_1$

where the letter in the centre of the square denotes a line lying in the same plane with the lines denoted by the letters of each vertical pair in the same square. Thus A_1 lies in the planes $A_1a_1\alpha_1$, $A_1b_2\beta_2$, $A_1c_3\gamma_3$ (and in the before-mentioned two planes $A_1A_2A_3$, $A_1B_1C_1$).

41. I find that one way in which this may be identified with the double-sixer notation is to represent the above arrangement by

$$\begin{array}{ccc|ccc|ccc} 1, & 2', & 12 & 3', & 4, & 34 & 13, & 24, & 56 \\ 14, & 25, & 36 & 2, & 6', & 26 & 1', & 16, & 6 \\ 4', & 5, & 45 & 23, & 46, & 15 & 3, & 35, & 5' \end{array}$$

and then the identification may apparently be effected in (720 × 36 =) 25920 ways, viz. we may first in any way permute the $\frac{1}{1'}$, $\frac{2}{2'}$, $\frac{3}{3'}$, $\frac{4}{4'}$, $\frac{5}{5'}$, $\frac{6}{6'}$, by this means not altering the double-sixer $\frac{1\ 2\ 3\ 4\ 5\ 6}{1'\ 2'\ 3'\ 4'\ 5'\ 6'}$, and then upon the arrangements so obtained make any of the substitutions which permute *inter se* the 36 double-sixers.

42. The equations of the 45 planes are obtained in my paper last referred to, viz. taking the equation of the surface to be

$$W\left(1,\ 1,\ 1,\ 1,\ mn+\frac{1}{mn},\ nl+\frac{1}{nl},\ lm+\frac{1}{lm},\ l+\frac{1}{l},\ m+\frac{1}{m},\ n+\frac{1}{n}\between X,\ Y,\ Z,\ W\right)^2+kXZY=0,$$

where $$k=\frac{p^2-\beta^2}{2(p-\alpha)},\ \alpha=lmn+\frac{1}{lmn},\ \beta=lmn-\frac{1}{lmn},$$

then the equations of the planes are:

$$W=0, \qquad [12'=w]$$

$$lX+mY+nZ+W\left[1+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0, \qquad [23'=\theta]$$

$$\frac{X}{l}+\frac{Y}{m}+\frac{Z}{n}+W\left[1-\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)\right]=0, \qquad [31'=\bar{\theta}]$$

$$X=0, \qquad [12.34.56=x]$$

$$Y=0, \qquad [42'=y]$$

$$Z=0, \qquad [14'=z]$$

$$X+\frac{1}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)W=0, \qquad [21'=\xi]$$

$$Y+\frac{1}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)W=0, \qquad [32'=\eta]$$

$$Z+\frac{1}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)W=0, \qquad [13'=\zeta]$$

$$lX+\frac{Y}{m}+\frac{Z}{n}+W=0, \qquad [41'=\mathrm{f}]$$

$$\frac{X}{l}+mY+\frac{Z}{n}+W=0, \qquad [34'=\mathrm{g}]$$

$$\frac{X}{l}+\frac{Y}{m}+Z+W=0, \qquad [13.24.56=\mathrm{h}]$$

$$\frac{X}{l}+mY+nZ+W=0, \qquad [24'=\bar{\mathrm{f}}]$$

$$lX+\frac{Y}{m}+nZ+W=0, \qquad [14.25.36=\bar{\mathrm{g}}]$$

$$lX+mY+\frac{Z}{n}+W=0, \qquad [43'=\bar{\mathrm{h}}]$$

$$X+\frac{l(p-\alpha)+2mn}{p+\beta}W=0, \qquad [12.35.46=\mathrm{x}]$$

$$Y+\frac{m(p-\alpha)+2nl}{p+\beta}W=0, \qquad [52'=\mathrm{y}]$$

$$Z+\frac{n(p-\alpha)+2lm}{p+\beta}W=0, \qquad [15'=\mathrm{z}]$$

$$X+\frac{\frac{1}{l}(p-\alpha)+\frac{2}{mn}}{p+\beta}W=0, \qquad [12.36.45=\bar{\mathrm{x}}]$$

$$Y+\frac{\frac{1}{m}(p-\alpha)+\frac{2}{nl}}{p+\beta}W=0, \qquad [62'=\bar{\mathrm{y}}]$$

$$Z+\frac{\frac{1}{n}(p-\alpha)+\frac{2}{lm}}{p+\beta}W=0, \qquad [16'=\bar{\mathrm{z}}]$$

$$-\frac{2n}{m(p-\alpha)}X+\frac{1}{m}Y+nZ+W=0, \qquad [56'=\mathrm{l}]$$

$$lX-\frac{2l}{n(p-\alpha)}Y+\frac{1}{n}Z+W=0, \qquad [45'=\mathrm{m}]$$

$$\frac{1}{l}X+mY-\frac{2m}{l(p-\alpha)}Z+W=0, \qquad [64'=\mathrm{n}]$$

$$-\frac{2m}{n(p-\alpha)}X+mY+\frac{1}{n}Z+W=0, \qquad [15.26.34=\overline{\mathrm{l}}]$$

$$\frac{1}{l}X-\frac{2n}{l(p-\alpha)}Y+nZ+W=0, \qquad [16.24.35=\overline{\mathrm{m}}]$$

$$lX+\frac{1}{m}Y-\frac{2l}{m(p-\alpha)}Z+W=0, \qquad [14.25.36=\overline{\mathrm{n}}]$$

$$-\frac{n(p-\alpha)}{2m}X+\frac{Y}{m}+nZ+W=0, \qquad [65'=\mathrm{l}_1]$$

$$lX-\frac{l(p-\alpha)}{2n}Y+\frac{1}{n}Z+W=0, \qquad [46'=\mathrm{m}_1]$$

$$\frac{1}{l}X+mY-\frac{m(p-\alpha)}{2l}Z+W=0, \qquad [54'=\mathrm{n}_1]$$

$$\frac{m(p-\alpha)}{2n}X+mY+\frac{1}{n}Z+W=0, \qquad [16.25.34=\overline{\mathrm{l}}_1]$$

$$\frac{1}{l}X-\frac{n(p-\alpha)}{2l}Y+nZ+W=0, \qquad [15.24.36=\overline{\mathrm{m}}_1]$$

$$lX+\frac{1}{m}Y-\frac{l(p-\alpha)}{2m}Z+W=0, \qquad [14.26.35=\overline{\mathrm{n}}_1]$$

$$-\frac{2X}{p-\alpha}+nY+mZ+\left(mn(p-\alpha)-2l\,(1-m^2-n^2)\right)\frac{W}{p+\beta}=0, \qquad [51'=\mathrm{p}]$$

$$nX-\frac{2Y}{p-\alpha}+lZ+\left(nl\,(p-\alpha)-2m(1-n^2-l^2)\right)\frac{W}{p+\beta}=0, \qquad [35'=\mathrm{q}]$$

$$mX+lY-\frac{2Z}{p-\alpha}+\left(lm\,(p-\alpha)-2n\,(1-l^2-m^2)\right)\frac{W}{p+\beta}=0, \qquad [13.25.46=\mathrm{r}]$$

$$-\frac{2X}{p-\alpha}+\frac{1}{n}Y+\frac{1}{m}Z+\left(\frac{1}{mn}(p-\alpha)-\frac{2}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [26'=\overline{\mathrm{p}}]$$

$$\frac{1}{n}X+\frac{2Y}{p-\alpha}+\frac{1}{l}Z+\left(\frac{1}{nl}(p-\alpha)-\frac{2}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [16.23.45=\overline{\mathrm{q}}]$$

$$\frac{1}{m}X+\frac{1}{l}Y-\frac{2Z}{p-\alpha}+\left(\frac{1}{lm}(p-\alpha)-\frac{2}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)\right)\frac{W}{p-\beta}=0, \qquad [36'=\overline{\mathrm{r}}]$$

$$-\frac{p-\alpha}{2}X+\frac{Y}{n}+\quad\frac{Z}{m}-lmn\left(\frac{1}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)(p-\alpha)-\frac{2}{mn}\right)\frac{W}{p+\beta}=0,\quad [25'=\mathrm{p}_1]$$

$$\frac{X}{n}-\frac{p-\alpha}{2}Y+\quad\frac{Z}{l}-lmn\left(\frac{1}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)(p-\alpha)-\frac{2}{nl}\right)\frac{W}{p+\beta}=0,\quad [15\,.\,23\,.\,46=\mathrm{q}_1]$$

$$\frac{X}{m}+\frac{Y}{l}-\frac{p-\alpha}{2}\quad Z-lmn\left(\frac{1}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)(p-\alpha)-\frac{2}{lm}\right)\frac{W}{p+\beta}=0,\quad [53'=\mathrm{r}_1]$$

$$-\frac{p-\alpha}{2}X+nY+mZ-\frac{1}{lmn}\left(l\,(1-m^2-n^2)(p-\alpha)-2mn\right)\frac{W}{p-\beta}=0,\quad [61'=\bar{\mathrm{p}}_1]$$

$$nX-\frac{p-\alpha}{2}Y+lZ-\frac{1}{lmn}\left(m\,(1-n^2-l^2)(p-\alpha)-2nl\right)\frac{W}{p-\beta}=0,\quad [36'=\bar{\mathrm{q}}_1]$$

$$mX-lY-\frac{p-\alpha}{2}Z-\frac{1}{lmn}\left(n\,(1-l^2-m^2)(p-\alpha)-2lm\right)\frac{W}{p-\beta}=0,\quad [13\,.\,26\,.\,45=\bar{\mathrm{r}}_1]$$

43. The coordinates of the 27 lines are then found to be as follows:

(*a*)	(*b*)	(*c*)
1	0	0
0	1	0
0	0	1
$1-\frac{1}{kl}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$	$-\frac{m}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$	$-\frac{n}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$
$-\frac{l}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$	$1-\frac{1}{km}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$	$-\frac{n}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$
$-\frac{l}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{m}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$1-\frac{1}{kn}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$
$1-\frac{l}{k}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$	$-\frac{1}{mk}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$	$-\frac{1}{nk}\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)$
$-\frac{1}{lk}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$	$1-\frac{m}{k}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$	$-\frac{1}{nk}\left(n-\frac{1}{n}\right)\left(l-\frac{1}{l}\right)$
$-\frac{1}{lk}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$-\frac{1}{mk}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$	$1-\frac{n}{k}\left(l-\frac{1}{l}\right)\left(m-\frac{1}{m}\right)$
1	0	0
0	1	0
0	0	1

(f)	(g)	(h)	
0	0	0	$(12 = a_1)$
0	0	0	$(\ 2' = b_1)$
0	0	0	$(\ 1 = c_1)$
0	$-n$	m	$(\ 2 = a_2)$
n	0	$-l$	$(23 = b_2)$
$-m$	l	0	$(\ 3' = c_2)$
0	$-\frac{1}{n}$	$\frac{1}{m}$	$(\ 1' = a_3)$
$\frac{1}{n}$	0	$-\frac{1}{l}$	$(\ 3 = b_3)$
$-\frac{1}{m}$	$\frac{1}{l}$	0	$(13 = c_3)$
0	$-\frac{1}{n}$	m	$(34 = a_4)$
n	0	$-\frac{1}{l}$	$(24 = b_4)$
$-\frac{1}{m}$	l	0	$(14 = c_4)$

(a)	(b)	(c)
1	0	0
0	1	0
0	0	1
$2m\left(l-\frac{1}{l}\right)$	$2\left(1+\frac{2mn}{l(p-\alpha)}\right)$	$-l(p-\alpha)\left(1+\frac{2mn}{l(p-\alpha)}\right)$
$-m(p-\alpha)\left(1+\frac{2nl}{m(p-\alpha)}\right)$	$2n\left(m-\frac{1}{m}\right)$	$2\left(1+\frac{2nl}{m(p-\alpha)}\right)$
$2\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$-n(p-\alpha)\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$2l\left(n-\frac{1}{n}\right)$
$2n\left(l-\frac{1}{l}\right)$	$-l(p-\alpha)\left(1+\frac{2mn}{l(p-\alpha)}\right)$	$2\left(1+\frac{2mn}{l(p-\alpha)}\right)$
$2\left(1+\frac{2nl}{m(p-\alpha)}\right)$	$2l\left(m-\frac{1}{m}\right)$	$-m(p-\alpha)\left(1+\frac{2nl}{m(p-\alpha)}\right)$
$-n(p-\alpha)\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$2\left(1+\frac{2lm}{n(p-\alpha)}\right)$	$2m\left(n-\frac{1}{n}\right)$
$\frac{2}{n}\left(l-\frac{1}{l}\right)$	$\frac{p-\alpha}{l}\left(1+\frac{2l}{mn(p-\alpha)}\right)$	$-2\left(1+\frac{2l}{mn(p-\alpha)}\right)$
$-2\left(1+\frac{2m}{nl(p-\alpha)}\right)$	$\frac{2}{l}\left(m-\frac{1}{m}\right)$	$\frac{p-\alpha}{m}\left(1+\frac{2m}{nl(p-\alpha)}\right)$
$\frac{p-\alpha}{n}\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$-2\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$\frac{2}{m}\left(n-\frac{1}{n}\right)$
$\frac{2}{m}\left(l-\frac{1}{l}\right)$	$-2\left(1+\frac{2l}{mn(p-\alpha)}\right)$	$\frac{p-\alpha}{l}\left(1+\frac{2l}{mn(p-\alpha)}\right)$
$\frac{p-\alpha}{m}\left(1+\frac{2m}{nl(p-\alpha)}\right)$	$\frac{2}{n}\left(m-\frac{1}{m}\right)$	$-2\left(1+\frac{2m}{nl(p-\alpha)}\right)$
$-2\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$\frac{p-\alpha}{n}\left(1+\frac{2n}{lm(p-\alpha)}\right)$	$\frac{2}{l}\left(n-\frac{1}{n}\right)$

(f)	(g)	(h)	
0	$-n$	$\frac{1}{m}$	$(56 = a_5)$
$\frac{1}{n}$	0	$-l$	$(4 = b_5)$
$-m$	$\frac{1}{l}$	0	$(4' = c_5)$
0	$-(p+\beta)$	$-\frac{2(p+\beta)}{l(p-\alpha)}$	$(35 = a_6)$
$-\frac{2(p+\beta)}{m(p-\alpha)}$	0	$-(p+\beta)$	$(25 = b_6)$
$-(p+\beta)$	$-\frac{2(p+\beta)}{n(p-\alpha)}$	0	$(15 = c_6)$
0	$\frac{2(p+\beta)}{l(p-\alpha)}$	$p+\beta$	$(46 = a_7)$
$p+\beta$	0	$\frac{2(p+\beta)}{m(p-\alpha)}$	$(5 = b_7)$
$\frac{2(p+\beta)}{n(p-\alpha)}$	$p+\beta$	0	$(5' = c_7)$
0	$-\frac{2l(p-\beta)}{p-\alpha}$	$-(p-\beta)$	$(36 = a_8)$
$-(p-\beta)$	0	$-\frac{2m(p-\beta)}{p-\alpha}$	$(26 = b_8)$
$-\frac{2n(p-\beta)}{p-\alpha}$	$-(p-\beta)$	0	$(16 = c_8)$
0	$p-\beta$	$\frac{2l(p-\beta)}{p-\alpha}$	$(45 = a_9)$
$\frac{2m(p-\beta)}{p-\alpha}$	0	$p-\beta$	$(6 = b_9)$
$p-\beta$	$\frac{2n(p-\beta)}{p-\alpha}$	0	$(6' = c_9)$

44. We have $X=0$, $Y=0$, $Z=0$, $W=0$ for the equations of the planes

$$(12.34.56=x),\quad (42'=y),\quad (14'=z),\quad (12'=w);$$

and representing by $\mathrm{f}=lX+\frac{1}{m}Y+\frac{1}{n}Z+W=0$ the equation of any other plane $(41'=\mathrm{f})$ the equation of the cubic surface may be presented in the several forms:

$$\begin{aligned}
0=U&=W\mathrm{f}\bar{\mathrm{f}} &&+\mathrm{k}\xi YZ,\\
&=W\mathrm{g}\bar{\mathrm{g}} &&+\mathrm{k}\eta ZX,\\
&=W\mathrm{h}\bar{\mathrm{h}} &&+\mathrm{k}\zeta XY,\\
&=W\theta\bar{\theta} &&+\mathrm{k}\xi\eta\zeta,\\
&=W\mathrm{l}\bar{\mathrm{l}}_1 &&+\mathrm{k}\mathrm{y}\bar{\mathrm{z}}\mathrm{x},\\
&=W\mathrm{m}\bar{\mathrm{m}}_1 &&+\mathrm{k}\mathrm{z}\bar{\mathrm{x}}\mathrm{y},\\
&=W\mathrm{n}\bar{\mathrm{n}}_1 &&+\mathrm{k}\mathrm{x}\bar{\mathrm{y}}\mathrm{z},\\
&=W\mathrm{l}_1\bar{\mathrm{l}} &&+\mathrm{k}\bar{\mathrm{y}}\mathrm{z}\mathrm{x},\\
&=W\mathrm{m}_1\bar{\mathrm{m}} &&+\mathrm{k}\bar{\mathrm{z}}\mathrm{x}\mathrm{y},\\
&=W\mathrm{n}_1\bar{\mathrm{n}} &&+\mathrm{k}\bar{\mathrm{x}}\mathrm{y}\mathrm{z},\\
&=W\mathrm{p}\mathrm{p}_1 &&+\mathrm{k}\xi\mathrm{y}\mathrm{z},\\
&=W\mathrm{q}\mathrm{q}_1 &&+\mathrm{k}\eta\mathrm{z}\mathrm{x},\\
&=W\mathrm{r}\mathrm{r}_1 &&+\mathrm{k}\zeta\mathrm{x}\mathrm{y},\\
&=W\bar{\mathrm{p}}\bar{\mathrm{p}}_1 &&+\mathrm{k}\xi\bar{\mathrm{y}}\bar{\mathrm{z}},\\
&=W\bar{\mathrm{q}}\bar{\mathrm{q}}_1 &&+\mathrm{k}\eta\bar{\mathrm{z}}\bar{\mathrm{x}},\\
&=W\bar{\mathrm{r}}\bar{\mathrm{r}}_1 &&+\mathrm{k}\zeta\bar{\mathrm{x}}\bar{\mathrm{y}},
\end{aligned}$$

which are the 16 forms containing W, out of the complete system of 120 trihedral-pair forms.

45. The 27 lines are each of them facultative; we have therefore $b'=\rho'=27$; $t'=45$; moreover each of the lines is a double tangent of the spinode curve, and therefore $\beta'\,(=2\rho')=54$.

46. The equation of the reciprocal surface is not here investigated; its form is

$$S^3-T^2=0,$$

where $S=(*\mathrel{\big)\!\big(}x, y, z, w)^4$, $T=(*\mathrel{\big)\!\big(}x, y, z, w)^6$; wherefore $n'=12$.

The nodal curve is composed of the lines which are the reciprocals of the original 27 lines ($b'=27$, $t'=45$ *ut suprà*). It may be remarked that the reciprocal of a double-sixer is a double-sixer. Hence the 27 lines of the reciprocal surface may be (and that in 36 different ways) represented by

$$1, 2, 3, 4, 5, 6$$

$$1', 2', 3', 4', 5', 6'$$

$$12, 13, \ldots. 56,$$

where 12 is now the line joining the points 12′ and 1′2; and so for the other lines. The lines 12, 34, 56 meet in a point 12 . 34 . 56; the 30 points 12′, 1′2 … 56′, 5′6, and the fifteen points 12 . 34 . 56 make up the 45 points t'.

The above equation, $S^3 - T^2 = 0$, shows that the cuspidal curve is a complete intersection 6×4; $c' = 24$.

Section $\text{II} = 12 - C_2$.

Article Nos. 47 to 59. Equation $W(a, b, c, f, g, h \between X, Y, Z)^2 + 2kXYZ = 0$.

47. It may be remarked that the system of lines and planes is at once deduced from that belonging to $\text{I} = 12$, by supposing that in the double-sixer the corresponding lines 1 and 1′, &c. severally coincide; the line 12, instead of being given as the intersection of the planes 12′, 1′2, is given as the third line in the plane 12, which in fact represents the coincident planes 12′ and 1′2.

48. The diagram is

Lines.

Planes.

		56	46	45	36	35	34	26	25	24	23	16	15	14	13	12	6	5	4	3	2	1
II = 12 − C_2		$\overline{21}$ $\overline{27}$							15 × 1 = 15											6 × 2 = 12		
12																•					•	•
13															•					•		•
14														•					•			•
15													•					•				•
16												•					•					•
23											•									•	•	
24										•									•		•	
25									•									•			•	
26	15 × 2 = 30							•									•				•	
34							•												•	•		
35						•												•		•		
36					•												•			•		
45				•														•	•			
46			•														•		•			
56		•															•	•				
12 . 34 . 56		•					•									•						
12 . 35 . 46			•			•										•						
12 . 36 . 45				•	•											•						
13 . 24 . 56		•								•					•							
13 . 25 . 46			•						•						•							
13 . 26 . 45				•				•							•							
14 . 23 . 56		•									•			•								
14 . 25 . 36	15 × 1 = 15				•				•					•								
14 . 26 . 35						•		•						•								
15 . 23 . 46			•								•		•									
15 . 24 . 36					•					•			•									
15 . 26 . 34							•	•					•									
16 . 23 . 45				•							•	•										
16 . 24 . 35						•				•		•										
16 . 25 . 34	$\overline{30}$ 45						•		•			•										

Biradial planes, through each pair of rays.

Planes each containing three mere lines.

Mere lines, one in each biradial ray.

Rays, through the node situate in a quadric cone.

49. Putting the equation of the surface in the form

$$W\left(1,\ 1,\ 1,\ l+\frac{1}{l},\ m+\frac{1}{m},\ n+\frac{1}{n}\between X,\ Y,\ Z\right)^2+\frac{\alpha\beta\gamma\delta}{p}XYZ=0,$$

where for shortness

$$\begin{aligned}\alpha&=mn-l,\\ \beta&=nl-m,\\ \gamma&=lm-n,\\ \delta&=lmn-1,\\ p&=lmn,\end{aligned}$$

then taking $X=0$ as the equation of the plane [12], $Y=0$ as that of the plane [34], $Z=0$ as that of the plane [56], the equations of the 30 distinct planes are found to be

$$\begin{array}{rl}
X=0, & [12]\\
Y=0, & [34]\\
Z=0, & [56]\\
m\ X+l\ Y+Z=0, & [23]\\
m^{-1}X+l\ Y+Z=0, & [24]\\
m\ X+l^{-1}Y+Z=0, & [13]\\
m^{-1}X+l^{-1}Y+Z=0, & [14]\\
X+n\ Y+m\ Z=0, & [45]\\
X+n^{-1}Y+m\ Z=0, & [46]\\
X+n\ Y+m^{-1}Z=0, & [35]\\
X+n^{-1}Y+m^{-1}Z=0, & [36]\\
n\ X+Y+l\ Z=0, & [16]\\
n^{-1}X+Y+l\ Z=0, & [15]\\
n\ X+Y+l^{-1}Z=0, & [26]\\
n^{-1}X+Y+l^{-1}Z=0, & [25]\\
W=0, & [12.34.56]\\
X+\beta\gamma\ W=0, & [12.36.45]\\
X-\alpha\delta\ W=0, & [12.35.46]\\
Y+\alpha\gamma\ W=0, & [16.25.34]\\
Y-\beta\delta\ W=0, & [15.26.34]\\
Z+\alpha\beta\ W=0, & [14.23.56]\\
Z-\gamma\delta\ W=0, & [13.24.56]\\
mnX+nl\ Y+lm\ Z+\alpha\beta\gamma\delta\ W=0, & [16.23.45]\\
pX+n\ Y+m\ Z+\beta\gamma\delta\ W=0, & [13.26.45]\\
nX+p\ Y+l\ Z+\gamma\alpha\delta\ W=0, & [16.24.35]\\
mX+l\ Y+p\ Z+\alpha\beta\delta\ W=0, & [15.23.46]\\
X+lm\ Y+ln\ Z-\beta\gamma\delta\ W=0, & [15.24.36]\\
lmX+\ Y+mn\ Z-\gamma\alpha\delta\ W=0, & [13.25.46]\\
nlX+mn\ Y+\ Z-\alpha\beta\delta\ W=0, & [14.26.35]\\
lX+m\ Y+n\ Z-\alpha\beta\gamma\ W=0, & [14.25.36]
\end{array}$$

50. And the coordinates of the 21 distinct lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be taken to be
l	0	0	0	-1	1	(1) $X=0,\ Y+\ lZ=0$
0	m	0	1	0	-1	(3) $Y=0,\ Z+mX=0$
0	0	n	-1	1	0	(5) $Z=0,\ X+\ nY=0$
l^{-1}	0	0	0	-1	1	(2) $X=0,\ Y+\ l^{-1}Z=0$
0	m^{-1}	0	1	0	-1	(4) $Y=0,\ Z+m^{-1}X=0$
0	0	n^{-1}	-1	1	0	(6) $Z=0,\ X+\ n^{-1}Y=0$
1	n	m	0	$\frac{m}{\beta\gamma}$	$-\frac{n}{\beta\gamma}$	(45) $X+\ nY+mZ=0,\ X+\beta\gamma W=0$
n	1	l	$-\frac{l}{\gamma\alpha}$	0	$\frac{n}{\gamma\alpha}$	(16) $Y+\ lZ+nX=0,\ Y+\gamma\alpha W=0$
m	l	1	$\frac{l}{\alpha\beta}$	$-\frac{m}{\alpha\beta}$	0	(23) $Z+mX+\ lY=0,\ Z+\alpha\beta W=0$
1	$\frac{1}{n}$	m	0	$-\frac{m}{\alpha\delta}$	$\frac{1}{n\alpha\delta}$	(46) $X+n^{-1}Y+mZ=0,\ X-\alpha\delta W=0$
n	1	$\frac{1}{l}$	$\frac{1}{l\beta\delta}$	0	$-\frac{n}{\beta\delta}$	(26) $Y+\ l^{-1}Z+nX=0,\ Y-\beta\delta W=0$
$\frac{1}{m}$	l	1	$-\frac{l}{\gamma\delta}$	$\frac{1}{m\gamma\delta}$	0	(24) $Z+m^{-1}X+\ lY=0,\ Z-\gamma\delta W=0$
1	n	$\frac{1}{m}$	0	$-\frac{1}{m\alpha\delta}$	$\frac{n}{\alpha\delta}$	(35) $X+\ nY+m^{-1}Z=0,\ X-\alpha\delta W=0$
$\frac{1}{n}$	1	l	$\frac{l}{\beta\delta}$	0	$-\frac{1}{n\beta\delta}$	(15) $Y+\ lZ+n^{-1}X=0,\ Y-\beta\delta W=0$
m	$\frac{1}{l}$	1	$-\frac{1}{l\gamma\delta}$	$\frac{m}{\gamma\delta}$	0	(13) $Z+mX+l^{-1}Y=0,\ Z-\gamma\delta W=0$
1	$\frac{1}{n}$	$\frac{1}{m}$	0	$\frac{1}{m\beta\gamma}$	$-\frac{1}{n\beta\gamma}$	(36) $X+\ n^{-1}Y+m^{-1}Z=0,\ X+\beta\gamma W=0$
$\frac{1}{n}$	1	$\frac{1}{l}$	$-\frac{1}{l\gamma\alpha}$	0	$\frac{1}{n\gamma\alpha}$	(25) $Y+\ l^{-1}Z+n^{-1}X=0,\ Y+\gamma\alpha W=0$
$\frac{1}{m}$	$\frac{1}{l}$	1	$\frac{1}{l\alpha\beta}$	$-\frac{1}{m\gamma\alpha}$	0	(14) $Z+m^{-1}X+\ l^{-1}Y=0,\ Z+\alpha\beta W=0$
1	0	0	0	0	0	(12) $X=0,\ W=0$
0	1	0	0	0	0	(34) $Y=0,\ W=0$
0	0	1	0	0	0	(56) $Z=0,\ W=0$

51. The six nodal rays are not, the fifteen mere lines are, facultative. Hence

$$b'=\rho'=15\,;\ t'=15.$$

52. Resuming the equation $W(a,\ b,\ c,\ f,\ g,\ h\between X,\ Y,\ Z)^2+2kXYZ=0$, the equation of the Hessian surface is found to be

$$\begin{aligned}&KW^2\,(a,\ b,\ c,\ f,\ g,\ h\between X,\ Y,\ Z)^2\\&+2kW\,\{(a,\ b,\ c,\ f,\ g,\ h\between X,\ Y,\ Z)^2\,(FX+GY+HZ)-3KXYZ\}\\&-k^2\,\{a^2X^4+b^2Y^4+c^2Z^4-2bcY^2Z^2-2caZ^2X^2-2abX^2Y^2\\&\qquad-4XYZ\,[(af+gh)\,X+(bg+hf\,)\,Y+(ch+fg)\,Z]\}=0,\end{aligned}$$

where

$$(A,\ B,\ C,\ F,\ G,\ H)=(bc-f^{\,2},\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch),$$
$$K=abc-af^{\,2}-bg^2-ch^2+2fgh.$$

The Hessian and the cubic intersect in an indecomposable curve, which is the spinode curve; that is, spinode curve is a complete intersection 3×4; $\sigma'=12$.

The equations of the spinode curve may be written in the simplified form

$$\begin{aligned}&W(a,\ b,\ c,\ f,\ g,\ h\between X,\ Y,\ Z)^2+2kXYZ=0,\\&-8KXYZW\\&+8kXYZ\,(afX+bgY+chZ)\\&-k^2\,\{a^2X^4+b^2Y^4+c^2Z^4-2bcY^2Z^2-2caZ^2X^2-2abX^2Y^2\}=0\,;\end{aligned}$$

and it appears hereby that the node C_2 is a sixfold point on the curve, the tangents of the curve in fact coinciding with the six rays.

Each of the 15 lines touches the spinode curve twice; in fact, for the line 12 we have $X=0$, $W=0$; and substituting in the equations of the spinode curve, we have $(bY^2-cZ^2)=0$; that is, we have the two points of contact $X=0$, $W=0$, $Y\sqrt{b}=\pm Z\sqrt{c}$. Hence $\beta'=30$.

Reciprocal Surface.

53. The equation is found by equating to zero the discriminant of the ternary cubic function

$$(Xx+Yy+Zz)\,(a,\ b,\ c,\ f,\ g,\ h\between X,\ Y,\ Z)^2-2kwXYZ,$$

viz. the discriminant contains the factor w^2 which is to be thrown out, thus reducing the order to $n'=10$.

The ternary cubic, multiplying by 3 to avoid fractions, is

X^3,	Y^3,	Z^3,	$3Y^2Z$,	$3Z^2X$,	$3X^2Y$,	$3YZ^2$,	$3ZX^2$,	$3XY^2$,	$6XYZ$,
$3ax$,	$3by$,	$3cz$,	$bz+2fy$,	$cx+2gz$,	$ay+2hx$,	$cy+2fz$,	$az+2gx$,	$bx+2hy$,	$fx+gy+hz-kw$.

Write as before (A, B, C, F, G, H) for the inverse coefficients $(A = bc - f^2$, &c.), and $K = abc - af^2 - bg^2 - ch^2 + 2fgh$; and moreover

$$\begin{aligned}
\Phi &= (A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2,\\
P &= Ax + Hy + Gz,\\
Q &= Hx + By + Fz,\\
R &= Gx + Fy + Cz,\\
t &= fx + gy + hz,\\
U &= afyz + bgzx + chxy,\\
V &= 2Kxyz - aPyz - bQzx - cRxy\\
&= -a\,Hy^2z - b\,Fz^2x - c\,Gx^2y\\
&\quad - a\,Gyz^2 - b\,Hzx^2 - c\,Fxy^2\\
&\quad + (-abc - af^2 - bg^2 - ch^2 + 4fgh)\,xyz,\\
W &= (A,\ B,\ C,\ F,\ G,\ H \between ayz,\ bzx,\ cxy)^2,\\
L &= k^2w^2 - 2ktw - \Phi,\\
M &= kw\,U + V,\\
N &= 2kabc\,xyzw + W:
\end{aligned}$$

54. Then the invariants of the ternary cubic are

$$\begin{aligned}
S &= L^2 - 12kwM,\\
T &= L^3 - 18kw\,LM - 54k^2w^2N;
\end{aligned}$$

and the required equation of the reciprocal surface is

$$\frac{1}{108w^2}\{(L^2 - 12kwM)^3 - (L^3 - 18kwLM - 54k^2w^2N)^2\} = 0,$$

viz. this is

$$\begin{aligned}
0 = \quad & L^3N && = (k^2w^2 - 2ktw - \Phi)^3\,(2kabc\,xyzw + W)\\
& + L^2M^2 && + (k^2w^2 - 2ktw - \Phi)^2\,(kwU + V)^2\\
& - 18kwLMN && - 18kw\,(k^2w^2 - 2ktw - \Phi)\,(kw\,U + V)\,(2kabc\,xyzw + W)\\
& - 16kwM^3 && - 16kw\,(kw\,U + V)^3\\
& - 27k^2w^2N^2 && - 27k^2w^2\,(2kabc\,xyzw + W)^2,
\end{aligned}$$

which, arranged in powers of kw, is as follows; viz. we have

$$\begin{aligned}
\text{Coeff. } (kw)^7 = \quad & 2abc\,xyz,\\
(kw)^6 = \quad & 2abc\,xyz\,(-6t) + W\\
& + U^2,\\
(kw)^5 = \quad & 2abc\,xyz\,(-3\Phi + 12t^2) + W\,(-6t)\\
& + U^2\,(-4t) + 2UV\\
& - 36abc\,xyz\,U,
\end{aligned}$$

$$\begin{aligned}
(kw)^4 = {} & 2abc\,xyz\,(12t\Phi - 8t^3) + W(-3\Phi + 12t^2) \\
& + U^2(-2\Phi + 4t^2) + 2UV(-4t) + V^2 \\
& - 36abc\,xyz\,V - 18UW + 72abc\,xyzt\,U \\
& - 16U^3 \\
& - 108a^2b^2c^2\,x^2y^2z^2,
\end{aligned}$$

$$\begin{aligned}
,,\quad (kw)^3 = {} & 2abc\,xyz\,(3\Phi^2 - 12t^2\Phi) + W(12t\Phi - 8t^3) \\
& + U^2 4t\Phi + 2UV(-2\Phi + 4t^2) + V^2(-4t) \\
& - 18VW + 72abc\,xyzt\,V + 36tUW - 36abc\,xyz\Phi U \\
& - 48U^2V \\
& - 108abc\,xyz\,W,
\end{aligned}$$

$$\begin{aligned}
,,\quad (kw)^2 = {} & 2abc\,xyz\,(-6t\Phi^2) + W(3\Phi^2 - 12t^2\Phi) \\
& + U^2\Phi^2 + 2UV(4t\Phi) + V^2(-2\Phi + 4t^2) \\
& + 36tVW - 36abc\,xyz\Phi V - 18\Phi UW \\
& - 48UV^2 \\
& - 27W^2,
\end{aligned}$$

$$\begin{aligned}
,,\quad (kw)^1 = {} & 2abc\,xyz\,(-\Phi^3) + W(-6t\Phi)^2 \\
& + 2UV\Phi^2 + V^2(4t\Phi) \\
& - 18\Phi VW \\
& - 16V^3,
\end{aligned}$$

$$\begin{aligned}
,,\quad (kw)^0 = {} & W(-\Phi^3) \\
& + V^2\Phi^2;
\end{aligned}$$

but I have not carried the ultimate reduction further than in Schläfli, viz. I give only the terms in $(kw)^7$, $(kw)^6$, $(kw)^5$, and $(kw)^0$.

55. I present the result as follows; the coefficients deducible from those which precede, by mere cyclical permutations of the letters a, b, c and f, g, h, are indicated by (,,).

$0 = (kw)^7 \,.\, 2abc\,xyz$

	y^2z^2	z^2x^2	x^2y^2	x^2yz	xy^2z	xyz^2
$+(kw)^6\,.$	$a^2bc + 1$	,,	,,	$abcf - 14$ $gcbh + 4$	,,	,,

	y^3z^2	y^2z^3	z^3x^2	z^2x^3	x^3y^2	x^2y^3	x^3yz	xy^3z	xyz^3	xy^2z^2	x^2yz^2	x^2y^2z
$+(kw)^5\,.$	$a^2bcg - 6$ $a^2cfh + 2$	$a^2bch - 6$ $a^2bfg + 2$	,,	,,	,,	,,	$ab^2c^2 - 6$ $abcf^2 + 42$ $b^2cg^2 + 2$ $bc^2h^2 + 2$ $bcfgh - 24$	,,	,,	$a^2bcf - 32$ $abcfgh + 64$ $abfg^2 - 24$ $acfh^2 - 24$ $af^2gh + 8$	,,	,,

$\vdots$

$$+ (kw)^0 \,.\, -K\,[(A, B, C, F, G, H)(x, y, z)^2]^2\,(cy^2 - 2fyz + bz^2)\,(az^2 - 2gzx + bx^2)\,(bx^2 - 2hxy + ay^2).$$

56. In explanation of the discussion of the reciprocal surface, it is convenient to remark that we have

Node C_2, $X=0,\ Y=0,\ Z=0$.

Tangent cone is

$$(a,\ b,\ c,\ f,\ g,\ h \between X,\ Y,\ Z)^2=0.$$

Nodal rays are sections of cone by planes $X=0$, $Y=0$, $Z=0$ respectively, viz. equations of the rays are

$$\begin{aligned} X&=0, & bY^2+2fYZ+cZ^2&=0,\\ Y&=0, & cZ^2+2gZX+aX^2&=0,\\ Z&=0, & aX^2+2hXY+bY^2&=0. \end{aligned}$$

Reciprocal plane is $w=0$.

Conic of contact is

$$(A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2=0,\ w=0.$$

Lines are tangents of this conic from points $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$ respectively, viz. equations are

$$\begin{aligned} w&=0, & cy^2-2fyz+bz^2&=0,\\ w&=0, & az^2-2gzx+cx^2&=0,\\ w&=0, & bx^2-2hxy+ay^2&=0. \end{aligned}$$

57. The equation shows that the section by the plane $w=0$ is made up of the conic $(A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2=0$, twice, and of the six lines, tangents to this conic, viz. the lines

$$\begin{aligned} w&=0, & cy^2-2fyz+bz^2&=0,\\ w&=0, & az^2-2gzx+cx^2&=0,\\ w&=0, & bx^2-2hxy+ay^2&=0, \end{aligned}$$

each once; the lines in question (reciprocals of the nodal rays) are thus mere scrolar lines on the reciprocal surface.

58. I do not attempt to put in evidence the nodal curve of the surface; by what precedes it is made up of 15 lines, intersecting 3 together in 15 points; and if we denote the six tangents of the conic just referred to by

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

then the fifteen lines are respectively lines passing through the intersections of each pair of these tangents; viz. through the intersection of the tangents 1 and 2, we have a line 12; and so in other cases; that is, the 15 lines are 12, 13 .... 56. The lines 12 and 34 meet; and the lines 12, 34, 56 meet in a point; we have thus the 15 points 12.34.56, triple points of the nodal curve.

59. As regards the cuspidal curve, the equation of the surface may be written

$$\begin{aligned} (L^2-12kwM)(4M^2+3LN)-(LM+9kwN)^2\\ =3(L^2M^2+L^3N-18kwLMN-16kwM^3-27k^2w^2N^2)=0, \end{aligned}$$

and we thus have

$$\begin{aligned} 4M^2+\ 3LN&=0,\\ LM+\ 9kwN&=0,\\ L^2-12kwM&=0, \end{aligned}$$

or, what is the same thing,

$$\left\| \begin{matrix} L, & 12M, & -9N \\ kw, & L, & M \end{matrix} \right\| = 0$$

(equivalent to two equations) for the equations of the cuspidal curve. Attending to the second and third equations, the cuspidal curve may be considered as the residual intersection of the quartic and quintic surfaces $L^2-12kwM=0$, $LM+9kwN=0$, which partially intersect in the conic $w=0$, $L=0$; or say it is a curve $4\times5-2$; $c'=18$.

Section III $= 12 - B_3$.

Article Nos. 60 to 72. Equation $2W(X+Y+Z)(lX+mY+nZ)+2kXYZ=0$.

60. The system of lines and planes is at once deduced from that belonging to II $= 12 - C_2$, by supposing the tangent cone to reduce itself to the pair of biplanes; 3 of the planes (a) of II $= 12 - C_2$ thus coming to coincide with the one biplane, and three of them with the other biplane.

61. The diagram is

Lines.

	III $= 12 - B_3$		36	35	34	26	25	24	16	15	14	6	5	4	3	2	1	
			$\overline{15}$ $\overline{27}$					$9 \times 1 = 9$						$6 \times 3 = 18$				
	123	$2 \times 6 = 12$													•	•	•	Biplanes.
	456											•	•	•				
	14	$9 \times 3 = 27$									•			•			•	Biradial planes each containing a ray 1, 2, or 3 of the one biplane, and a ray 4, 5, or 6 of the other biplane.
	15									•			•				•	
	16								•			•					•	
	24							•						•		•		
Planes.	25						•						•			•		
	26					•						•				•		
	34				•									•	•			
	35			•									•		•			
	36		•									•			•			
	14.25.36	$6 \times 1 = 6$	•				•				•							Planes each containing three mere lines.
	14.26.35			•		•					•							
	15.26.34				•	•				•								
	15.24.36		•					•		•								
	16.24.35			•				•	•									
	16.25.34	$\overline{17}$ $\overline{45}$			•		•		•									
			Mere lines, in each biradial plane, one.									Rays 1, 2, 3 and 4, 5, 6, in the two biplanes respectively.						

62. Taking $X+Y+Z=0$ for the biplane that contains the rays 1, 2, 3, and $lX+mY+nZ=0$ for that which contains the rays 4, 5, 6, we may take $X=0$, $Y=0$, $Z=0$ for the equations of the planes [14], [25], [36] respectively; and then writing for shortness

$$m-n,\ n-l,\ l-m=\lambda,\ \mu,\ \nu,$$

and assuming, as we may do, $k=\lambda\mu\nu$, so that the equation of the surface is

$$W(X+Y+Z)(lX+mY+nZ)+(m-n)(n-l)(l-m)XYZ=0,$$

the equations of the 17 distinct planes are

$$\begin{array}{rl}
X=0, & [14] \\
Y=0, & [25] \\
Z=0, & [36] \\
X+\ \ Y+\ \ Z=0, & [123] \\
lX+mY+nZ=0, & [456] \\
lX+\ nY+nZ=0, & [15] \\
lX+\ nY+nZ=0, & [16] \\
lX+mY+\ lZ=0, & [25] \\
nX+mY+nZ=0, & [26] \\
mX+mY+nZ=0, & [35] \\
lX+\ \ lY+nZ=0, & [36] \\
W=0, & [14\,.\,25\,.\,36] \\
W+\ l\lambda X=0, & [14\,.\,26\,.\,35] \\
W+m\mu Y=0, & [16\,.\,25\,.\,34] \\
W+\ n\nu Z=0, & [15\,.\,24\,.\,36] \\
lmX+mnY+\ nlZ+W=0, & [15\,.\,26\,.\,34] \\
nlX+\ lmY+mnZ-W=0, & [16\,.\,24\,.\,35]
\end{array}$$

63. And the coordinates of the fifteen distinct lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be written
0	0	0	0	-1	1	(1) $X=0,\ Y+Z=0$
0	0	0	1	0	-1	(2) $Y=0,\ Z+X=0$
0	0	0	-1	1	0	(3) $Z=0,\ X+Y=0$
0	0	0	0	$-n$	m	(4) $X=0,\ mY+nZ=0$
0	0	0	n	0	$-l$	(5) $Y=0,\ nZ+lX=0$
0	0	0	$-m$	l	0	(6) $Z=0,\ lX+mY=0$
1	0	0	0	0	0	(14) $X=0,\ W=0$
0	1	0	0	0	0	(25) $Y=0,\ W=0$
0	0	1	0	0	0	(36) $Z=0,\ W=0$
l	n	n	$n^2\nu$	$-nl\nu$	0	(15) $lX+nY+nZ=0,\ W+n\nu Z=0$
l	m	m	$-m^2\mu$	0	$lm\mu$	(16) $lX+mY+mZ=0,\ W+m\mu Y=0$
l	m	l	0	$l^2\lambda$	$-lm\mu$	(26) $lX+mY+lZ=0,\ W+l\lambda X=0$
n	m	n	$mn\nu$	$-n^2\nu$	0	(24) $nX+mY+nZ=0,\ W+n\nu Z=0$
m	m	n	$-mn\mu$	0	$m^2\mu$	(34) $mX+mY+nZ=0,\ W+m\mu Y=0$
l	l	n	0	$nl\lambda$	$-l^2\lambda$	(35) $lX+lY+nZ=0,\ W+l\lambda X=0$

64. The rays are not, the mere lines are, facultative; hence $b'=\rho'=9:\ t'=6$.

65. The equation of the Hessian surface is

$$\begin{aligned}&-W(X+Y+Z)(lX+mY+nZ)(\mu\nu X+\nu\lambda Y+\lambda\mu Z)\\ &-k(l^2X^4+m^2Y^4+n^2Z^4-2mnY^2Z^2-2nlZ^2X^2-2lmX^2Y^2)\\ &+kXYZ\{(l^2+3lm+3ln+mn)X+(m^2+3mn+3ml+nl)Y+(n^2+3nl+3nm+lm)Z\}=0.\end{aligned}$$

The Hessian and cubic surfaces intersect in an indecomposable curve, which is the spinode curve; that is, spinode curve is a complete intersection 3×4; $\sigma'=12$.

The equations may be written in the simplified form

$$\begin{aligned}&W(X+Y+Z)(lX+mY+nZ)+kXYZ=0,\\ &l^2X^4+m^2Y^4+n^2Z^4-2mnY^2Z^2-2nlZ^2X^2-2lmX^2Y^2\\ &\qquad-4XYZ\{l(m+n)X+m(n+l)Y+n(l+m)Z\}=0.\end{aligned}$$

We may also obtain the equation

$$\begin{aligned}&k^2(X+Y+Z)(lX+mY+nZ)\{lX^2+mY^2+nZ^2-(m+n)YZ-(n+l)ZX-(l+m)XY\}\\ &\qquad+\lambda^2Y^2Z^2+\mu^2Z^2X^2+\nu^2X^2Y^2-2XYZ(\mu\nu X+\nu\lambda Y+\lambda\mu Z)=0,\end{aligned}$$

which shows that there is at B_3 an eightfold point, the tangents being given by

$$(X+Y+Z)(lX+mY+nZ)=0,$$

$$(\lambda^2,\ \mu^2,\ \nu^2,\ -\mu\nu,\ -\nu\lambda,\ -\lambda\mu \between YZ,\ ZX,\ XY)^2=0.$$

Each of the facultative lines is a double tangent of the spinode curve; whence $\beta'=18$.

Reciprocal Surface.

66. The equation may be deduced from that for $\Pi=12-C_2$, viz. writing

$$(a,\ b,\ c,\ f,\ g,\ h \between X,\ Y,\ Z)^2=2(X+Y+Z)(lX+mY+nZ),$$

that is

$$(a,\ b,\ c,\ f,\ g,\ h)\quad=\quad(2l,\ 2m,\ 2n,\ m+n,\ n+l,\ l+m),$$

we have

$$(A,\ B,\ C,\ F,\ G,\ H)=-(\lambda^2,\ \ \mu^2,\ \ \nu^2,\ \ \mu\nu\ \ ,\ \ \nu\lambda\ ,\ \ \lambda\mu\ \ );\ \ K=0.$$

Writing also

$$\lambda,\ \mu,\ \nu=m-n,\ n-l,\ l-m \text{ as before,}$$

$$\lambda x+\mu y+\nu z=\sigma,$$

$$lmn\, xyz=\theta,$$

$$l(m+n)\,yz+m(n+l)\,zx+n(l+m)\,xy=\upsilon,$$

$$l\lambda\,yz+\quad m\mu\,zx+\quad n\nu\,xy=\psi,$$

$$(m+n)\,x+\quad(n+l)\,y+\quad(l+m)\,z=t,$$

we have

$$U=2\upsilon,\quad V=2\sigma\psi,\quad W=-4\psi^2,$$

and then

$$L=k^2w^2-2ktw+\sigma^2,\quad M=2(kw\upsilon+\sigma\psi),\quad N=4(4lmnkxyzw-\psi^2);$$

so that the equation is

$$\begin{array}{rll}
0= & L^3N= & 4(k^2w^2-2kwt+\sigma^2)^3(4kw\theta-\psi^2)\\
 & +L^2M^2 & +\ \ 4(k^2w^2-2kwt+\sigma^2)^2(kw\upsilon+\sigma\psi)^2\\
 & -18kwLMN & -144\ kw(k^2w^2-2kwt+\sigma^2)(kw\upsilon+\sigma\psi)(kw\theta-\psi^2)\\
 & -16kw\ \ M^3 & -128\ kw(kw\upsilon+\sigma\psi)^3\\
 & -27k^2w^2\ \ N^2 & -432\ k^2w^2(kw\theta-\psi^2);
\end{array}$$

or reducing the first two terms so as to throw out from the whole equation the factor *kw*, the equation is

$$4L^2\{\theta L+(\upsilon^2-\psi^2)\,kw+2\psi(t\psi+\upsilon\sigma)\}-18LMN-16M^3-27kwN^2=0$$

or, what is the same thing, it is

$$\begin{aligned}&(k^2w^2 - 2kwt + \sigma^2)^2 \{k^2w^2\theta + kw\,(-2t\theta + \upsilon^2 - \psi^2) + \sigma^2\theta + 2\sigma\upsilon\psi + 2t\psi^2\}\\ &-\ 36\,(k^2w^2 - 2kwt + \sigma^2)\,(kw\upsilon + \sigma\psi)\,(4kw\theta - \psi^2)\\ &-\ 32\,(kw\upsilon + \sigma\psi)^3\\ &-108kw\,(4kw\theta - \psi^2)^2 = 0.\end{aligned}$$

67. This is

$$\begin{aligned}&(kw)^6\,.\quad \theta\\ &+(kw)^5\,.-\psi^2 - 6t\theta + \upsilon^2\\ &+(kw)^4\,.\quad \sigma^2\,.\,3\theta + \sigma\psi\,.\,2\upsilon + \psi^2\,.\,6t + 12t^2\theta - 4t\upsilon^2 - 144\theta\upsilon\\ &+(kw)^3\,.-2\sigma^2\psi^2 + \sigma^2\,(2\upsilon^2 - 10t\theta) + \sigma\psi\,(-8t\upsilon - 144\theta) + \psi^2\,(-12t^2 + 36\upsilon)\\ &\qquad -8t^3\theta + 4t^2\upsilon^2 + 288t\upsilon\theta - 32\upsilon^3 - 1728\theta^2\\ &+(kw)^2\,.\quad \sigma^4\,.\,3\theta + \sigma^3\psi\,.\,4\upsilon + \sigma^2\psi^2\,.\,12t + \sigma\psi^3\,.\,37\\ &\qquad +\sigma^2\,(12t^2\theta - 4t\upsilon^2 - 144\theta\upsilon) + \sigma\psi\,(+8t^2\upsilon + 288t\theta - 96\upsilon^2) + \psi^2\,(8t^3 - 72t\upsilon + 864\theta)\\ &+(kw)\ .-\sigma^4\psi^2 + \sigma^4\,(-6t\theta + \upsilon^2) + \sigma^3\psi\,(-8t\upsilon - 144\theta)\\ &\qquad +\sigma^2\psi^2\,(-8t - 90\upsilon) + \sigma\psi^3\,.-72 + \psi^4\,.-108\\ &+(kw)^0\,.\quad \sigma^3\,(\theta,\ 2\upsilon,\ 2t,\ 4\,)\!(\,\sigma,\ \psi)^3 = 0,\end{aligned}$$

which, reducing the last term, is

$$\begin{aligned}&(kw)^6\,lmnxyz\\ &\quad\vdots\\ &-4\sigma^3\lambda\mu\nu\,(y-z)\,(z-x)\,(x-y)\,(ny-mz)\,(lz-nx)\,(mx-ly) = 0.\end{aligned}$$

68. I verify the last term in the particular case $z=0$ as follows: the coefficient of σ^3 is

$$\big(0,\ 2n\,(l+m)\,xy,\ 2\,(m+n)\,x + 2\,(n+l)\,y,\ 4\,\big)\!\big(\,\lambda x + \mu y,\ n\nu xy\big)^3,$$

which is

$$\begin{aligned}&= 2n^2\nu x^2y^2\,\{(l+m)\,(\lambda x + \mu y)^2 + [(m+n)\,x + (n+l)\,y]\,(\nu\lambda x + \mu\nu y) + 2n\nu^2xy\}\\ &= 2n^2\nu x^2y^2\,\{[(l+m)\,\lambda + (m+n)\,\nu]\,\lambda x^2\\ &\qquad + [2\,(l+m)\,\lambda\mu + (m+n)\,\mu\nu + (n+l)\,\nu\lambda + 2n\nu]\,xy\\ &\qquad + [(l+m)\,\mu + (n+l)\,\nu]\,\mu y^2\},\end{aligned}$$

which, substituting for λ, μ, ν their values $m-n$, $n-l$, $l-m$, is

$$= 2n^2\nu x^2y^2\,.-2\lambda\mu\,(x-y)\,(mx-ly);$$

or for $z=0$ the coefficient of σ^3 is

$$= -4\lambda\mu\nu\,n^2x^2y^2\,(x-y)\,(mx-ly),$$

agreeing with the general value

$$-4\lambda\mu\nu\,(y-z)\,(z-x)\,(x-y)\,(ny-mz)\,(lz-nx)\,(lx-my).$$

69. In the discussion of the equation it is convenient to write down the relations of the two surfaces, thus:

Cubic surface.	Reciprocal surface.
B_3, $X=0,\ Y=0,\ Z=0$	Plane $w=0$,
Biplanes $X+\ Y+\ Z=0$ $lX+mY+nZ=0$, intersecting in edge.	Points in $w=0$, viz. $x=y=z$ and $x:y:z=l:m:n$, in line $(m-n)x+(n-l)y+(l-m)z=0$, that is, $\lambda x+\mu y+\nu z=0$, or $\sigma=0$.
Rays in first biplane, $X=0,\ Y+Z=0;\ Y=0,\ Z+X=0,$ $Z=0,\ X+Y=0;$	Lines in plane $w=0$, and through first point, viz. $y-z=0,\ z-x=0,\ x-y=0;$
rays in second biplane, $X=0,\ mY+nZ=0;\ Y=0,\ nZ+lX=0,$ $Z=0,\ lX+mY=0.$	lines through second point, viz. $ny-mz=0,\ nz-lx=0,\ lx-my=0.$

70. The equation puts in evidence the section by the plane $w=0$, viz. this is the line $\sigma=0$ (reciprocal of the edge) three times, and the six lines (reciprocals of the rays) each once. Observe that the edge is *not* a line on the cubic; but its reciprocal is a line, and that an oscular line on the reciprocal surface; the six lines (reciprocals of the rays) are mere scrolar lines on the reciprocal surface; they pass, three of them, through the point $x=y=z$, and the other three through the point $x:y:z=l:m:n$; that is, they are six tangents of the point-pair (reciprocal of the pair of biplanes) formed by these two points.

71. I do not attempt to put in evidence the nodal curve on the surface; by what precedes it consists of 9 lines, reciprocals of the mere lines. If we denote by 1, 2, 3 and 4, 5, 6 the lines which pass through the points $x=0$, $y=0$, $z=0$ and through the point $x:y:z=l:m:n$ respectively, then these intersect in the nine points 14, 15, 16, 24, 25, 26, 34, 35, 36; and through each of these there passes a nodal line which may be represented by the same symbol; that is, we have the nodal lines 14, ... 36. Two lines such as 14, 25 meet; and three lines such as 14, 25, 36 meet in a point; we have thus the six points 14.25.36 &c. triple points on the nodal curve; as before, $b'=9$, $t'=6$.

72. The cuspidal curve is given by the equations

$$\left|\begin{matrix} k^2w^2-2kwt+\sigma^2, & 24\,(kwv+\sigma\psi), & -36\,(4lmnk\,xyzw-\psi^2) \\ kw \quad , & k^2w^2-2kwt+\sigma^2, & 2\,(kwv+\sigma\psi) \end{matrix}\right\| = 0.$$

Writing down the two equations,

$$(k^2w^2-2kwt+\sigma^2)^2-24kw\,(kwv+\sigma\psi)=0,$$
$$(k^2w^2-2kwt+\sigma^2)\,(kwv+\sigma\psi)+18w\,(lmnk\,xyzw-\psi^2)=0,$$

these are respectively of the orders 4 and 5; but they intersect in the line $w=0$, $\sigma=0$ taken four times, or say, the cuspidal curve is a partial intersection $4.5-4$; $c'=16$.

Section IV $= 12 - 2C_2$.

Article Nos. 73 to 84. Equation $WXZ + Y^2(\gamma Z + \delta W) + (a, b, c, d \between X, Y)^3 = 0$.

73. The diagram of the lines is

													Lines.						
	IV $=12-2C_2$.		34 . 1′2′	24 . 1′3′	23 . 1′4′	14 . 2′3′	13 . 2′4′	12 . 3′4′	4′	3′	2′	1′	4	3	2	1	5	0	
			$\overline{16}$ $\overline{27}$			6 × 1 = 6							8 × 2 = 16				1 × 1 = 1	1 × 4 = 4	
Planes.	[0]	1 × 2 = 2															• •	• •	Plane touching along axis.
	11′	4 × 4 = 16										• •				• •		•	Planes through axis, each containing a ray of the one node and a ray of the other node.
	22′										• •				• •			•	
	33′									• •				• •				•	
	44′								• •				• •					•	
	12	12 × 2 = 24						• •							•	•			Biradial planes, each containing two rays of the one node or two rays of the other node.
	13						• •							•		•			
	14					• •							•			•			
	23				• •									•	•				
	24			• •									•		•				
	34		• •										•	•					
	1′2′		• •								•	•							
	1′3′			• •						•		•							
	1′4′				• •				•			•							
	2′3′					• •				•	•								
	2′4′						• •		•		•								
	3′4′							• •	•	•									
	12 . 34	3 × 1 = 3	•					•									•		Planes each containing three mere lines.
	13 . 24			•			•										•		
	14 . 23	$\overline{20}$ $\overline{45}$			•	•											•		
			Mere lines, each the intersection of a biradial plane of the one node and a biradial plane of the other node.										Rays 1, 2, 3, 4, and 1′, 2′, 3′, 4′ through the two nodes respectively.				Transversal, in the tangent plane along the axis.	Axis, through the two nodes.	

74. Writing $X(a, b, c, d\,)\!(\,X, Y)^3 - \gamma\delta Y^4 = -\gamma\delta(f_1X - Y)(f_2X - Y)(f_3X - Y)(f_4X - Y)$, the 20 planes are

$$X = 0, \qquad [\ 0]$$

$$X - f_1Y = 0, \qquad [11']$$

$$X - f_2Y = 0, \qquad [22']$$

$$X - f_3Y = 0, \qquad [33']$$

$$X - f_4Y = 0, \qquad [44']$$

$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_2)Y\} - \mathrm{f}_1\mathrm{f}_2Z = 0, \qquad [12]$$

$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_3)Y\} - \mathrm{f}_1\mathrm{f}_3Z = 0, \qquad [13]$$

$$\delta\{X - (\mathrm{f}_1 + \mathrm{f}_4)Y\} - \mathrm{f}_1\mathrm{f}_4Z = 0, \qquad [14]$$

$$\delta\{X - (\mathrm{f}_2 + \mathrm{f}_3)Y\} - \mathrm{f}_2\mathrm{f}_3Z = 0, \qquad [23]$$

$$\delta\{X - (\mathrm{f}_2 + \mathrm{f}_4)Y\} - \mathrm{f}_2\mathrm{f}_4Z = 0, \qquad [24]$$

$$\delta\{X - (\mathrm{f}_3 + \mathrm{f}_4)Y\} - \mathrm{f}_3\mathrm{f}_4Z = 0, \qquad [34]$$

$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_2)Y\} - \mathrm{f}_1\mathrm{f}_2W = 0, \qquad [1'2']$$

$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_3)Y\} - \mathrm{f}_1\mathrm{f}_3W = 0, \qquad [1'3']$$

$$\gamma\{X - (\mathrm{f}_1 + \mathrm{f}_4)Y\} - \mathrm{f}_1\mathrm{f}_4W = 0, \qquad [1'4']$$

$$\gamma\{X - (\mathrm{f}_2 + \mathrm{f}_3)Y\} - \mathrm{f}_2\mathrm{f}_3W = 0, \qquad [2'3']$$

$$\gamma\{X - (\mathrm{f}_2 + \mathrm{f}_4)Y\} - \mathrm{f}_2\mathrm{f}_4W = 0, \qquad [2'4']$$

$$\gamma\{X - (\mathrm{f}_3 + \mathrm{f}_4)Y\} - \mathrm{f}_3\mathrm{f}_4W = 0, \qquad [3'4']$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_2} + \frac{1}{\mathrm{f}_3\mathrm{f}_4}\right)X + dy + \gamma Z + \delta W = 0, \qquad [12\,.\,34]$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_3} + \frac{1}{\mathrm{f}_2\mathrm{f}_4}\right)X + dy + \gamma Z + \delta W = 0, \qquad [13\,.\,24]$$

$$-\gamma\delta\left(\frac{1}{\mathrm{f}_1\mathrm{f}_4} + \frac{1}{\mathrm{f}_2\mathrm{f}_3}\right)X + dy + \gamma Z + \delta W = 0, \qquad [14\,.\,23]$$

75. And the 16 lines are

(a)	(b)	(c)	(f)	(g)	(h)	whence equations may be written	
0	0	0	0	0	1	(0)	$X = 0,\ Y = 0$
δ	0	0	0	$-\gamma$	d	(5)	$X = 0,\ dY + \gamma Z + \delta W = 0$
0	0	0	f_1^2	f_1	$-\delta$	(1)	$X - \mathrm{f}_1Y = 0,\ \delta Y + \mathrm{f}_1Z = 0$
0	0	0	f_2^2	f_2	$-\delta$	(2)	,, ,,
0	0	0	f_3^2	f_3	$-\delta$	(3)	,, ,,
0	0	0	f_4^2	f_4	$-\delta$	(4)	,, ,,

(a)	(b)	(c)	(f)	(g)	(h)		
f_1	$-f_1^2$	0	0	0	γ	$(1')$	$X-f_1Y=0,\ \gamma Y+f_1W=0$
f_2	$-f_2^2$	0	0	0	γ	$(2')$	,, ,,
f_3	$-f_3^4$	0	0	0	γ	$(3')$	,, ,,
f_4	$-f_4^2$	0	0	0	γ	$(4')$	,, ,,
$-\dfrac{\delta}{f_1f_2}$	$\delta\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}\right)$	1	$-\gamma\left(\dfrac{1}{f_3}+\dfrac{1}{f_4}\right)$	$-\dfrac{\gamma}{f_3f_4}$	$\dfrac{1}{f_3f_4}\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}\right)-\dfrac{1}{f_1f_2}\left(\dfrac{1}{f_3}+\dfrac{1}{f_4}\right)$	$(12\,.\,3'4')$*	
$-\dfrac{\delta}{f_1f_3}$	$\delta\left(\dfrac{1}{f_1}+\dfrac{1}{f_3}\right)$	1	$-\gamma\left(\dfrac{1}{f_2}+\dfrac{1}{f_4}\right)$	$-\dfrac{\gamma}{f_2f_4}$	$\dfrac{1}{f_2f_4}\left(\dfrac{1}{f_1}+\dfrac{1}{f_3}\right)-\dfrac{1}{f_1f_3}\left(\dfrac{1}{f_2}+\dfrac{1}{f_4}\right)$	$(13\,.\,2'4')$	,,
$-\dfrac{\delta}{f_1f_4}$	$\delta\left(\dfrac{1}{f_1}+\dfrac{1}{f_4}\right)$	1	$-\gamma\left(\dfrac{1}{f_2}+\dfrac{1}{f_3}\right)$	$-\dfrac{\gamma}{f_2f_3}$	$\dfrac{1}{f_2f_3}\left(\dfrac{1}{f_1}+\dfrac{1}{f_4}\right)-\dfrac{1}{f_1f_4}\left(\dfrac{1}{f_2}+\dfrac{1}{f_3}\right)$	$(14\,.\,2'3')$	,,
$-\dfrac{\delta}{f_2f_3}$	$\delta\left(\dfrac{1}{f_2}+\dfrac{1}{f_3}\right)$	1	$-\gamma\left(\dfrac{1}{f_1}+\dfrac{1}{f_4}\right)$	$-\dfrac{\gamma}{f_1f_4}$	$\dfrac{1}{f_1f_4}\left(\dfrac{1}{f_2}+\dfrac{1}{f_3}\right)-\dfrac{1}{f_2f_3}\left(\dfrac{1}{f_1}+\dfrac{1}{f_4}\right)$	$(23\,.\,1'4')$	,,
$-\dfrac{\delta}{f_2f_4}$	$\delta\left(\dfrac{1}{f_2}+\dfrac{1}{f_4}\right)$	1	$-\gamma\left(\dfrac{1}{f_1}+\dfrac{1}{f_3}\right)$	$-\dfrac{\gamma}{f_1f_3}$	$\dfrac{1}{f_1f_3}\left(\dfrac{1}{f_2}+\dfrac{1}{f_4}\right)-\dfrac{1}{f_2f_4}\left(\dfrac{1}{f_1}+\dfrac{1}{f_3}\right)$	$(24\,.\,1'3')$	,,
$-\dfrac{\delta}{f_3f_4}$	$\delta\left(\dfrac{1}{f_3}+\dfrac{1}{f_4}\right)$	1	$-\gamma\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}\right)$	$-\dfrac{\gamma}{f_1f_2}$	$\dfrac{1}{f_1f_2}\left(\dfrac{1}{f_3}+\dfrac{1}{f_4}\right)-\dfrac{1}{f_3f_4}\left(\dfrac{1}{f_1}+\dfrac{1}{f_2}\right)$	$(34\,.\,1'2')$	,,

*equations are

$$\delta\{X-(f_1+f_2)\,Y\}-f_1f_2Z=0,\quad \gamma\{X-(f_3+f_4)\,Y\}-f_3f_4W=0,$$

[and similarly for each of the remaining five lines].

76. To verify the equations of the line 12 . 3′4′, observe that the two equations give

$$\gamma Z+\delta W=\gamma\delta\left\{X\left(\frac{1}{f_1f_2}+\frac{1}{f_3f_4}\right)-Y\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right)\right\},$$

$$ZW=\frac{\gamma\delta}{f_1f_2f_3f_4}\{X-(f_1+f_2)\,YX-(f_3+f_4)\,Y\}:$$

the equation of the surface, multiplying by X and observing that $-\gamma\delta=\alpha f_1f_2f_3f_4$, becomes

$$X^2ZW+XY^2(\gamma Z+\delta W)+\gamma\delta Y^4-\frac{\gamma\delta}{f_1f_2f_3f_4}(X-f_1Y)(X-f_2Y)(X-f_3Y)(X-f_4Y)=0;$$

and substituting the values just obtained, this is

$$X^2[X-(f_1+f_2)\,Y][X-(f_3+f_4)\,Y]+XY^2[X(f_1f_2+f_3f_4)-Y(f_1f_2f_3+f_1f_2f_4+f_1f_3f_4+f_2f_3f_4)]$$
$$+f_1f_2f_3f_4Y_4-(X-f_1Y)(X-f_2Y)(X-f_3Y)(X-f_4Y)=0,$$

which is in fact an identity.

77. The facultative lines are the transversal and the six mere lines; $b'=\rho'=7$; $t'=3$.

78. The equation of the Hessian surface is found to be

$$\begin{aligned}&(\gamma Z+\delta W)\,XZW+Y^2(\gamma Z-\delta W)^2+3\,(cX+dY)\,XZW+12\gamma\delta XY^2(aX+bY)\\&-(\gamma Z+\delta W)\,(3aX^3+9bX^2Y+6cXY^2)\\&-9X^2\{(ac-b^2)\,X^2+(ad-bc)\,XY+(bd-c^2)\,Y^2\}=0.\end{aligned}$$

79. Combining with the foregoing the equation of the surface

$$XZW+Y^2(\gamma Z+\delta W)+(a,\ b,\ c,\ d \between X,\ Y)^3=0,$$

it appears that these have along the line $X=0$, $Y=0$ the common tangent plane $X=0$, or, what is the same thing, that they meet in the line $X=0$, $Y=0$ (the axis) twice, and in a residual curve of the tenth order, which is the spinode curve; the equations may be presented in the somewhat more simple form

$$\begin{aligned}&XZW+Y^2(\gamma Z+\delta W)+(a,\ b,\ c,\ d \between X,\ Y)^3=0,\\&-4\gamma\delta Y^2ZW-4\,(\gamma Z+\delta W)\,(a,\ b,\ c,\ d \between X,\ Y)^3+12\gamma\delta XY^2\,(aX+bY)\\&\qquad+X^4(-12ac+9b^2)-3d\,(4aX^3Y+6bX^2Y^2+4cXY^3+dY^4)=0,\end{aligned}$$

which, however, still contain the line $X=0$, $Y=0$ twice. The spinode curve, as just mentioned, is of the tenth order; that is, we have $\sigma'=10$.

Each of the 6 mere lines is a double tangent to the spinode curve, but the transversal is only a single tangent: to show this, observe that the equations of the transversal are $X=0$, $\gamma Z+\delta W+dY=0$; substituting in the equations of the curve the first equation, that of the cubic surface is of course satisfied identically; for the second equation, writing $X=0$, this becomes $Y^2\{-4\gamma\delta ZW-4dY(\gamma Z+\delta W)-3d^2Y^2\}=0$; or writing herein $dY=-(\gamma Z+\delta W)$, it becomes $Y^2(\gamma Z-\delta W)^2=0$. The value $Y^2=0$ gives $X=0$, $Y=0$, $\gamma Z+\delta W=0$, viz. this is a point on the axis $X=0$, $Y=0$ not belonging to the spinode curve; the value $(\gamma Z-\delta W)^2=0$ gives a point of contact $X=0$, $\gamma Z+\delta W+dY=0$, $\gamma Z-\delta W=0$; and the transversal is thus a single tangent. Hence the number of contacts is $2\,.\,6+1$, $=13$; that is, we have $\beta'=13$.

Reciprocal Surface.

80. The equation is found by equating to zero the discriminant of the binary quartic

$$\{xX^2+yXY-(\delta z+\gamma w)\,Y^2\}+4Zw\,\{X(a,\ b,\ c,\ d \between X,\ Y)^3-\gamma\delta Y^4\},$$

or say this is $(* \between X,\ Y)^4$, where the coefficients are

$$\begin{array}{r}6x^2\qquad\qquad\ \ +24azw,\\3xy\qquad\qquad\ \ +18bzw,\\y^2-2\,(\delta z+\gamma w)\,x+12czw,\\-3\,(\delta z+\gamma w)\,y+\ \ 6dzw,\\6\,(\delta z-\gamma w)^2.\qquad\qquad\end{array}$$

81. Forming the invariants, these are

$$\begin{aligned} \tfrac{1}{3}I &= \Lambda^2 + 24Uzw + 144\mu z^2w^2, \\ -J &= \Lambda^3 + 36\Lambda Uzw + 216Vz^2w^2 + 864\nu z^3w^3, \end{aligned}$$

where

$$\begin{aligned} \Lambda &= y^2 + 4(\delta z + \gamma w)x, \\ U &= 2\gamma\delta x^2 + 2a(\delta z - \gamma w)^2 + 3by(\delta z + \gamma w) + c[y^2 - 2(\delta z + \gamma w)x] - dxy, \\ V &= (-8ac + 9b^2)(\delta z - \gamma w)^2 \\ &\quad + (2c^2 - bd)[y^2 - 2(\delta z + \gamma w)x] \\ &\quad + (-4ad + 6bc)y(\delta z + \gamma w) \\ &\quad - 2cd\,xy \\ &\quad + d^2x^2 \\ &\quad + 4\gamma\delta(2cx^2 - 3bxy + ay^2), \\ \mu &= c^2 - bd, \\ \nu &= ad^2 - 2bcd + 2c^3, \end{aligned}$$

and the equation is

$$\frac{1}{432z^2w^2}\{(\Lambda^2 + 24Uzw + 144\mu z^2w^2)^3 - (\Lambda^3 + 36\Lambda Uzw + 216Vz^2w^2 + 864\nu z^3w^3)^2\} = 0;$$

or, expanding, this is

$$\begin{array}{lll} & & \Lambda^4\mu - \Lambda^3V + \Lambda^2U^2 \\ + & 4zw\,(& -\Lambda^3\nu + 12\Lambda^2U\mu - 9\Lambda UV + 8U^3) \\ + & 36z^2w^2\,(& 4\Lambda^2\mu^2 - 4\Lambda U\nu + 16U^2\mu - 3V^2) \\ + & 864z^3w^3\,(& 4U\mu^2 - V\nu) \\ + & 1728z^4w^4\,(& 4\mu^3 - \nu^2) = 0, \end{array}$$

where observe that the value of

$$4\mu^3 - \nu^2, = 4(bd - c^2)^3 - (ad^2 - 3bcd + 2c^3)^2 \text{ is } = -d^2(a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd).$$

82. It is convenient to modify the form of the equation as follows; write

$$U_1 = U + 8a\gamma\delta zw,\ \ V_1 = V + (-8ac + 9b^2)\gamma\delta zw,$$

so that

$$\begin{aligned} \Lambda &= y^2 + 4(\delta z + \gamma w)x, \\ U_1 &= -2\gamma\delta x^2 + 2a(\delta z + \gamma w)^2 + 3by(\delta z + \gamma w) + c[y^2 - 2(\delta z + \gamma w)x] - dxy, \\ V_1 &= (-8ac + 9b^2)(\delta z + \gamma w)^2 \\ &\quad + (2c^2 - bd)[y^2 - 2(\delta z + \gamma w)x] \\ &\quad + (-4ad + 6bc)y(\delta z + \gamma w) \\ &\quad - 2cdxy \\ &\quad + d^2x^2 \\ &\quad + 4\gamma\delta(2cx^2 - 3bxy + ay^2), \\ \mu &= c^2 - bd, \\ \nu &= ad^2 - 2bcd + 2c^3, \end{aligned}$$

Λ, U_1, V_1 being, it will be observed, functions of x, y, $\delta z + \gamma w$. The transformed equation is

$$\Lambda^2 (\Lambda^2 \mu - \Lambda V_1 + U_1^2) + \Omega z w = 0,$$

where the term Ω may be calculated without difficulty: the first term of this is

$$= \{y^2 + 4 (\delta z + \gamma w) x\}^2 . 4\gamma^2\delta^2 [x + \mathrm{f}_1 y - \mathrm{f}_1^2 (\delta z + \gamma w)] .. [x + \mathrm{f}_4 y - \mathrm{f}_4^2 (\delta z + \gamma w)],$$

the developed expressions of $\frac{1}{4} (\Lambda^2\mu - \Lambda V_1 + U_1^2)$ and of $\gamma^2\delta^2$ into the product of the linear factors being in fact each

$$\begin{aligned} = \quad & x^4 . \gamma^2\delta^2 + x^3 y . d\gamma\delta + x^2y^2 . -3c\gamma\delta + xy^3 . 3b\gamma\delta + y^4 . -a\gamma\delta \\ & + [x^3 (-d^2 - 6c\gamma\delta) + x^2 y (3cd + 9b\gamma\delta) + xy^2 (-3bd - 4a\gamma\delta) + y^3 . ad] (\delta z + \gamma w) \\ & + [x^2 (9c^2 - 6bd - 2a\gamma) + xy (3ad - 9bc) + y^2 . 3ac] (\delta z + \gamma w)^2 \\ & + [x (6ac - 9b^2) + y . 3ab] (\delta z + \gamma w)^3 \\ & + a^2\delta^4 . (\delta z + \gamma w)^4. \end{aligned}$$

The form puts in evidence the section by the plane $w = 0$, which is the reciprocal of the node D, viz. this is a conic (the reciprocal of the tangent cone) twice, and four lines, the reciprocals of the nodal rays, each once. And similarly for the section by the plane $z = 0$.

83. The nodal curve is made up of the lines which are the reciprocals of the six mere lines and the transversal; viz. we have three pairs of lines and a seventh line, the lines of each pair intersecting at a point of the seventh line, and these three points being the triple points of the nodal curve; $t' = 3$ as before.

84. The equations of the cuspidal curve are at once reduced to the form

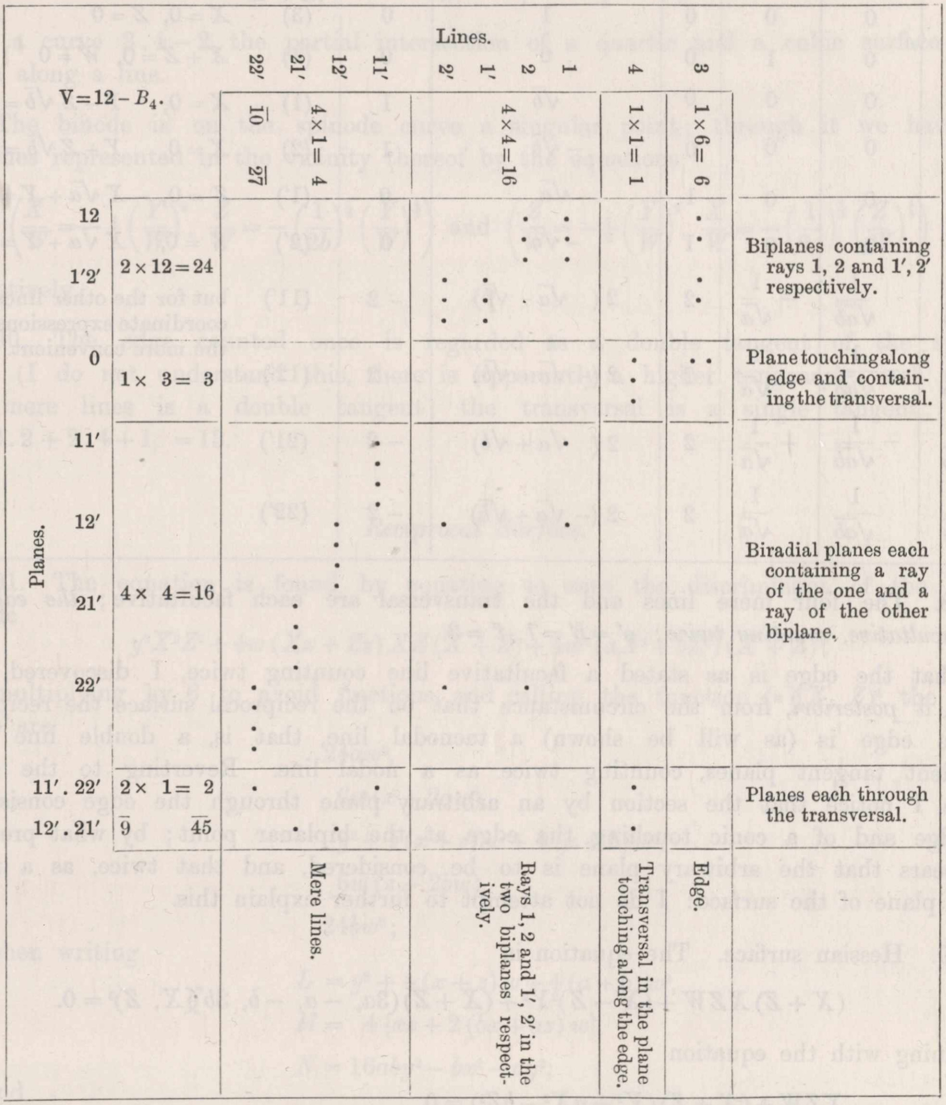
$$\begin{aligned} \Lambda^2 \quad & + 24 U z w + 144 \mu z^2 w^2 = 0, \\ \Lambda U \quad & + (18 V - 12 \mu \Lambda) z w + 72 \nu z^2 w^2 = 0, \end{aligned}$$

which are two quartic surfaces having in common the conics $z = 0$, $\Lambda = 0$, and $w = 0$, $\Lambda = 0$; or we may say that the cuspidal curve is a curve $4 . 4 - 2 - 2$; that is $c' = 12$.

Section V $= 12 - B_4$.

Article Nos. 85 to 94. Equation $WXZ + (X+Z)(Y^2 - aX^2 - bZ^2) = 0$.

85. The diagram of the lines and planes is



86. The planes are

$$\begin{aligned}
X &= 0, & &[12] \\
Z &= 0, & &[1'2'] \\
X + Z &= 0, & &[0] \\
-X\sqrt{a} + Y - Z\sqrt{b} &= 0, & &[11'] \\
X\sqrt{a} + Y - Z\sqrt{b} &= 0, & &[12'] \\
-X\sqrt{a} + Y + Z\sqrt{b} &= 0, & &[21'] \\
X\sqrt{a} + Y + Z\sqrt{b} &= 0, & &[22'] \\
\sqrt{ab}\,(X+Z) + W &= 0, & &[11'.\,22'] \\
-2\sqrt{ab}\,(X+Z) + W &= 0, & &[12'.\,21'].
\end{aligned}$$

87. And the lines are

a	b	c	f	g	h	equations may be written
0	0	0	0	1	0	(3) $X=0,\ Z=0$
1	0	1	0	0	0	(4) $X+Z=0,\ W=0$
0	0	0	0	$\sqrt{b}$	1	(1) $X=0,\quad Y-Z\sqrt{b}=0$
0	0	0	0	$-\sqrt{b}$	1	(2) $X=0,\quad Y+Z\sqrt{b}=0$
0	0	0	1	$\sqrt{a}$	0	(1′) $Z=0,\ -X\sqrt{a}+Y=0$
0	0	0	1	$-\sqrt{a}$	0	(2′) $Z=0,\quad X\sqrt{a}+Y=0$
$-\frac{1}{\sqrt{b}}$	$\frac{1}{\sqrt{ab}}$	$-\frac{1}{\sqrt{a}}$	2	$2(\sqrt{a}-\sqrt{b})$	-2	(11′) but for the other lines the coordinate expressions are the more convenient.
$-\frac{1}{\sqrt{b}}$	$-\frac{1}{\sqrt{ab}}$	$\frac{1}{\sqrt{a}}$	2	$2(-\sqrt{a}-\sqrt{b})$	-2	(12′)
$\frac{1}{\sqrt{b}}$	$-\frac{1}{\sqrt{ab}}$	$-\frac{1}{\sqrt{a}}$	2	$2(\sqrt{a}+\sqrt{b})$	-2	(21′)
$\frac{1}{\sqrt{b}}$	$\frac{1}{\sqrt{ab}}$	$\frac{1}{\sqrt{a}}$	2	$2(-\sqrt{a}+\sqrt{b})$	-2	(22′)

88. The four mere lines and the transversal are each facultative; *the edge is also facultative, counting twice*; $\rho'=b'=7,\ t'=3$.

That the edge is as stated a facultative line counting twice, I discovered, and accept, *à posteriori*, from the circumstance that on the reciprocal surface the reciprocal of the edge is (as will be shown) a tacnodal line, that is, a double line with coincident tangent planes, counting twice as a nodal line. Reverting to the cubic surface, I notice that the section by an arbitrary plane through the edge consists of the edge and of a conic touching the edge at the biplanar point; by what precedes it appears that the arbitrary plane is to be considered, and that twice, as a node-couple plane of the surface: I do not attempt to further explain this.

89. Hessian surface. The equation is

$$(X+Z)XZW+(X-Z)^2Y^2+(X+Z)(3a,\ -a,\ -b,\ 3b\between X,\ Z)^3=0.$$

Combining with the equation

$$XZW+(X+Z)(Y^2-aX^2-bZ^2)=0,$$

and observing that from the two equations we deduce

$$-XZY^2+(X+Z)(aX^3+bZ^3)=0,$$

it appears that the complete intersection of the Hessian and the surface is made up of the line $X=0$, $Z=0$ (the edge) twice (that is, the two surfaces touch along the edge), and of a curve of the tenth order, which is the spinode curve; $c'=10$.

The equations of the spinode curve may be presented in the form

$$\left\| \begin{matrix} XZ, & aX^2+bZ^2-Y^2, & aX^3+bZ^3 \\ X+Z, & W \quad , & Y^2 \end{matrix} \right\| = 0\,;$$

it is a curve $3 \,.\, 4-2$, the partial intersection of a quartic and a cubic surface which touch along a line.

The binode is on the spinode curve a singular point; through it we have two branches represented in the vicinity thereof by the equations

$$\left(\frac{X}{W} = -\tfrac{1}{2}\left(\frac{Y}{W}\right)^2,\ \frac{Z}{W} = -\left(\frac{1}{2b}\right)^{\frac{1}{3}}\left(\frac{Y}{W}\right)^{\frac{4}{3}}\right) \text{ and } \left(\frac{Z}{W} = -\tfrac{1}{2}\left(\frac{Y}{W}\right)^2,\ \frac{X}{W} = -\left(\frac{1}{2a}\right)^{\frac{1}{3}}\left(\frac{Z}{W}\right)^{\frac{4}{3}}\right)$$

respectively.

90. The edge counted once is regarded as a double tangent of the spinode curve (I do not understand this, there is apparently a higher tangency); each of the four mere lines is a double tangent; the transversal is a single tangent; hence $\beta' = 2 \,.\, 2 + 2 \,.\, 4 + 1, \ = 13$.

Reciprocal Surface.

91. The equation is found by equating to zero the discriminant of the binary quartic

$$y^2X^2Z^2 + 4w\,(Xx + Zz)\,XZ\,(X+Z) + 4w^2\,(aX^2 + bZ^2)\,(X+Z)^2,$$

viz. multiplying by 6 to avoid fractions, and calling the function $(*\between X, Z)^4$, the coefficients are

$$\begin{aligned} &24aw^2, \\ &6w\,(x + 2aw), \\ &y^2 + 4\,(x+z)\,w + 4\,(a+b)\,w^2, \\ &6w\,(z + 2bw), \\ &24bw^2\,; \end{aligned}$$

and then writing

$$\begin{aligned} L &= y^2 + 4\,(x+z)\,w + 4\,(a+b)\,w^2, \\ M &= \ \ 4\,\{xz + 2\,(bx + az)\,w\}, \\ N &= 16aby^2 - bx^2 - ay^2, \end{aligned}$$

we find

$$\begin{aligned} \tfrac{1}{3}I &= L^2 - 12w^2M, \\ -J &= L^3 - 18w^2LM - 54w^4N, \end{aligned}$$

and then the equation is

$$\frac{1}{108w^4}\,\{(L^2 - 12w^2M)^3 - (L^3 - 18w^2LM - 54w^4N)^2\} = 0,$$

viz. it is

$$L^3N + L^2M^2 - 18w^2LMN - 16w^2M^3 - 27w^4N^2 = 0.$$

92. This, completely developed, is

$$\begin{aligned}
& 64w^6 . ab\,(a+b)^2\,\{(a+b)\,y^2-(x-z)^2\} \\
& + 32w^5 . 2ab \left\{\begin{array}{l} 3\,(a+b)\,[(a-2b)\,x+(-2a+b)\,z]\,y^2 \\ +(x-z)^2\,[(-3a+5b)\,x+(5a-3b)\,z] \end{array}\right\} \\
& + 16w^4 \left\{\begin{array}{l} \quad 3ab\,(a^2-\ 7ab+b^2)\,y^4 \\ +[b\,(9a^2+26ab-b^2)\,x^2-26ab\,(a+b)\,xz+a\,(-a^2+26ab+9b^2)\,z^2]\,y^2 \\ +(x-z)^2\,[b\,(-12a+b)\,x^2+22abxz+a\,(a-12b)\,z^2] \end{array}\right\} \\
& + \ 8w^3 \left\{\begin{array}{l} \quad 3ab\,[(2a-b)\,x+(-a+2b)\,z]\,y^4 \\ +[b\,(-2a+5b)\,x^3+b\,(3a-2b)\,x^2z+a\,(-2a+3b)\,xz^2+a\,(5a-2b)\,z^3]\,y^2 \\ +2\,(x-z)^2\,[-2bx^3+bx^2z+axz^2-2ay^3] \end{array}\right\} \\
& + \ 4w^2 \left\{\begin{array}{l} \quad 3ab\,(a+b)\,y^6 \\ + \quad [b\,(9a-2b)\,x^2+8abxz+a\,(-2a+9b)\,z^2]\,y^4 \\ +2\,[-6bx^4+bx^3z-(a+b)\,x^2z^2+axz^3-6az^4]\,y^2 \\ +4x^2z^2\,(x-z)^2 \end{array}\right\} \\
& + \ 2w \left\{\begin{array}{l} \quad 2ab\,(x+z)\,y^6 \\ -[3bx^3+2bx^2z+2axz+3az^3]\,y^4 \\ +4x^2z^2\,(x+z)\,y^2 \end{array}\right\} \\
& + \qquad y^4\,(ay^2-x^2)\,(cy^2-z^2)=0,
\end{aligned}$$

where we see that the section by the plane $w=0$ (reciprocal of B_4) is made up of the line $w=0$, $y=0$ (reciprocal of the edge) four times, and of the lines $w=0$, $ay^2-x^2=0$; $w=0$, $by^2-z^2=0$ (reciprocals of the rays) each once.

93. The surface contains the line $y=0$, $w=0$ (reciprocal of the edge); and if we attend only to the terms of the lowest order in y, w, viz.

$$x^2z^2\,\{16\,(x-z)^2\,w^2+8\,(x+z)\,y^2w+y^4\},$$

which terms equated to zero give

$$w=-\tfrac{1}{4}\,\frac{1}{(\sqrt{x}\pm\sqrt{z})^2}\,y^2,$$

we see that the line in question ($y=0$, $w=0$) is a tacnodal line on the surface, the tacnodal plane being $w=0$, *a fixed plane for all points of the line*: it has already been seen that this plane meets the surface in the line taken 4 times; every other plane through the line meets the surface in the line taken twice. We have in what precedes the *à posteriori* proof that in the cubic surface the edge is a facultative line to be counted twice.

94. Cuspidal curve. The equation of the surface may be written

$$(L^2-12w^2M)\,(4M^2+3LN)-(LM+9w^2N)^2=0,$$

and we thus have

$$\begin{aligned} 4M^2 + 3LN &= 0, \\ LM + w^2N &= 0, \\ L^2 - 12w^2M &= 0, \end{aligned}$$

or, what is the same thing,

$$\left\| \begin{array}{ccc} L, & 12M, & -9N \\ w^2, & L, & M \end{array} \right\| = 0$$

for the equation of the cuspidal curve. Attending to the second and third equations, these are quartics having in common $w^2 = 0$, $L = 0$, that is, the line $y = 0$, $w = 0$ four times; or the cuspidal curve is a partial intersection $4 \times 4 - 4$: $c' = 12$.

Section VI $= 12 - B_3 - C_2$.

Article Nos. 95 to 102. Equation $WXZ + Y^2Z + (a, b, c, d \between X, Y)^3 = 0$.

95. The diagram of the lines and planes is

			Lines.											
			14 . 2'3'	13 . 2'4'	12 . 3'4'	4'	3'	2'	4	3	2	1	0	
	VI=12 − B_3 − C_2.		$\overline{11}$ $\overline{27}$	3 × 1 = 3		3 × 2 = 6			3 × 3 = 9			1 × 3 = 3	1 × 6 = 6	
	0	1 × 6 = 6										• •	• • •	Biplane touching along axis, and containing transversal ray.
	00	1 × 6 = 6							• •	• •	• •			Other biplane.
	22'							• • •			• •		•	
	33'	3 × 6 = 18					• • •			• •			•	Planes each through the axis and containing a ray of the binode and a ray of the cnicnode.
	44'					• • •			• •				•	
Planes.	12				• • •						•	•		
	13	3 × 3 = 9		• • •						•		•		Biradial planes of the binode, each containing ray of axial biplane and a ray of other biplane.
	14		• • •						•			•		
	2'3'		• •				•	•						
	2'4'	3 × 2 = 6		• •		•		•						Biradial planes of the cnicnode.
	3'4'	$\overline{11}$ $\overline{45}$			• •	•	•							
			Mere lines.			Cnicnodal rays.			Biplanar rays of other biplane.			Biplanar ray of axial biplane, being a transversal ray.	Axis, joining the two nodes.	

96. Writing $(a, b, c, d \between X, Y)^3 = -d(\theta_2 X - Y)(\theta_3 X - Y)(\theta_4 X - Y)$, the planes are

$$\begin{array}{rl}
X = 0, & [0] \\
Z = 0, & [00] \\
\theta_2 X - Y = 0, & [22'] \\
\theta_3 X - Y = 0, & [33'] \\
\theta_4 X - Y = 0, & [44'] \\
d(\theta_2 X - Y) - Z = 0, & [12] \\
d(\theta_3 X - Y) - Z = 0, & [13] \\
d(\theta_4 X - Y) - Z = 0, & [14] \\
X\theta_2\theta_3 - Y(\theta_2 + \theta_3) - W = 0, & [2'3'] \\
X\theta_2\theta_4 - Y(\theta_2 + \theta_4) - W = 0, & [2'4'] \\
X\theta_3\theta_4 - Y(\theta_3 + \theta_4) - W = 0, & [3'4']
\end{array}$$

97. And the lines are

a	b	c	f	g	h	equations may be written	
0	0	0	0	0	1	(0)	$X = 0, Y = 0$
0	0	0	0	-1	d	(1)	$X = 0, dY + Z = 0$
0	0	0	1	θ_2	0	(2)	$\theta_2 X - Y = 0, Z = 0$
0	0	0	1	θ_3	0	(3)	$\theta_3 X - Y = 0, Z = 0$
0	0	0	1	θ_4	0	(4)	$\theta_4 X - Y = 0, Z = 0$
θ_2	-1	0	0	0	$\theta_2{}^2$	(2′)	$\theta_2 X - Y = 0, \theta_2{}^2 X + W = 0$
θ_3	-1	0	0	0	$\theta_3{}^2$	(3′)	$\theta_3 X - Y = 0, \theta_3{}^2 X + W = 0$
θ_4	-1	0	0	0	$\theta_4{}^2$	(4′)	$\theta_4 X - Y = 0, \theta_4{}^2 X + W = 0$
$-d\theta_2$	d	1	$-(\theta_3 + \theta_4)$	$-\theta_3\theta_4$	$d(\theta_3\theta_4 - \theta_2\theta_3 - \theta_2\theta_4)$	(12 . 3′4′)	but for the remaining lines the coordinate expressions are more convenient.
$-d\theta_3$	d	1	$-(\theta_2 + \theta_4)$	$-\theta_2\theta_4$	$d(\theta_2\theta_4 - \theta_3\theta_2 - \theta_3\theta_4)$	(13 . 2′4′)	
$-d\theta_4$	d	1	$-(\theta_2 + \theta_3)$	$-\theta_2\theta_3$	$d(\theta_2\theta_3 - \theta_4\theta_2 - \theta_4\theta_3)$	(14 . 2′3′)	

The mere lines are each of them facultative; $b' = \rho' = 3$; $t' = 0$.

98. Hessian surface. The equation is

$$\begin{aligned}
\{Z + 3(cX + dY)\}\,\{XZW + Y^2 Z + (a, b, c, d \between X, Y)^3\} \\
- 4Z(a, b, c, d \between X, Y)^3 \\
- 3(4ac - 3b^2, ad, bd, cd, d^2 \between X, Y)^4 = 0;
\end{aligned}$$

and it is thence easy to see that the complete intersection is made up of the line $X = 0, Y = 0$ (the axis) three times, and of a curve of the ninth order, which is the spinode curve; $\sigma' = 9$.

99. The equations of the spinode curve may be written in the simplified form

$$XZW + Y^2Z + (a,\ b,\ c,\ d \between X,\ Y)^3 = 0,$$

$$4Z(a,\ b,\ c,\ d \between X,\ Y)^3 + 3(4ac - 3b^2,\ ad,\ bd,\ cd,\ d^2 \between X,\ Y)^4 = 0,$$

the line $X = 0$, $Y = 0$ here appearing as a triple line on the second surface; the curve is a partial intersection, $3 \times 4 - 3$.

The node C_2 is a triple point on the curve, the tangents being the nodal rays.

The node B_3 is a quintuple point, one tangent being $X = 0$, $3dY + 4Z = 0$, and the other tangents being given by $Z = 0$, $(4ac - 3b^2,\ ad,\ bd,\ cd,\ d^2 \between X,\ Y)^4 = 0$.

Each of the facultative lines is a double tangent to the curve, or we have $\beta' = 6$.

Reciprocal Surface.

100. Comparing the equation of the cubic surface with that for $\text{IV} = 12 - 2C_2$, it appears that the equation of $\text{VI} = 12 - B_3 - C_2$ is obtained by substituting in that equation the values $\delta = 0$, $\gamma = 1$. But instead of making this substitution in the final formula, it is convenient to make it in the binary quartic $(* \between X,\ Y)^4$, thus in fact working out the reciprocal surface by means of the function

$$(xX^2 + yXY - wY^2)^2 + 4zwX(a,\ b,\ c,\ d \between X,\ Y)^3,$$

the coefficients whereof (multiplying by 6 to avoid fractions) are

$$\begin{array}{l} 6x^2 + 24azw, \\ 3xy + 18bzw, \\ y^2 - 2xw + 12czw, \\ -3yw + \ \ 6dzw, \\ 6w^2. \end{array}$$

We find

$$\tfrac{1}{3}I = L^2 - 12zwM,$$

$$-J = L^3 - 18zwLM - 54z^2w^2N,$$

where

$$L = y^2 + 6(x + 3cz)w,$$

$$M = 2dxy + 6(2cx - by + 2bdz)w - 4aw^2,$$

$$N = -4d^2x^2 - 8d(3bx - 2ay + 2adz)w - 12(3b^2 - 4ac)w^2.$$

The equation is

$$\frac{1}{108z^2w^2}\{(L^2 - 12zwM)^3 - (L^3 - 18zwLM - 54z^2w^2N)^2\} = 0,$$

viz. it is

$$L^2(LN + M^2) - 18zwLMN - 16zwM^3 - 27z^2w^2N^2 = 0,$$

where however $LN + M^2$ contains the factor w, $= wP$ suppose; the equation thus is

$$L^2P - 18zLMN - 16zM^3 - 27z^2wN^2 = 0.$$

Write

$$A = 4x + 12cz,$$

$$B = 6cx - 3by + 6bdz - 2aw,$$

$$C = 6bdx - 4ady + 4ad^2z + 3(3b^2 - 4ac)w,$$

and therefore

$$\begin{aligned} L &= \quad y^2 + Aw, \\ M &= \quad 2dxy + 2Bw, \\ N &= -4d^2x^2 - 4Cw, \end{aligned}$$

then we have

$$\begin{aligned} P &= \frac{1}{w}\{-(y^2 + Aw)(4d^2x^2 + 4Cw) + (2dxy + 2Bw)^2\} \\ &= -4\{Cy^2 - 2Bdxy + Ad^2x^2 + w(AC - B^2)\}, \end{aligned}$$

or the equation is

$$4L^2\{Cy^2 - 2Bdxy + Ad^2x^2 + w(AC - B^2)\} + 18zLMN + 16zM^3 + 27z^2wN^2 = 0.$$

101. Consider the section by the plane $w = 0$, we have $L = y^2$, $M = 2dxy$, $N = -4d^2x^2$, and the equation becomes $4y^4(Cy^2 - 2Bdxy + Ad^2x^2) + (128 - 144 =) - 16d^3x^3y^3z = 0$; which substituting for A, B, C the values

$$\begin{aligned} A &= 4x + 12cz, \\ B &= 6cx - 3by + 6bdz, \\ C &= 6bdx - 4ady + 4ad^2z, \end{aligned}$$

becomes $16dy^3(y - dz)(dx^3 - 3cx^2y + 3cxy^2 - ay^3) = 0$; which is in fact the line $w = 0$, $y = 0$ (reciprocal of the edge) three times, and the lines $w = 0$, $(y - dz)(d, -c, b, -a \between x, y)^3 = 0$ (reciprocals of the biplanar rays) each once. Observe that the edge $(X = 0, Z = 0)$ is not a line of the cubic surface, but the reciprocal line $y = 0$, $w = 0$ presents itself as an oscular line of the reciprocal surface.

102. The equations of the cuspidal curve are in the first instance obtained in the form

$$\left\| \begin{matrix} L, & M, & 3N \\ 12zw, & L, & -4M \end{matrix} \right\| = 0.$$

Consider the two equations

$$\begin{aligned} L^2 &- 12zwM = 0, \\ LM &+ \ \ 9zwN = 0, \end{aligned}$$

each of the fourth order, but which are satisfied by $zw = 0$, $L = 0$; that is, by $(w = 0,\ y^2 = 0)$, $(z = 0,\ y^2 + 4xw = 0)$. The line $(w = 0, y = 0)$ however presents itself in the intersection of the two surfaces, not twice only, but 4 times. To show this, observe that the line in question is a nodal line on the surface $L^2 - 12zwM = 0$; in fact, attending only to the terms of the second order in y, w, we find

$$\{(4x + 12cz)^2 - 144cxz - 144bdz^2\}w^2 - 24dxzyw = 0,$$

giving the two sheets

$$\{(4x + 12cz)^2 - 144cxz - 144bdz^2\}w - 24dxzy \quad = 0 \text{ and } w = 0;$$

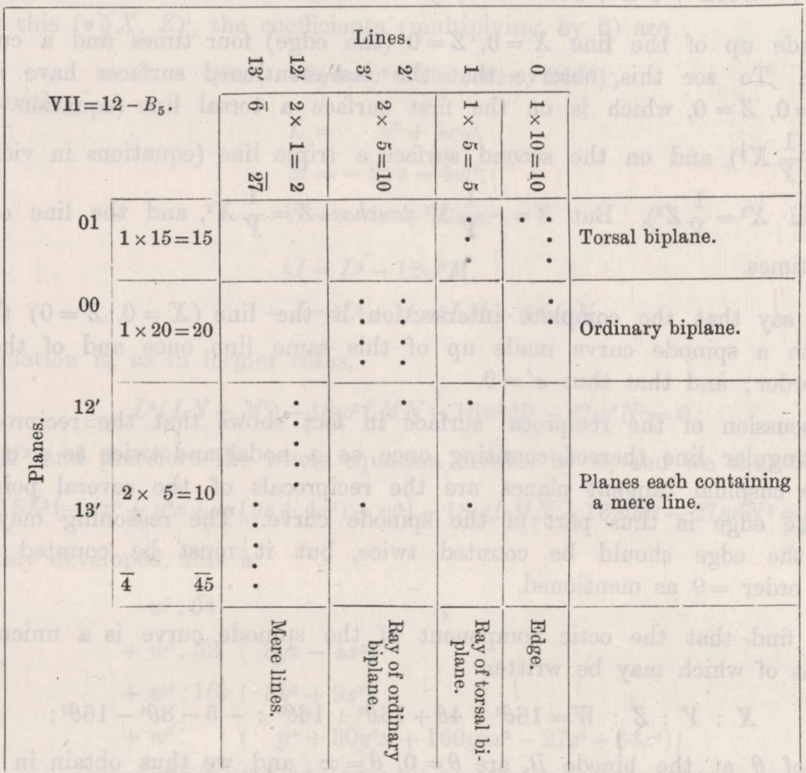
in regard to the last-mentioned sheet the form in the vicinity thereof is given by $w = Ay^3$, viz. we have approximately $L = y^2$, $M = 2dxy$, and thence $y^4 - 12z\,.\,Ay^3\,.\,2dxy = 0$, that is, $A = \dfrac{1}{24dxz}$ or $w = \dfrac{1}{24dxz}y^3$; the line is thus a flecnodal line on the surface

$L^2 - 12zwM = 0$. Next as regards the surface $LM + 9zwN = 0$; the line $y = 0$, $w = 0$ is a simple line on the surface, the terms of the lowest order being $9zw\,(-4d^2x^2) = 0$; that is, we have $w = 0$, and for a next approximation $w = Ay^3$, viz. $L = y^2$, $M = -2dxy$, $N = -4d^2x^2$, and therefore $-2dxy^3 + 9z\,.\,Ay^3(-4d^2x^2) = 0$, that is, $A = -\dfrac{1}{18dxz}$, or $w = -\dfrac{1}{18dxz}\,y^3$; there is thus a threefold intersection with one sheet and a simple intersection with the other sheet of the surface $L^2 - 12zwM = 0$. The surfaces intersect, as has been mentioned in the conic $z = 0$, $y^2 + 4xw = 0$; or we have the line $y = 0$, $w = 0$ four times, the conic once, and a residual cuspidal curve of the order $4\,.\,4 - 4 - 2, = 10$; that is, $c' = 10$.

Section VII $= 12 - B_5$.

Article Nos. 103 to 116. Equation $WXZ + Y^2Z + YX^2 - Z^3 = 0$.

103. The diagram of lines and planes[1] is



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[1] The marginal symbols in the preceding diagrams constitute a real notation of the lines and planes; but here, and still more so in some of the following diagrams, they are mere marks of reference, showing which are the lines and planes to which the several equations respectively belong.

104. The planes are		The lines are	
$Z=0,$	[10]	$X=0,\quad Z=0,$	(0)
$X=0,$	[00]	$Y=0,\quad Z=0,$	(1)
$Y+Z=0,$	[12′]	$X=0,\ Y+Z=0,$	(2′)
$Y-Z=0,$	[13′]	$X=0,\ Y-Z=0,$	(3′)
		$X-W=0,\ Y+Z=0,$	(12′)
		$X+W=0,\ Y-Z=0,$	(13′).

105. The two mere lines are facultative, and the edge is also facultative; $\rho'=b'=3$; $t'=0$.

106. Hessian surface. The equation is

$$Z(WXZ+Y^2Z+YX^2-Z^3)-4X^2YZ+X^4+4Z^4=0.$$

The complete intersection with the surface is thus given by the equations

$$WXZ+Y^2Z+YX^2-Z^3=0,\ -4X^2YZ+X^4+4Z^4=0,$$

which is made up of the line $X=0$, $Z=0$ (the edge) four times and a curve of the eighth order. To see this, observe that the last-mentioned surfaces have in common the line $X=0$, $Z=0$, which is on the first surface a torsal line (equation in vicinity being $Z=-\frac{1}{Y}X^2$), and on the second surface a triple line (equations in vicinity being $Z=\frac{1}{Y}X^2$ and $X^2=\frac{1}{Y}Z^3$). But $Z=-\frac{1}{Y}X^2$ *touches* $Z=\frac{1}{Y}X^2$, and the line counts thus $(2+2=)$ 4 times.

107. I say that the complete intersection is the line $(X=0,\ Z=0)$ three times together with a spinode curve made up of this same line once and of the curve of the eighth order; and that thus $\sigma'=9$.

The discussion of the reciprocal surface in fact shows that the reciprocal of the edge is a singular line thereof, counting once as a nodal and twice as a cuspidal line thereof; the cuspidal tangent planes are the reciprocals of the several points of the edge, and the edge is thus part of the spinode curve. The reasoning may appear to show that the edge should be counted twice, but it must be counted once only, making the order $=9$ as mentioned.

108. I find that the octic component of the spinode curve is a unicursal curve, the equations of which may be written

$$X:Y:Z:W=16\theta^2:4\theta+16\theta^5:16\theta^3:-5-8\theta^4-16\theta^8;$$

the values of θ at the binode B_5 are $\theta=0$, $\theta=\infty$, and we thus obtain in the neighbourhood thereof the two branches

$$\frac{Y}{W}=-5\left(\frac{Z}{W}\right)^2,\ \frac{X}{W}=\tfrac{25}{4}\left(\frac{Z}{W}\right)^3 \text{ and } \frac{X}{W}=\left(\frac{Z}{W}\right)^{\frac{5}{3}},\ \frac{Y}{W}=-\left(\frac{Z}{W}\right)^2.$$

109. Each of the lines $(X-W=0,\ Y+Z=0)$ and $(X+W=0,\ Y-Z=0)$ is a double tangent of the spinode octic; in fact for the first of these lines we have

$$16\theta^8+8\theta^4+16\theta^2+5=0,\quad 16\theta^5+16\theta^3+4\theta=0,$$

that is,

$$(2\theta^2+1)^2(4\theta^4-4\theta^2+5)=0,\quad 4\theta\,(2\theta^2+1)^2=0,$$

so that the line touches at the two points given by $2\theta^2+1=0$; and similarly the other line touches at the two points given by $2\theta^2-1=0$.

The edge $X=0$, $Z=0$ has apparently a higher contact with the spinode octic, viz. the equations $X=0$, $Z=0$ are satisfied by $\theta=0$ twice, $\theta=\infty$ five times; but it must be reckoned only as a double tangent. Hence $\beta'=2\,.\,2+2,=6$.

Reciprocal Surface.

110. The equation is obtained by equating to zero the discriminant of the binary quartic

$$X^2(yZ-wX)^2+4wZ^2(wZ^2+zZX+xX^2),$$

viz. calling this $(*\between X,\ Z)^4$, the coefficients (multiplying by 6) are

$$(6w^2,\ -3yw,\ y^2+4xw,\ 6zw,\ 24w^2);$$

and then writing

$$\begin{aligned} L &= \quad y^2+4xw,\\ M &= -2yz-4w^2,\\ N &= -4z^2+16xw,\end{aligned}$$

we have

$$\begin{aligned} \tfrac{1}{3}I &= L^2-12w^2M,\\ -J &= L^3-18w^2LM-54w^4N,\end{aligned}$$

and the equation is, as in former cases,

$$L^2(LN+M^2)-18w^2LMN-16w^2M^3-27w^4N^2=0;$$

but $LN+M^2$ and therefore the whole equation divides by w, and we thus obtain

$$16L^2\big(-xz^2+y^2x+w\,(yz+4x^2)+w^3\big)-18wLMN-16wM^3-27w^3N^2=0;$$

or, completely developed, this is

$$\begin{array}{lll} & w^7\,.\,64 & \\ + & w^5\,.\,32 & (\ 3yz-4x^2) \\ + & w^4\,.\,16x & (\ 5y^2+9z^2) \\ + & w^3\,. & (\ y^4+30y^2z^2+160yzx^2-27z^4+64x^4) \\ + & w^2\,.\ \ 4x & (11y^3z+12y^2x^2-9yz^3-4z^2x^2) \\ + & w\ \ .\ \ y^2 & (\ y^3z+12y^2x^2-yz^3-8z^2x^2) \\ + & \quad y^4x & (\ y^2-z^2)=0. \end{array}$$

111. To transform the equation so as to put in evidence the nodal curve, I collect the terms according to their degrees in (y, z) and (x, w); viz. the equation thus becomes

$$\begin{aligned}
&64x^4w^3-128x^2w^5+64w^7\\
&+z^2(\;-16x^3w^2+144xw^4)\\
&+zy(\;\;160x^2w^3+\;\;96w^5)\\
&+y^2(\;\;\;48x^3w^2+\;\;80xw^4)\\
&+z^4\;.-27w^3\\
&+z^3y\;.-36xw^2\\
&+z^2y^2\,.-8x^2w+30w^3\\
&+zy^3\;.\;44xw^2\\
&+y^4\;.\;12x^2w+w^3\\
&+z^3y^3\,.-w\\
&+z^2y^4\,.-x\\
&+zy^5\;.\;w\\
&+y^6\;\;.\;x\qquad\qquad=0\,;
\end{aligned}$$

and if for a moment we write $z=\alpha+\gamma$, $y=\alpha-\gamma$ and collect, ultimately replacing α, γ by their values $\frac{1}{2}(z+y)$, $\frac{1}{2}(z-y)$, the equation can be expressed in the form

$$\begin{aligned}
&\;\;\;64w^3(x^2-w^2)^2\\
&+\;\;8w^2(z+y)^2(x+w)^2(x+3w)\\
&+\;\;8w^2(z-y)^2(x-w)^2(x-3w)\\
&-32w^2(z^2-y^2)\,(x^2-w^2)\,x\\
&+\;\;\tfrac{1}{4}w\;(z+y)^4(x+w)^2\\
&-\quad w\;(z+y)^3(z-y)(x+w)(3x+7w)\\
&+\;\;\tfrac{1}{2}w\;(z^2-y^2)^2(11x^2-27w^2)\\
&-\quad w\;(z+y)\;(z-y)^3(x-w)(3x-7w)\\
&+\;\;\tfrac{1}{4}w\;(z-y)^4(x-w)^2\\
&-\quad y^3\;(z^2-y^2)\;(zw+xy)=0,
\end{aligned}$$

and observing that we have

$$\begin{aligned}
zw+xy&=-z\,(x-w)+x\,(z+y)\\
&=\;\;z\,(x+w)-x\,(z-y),
\end{aligned}$$

we see that every term of the equation is at least of the second order in $z+y$ and $x-w$ conjointly; and also at least of the second order in $z-y$ and $x+w$ conjointly;

that is, the surface has the nodal lines $(z+y=0,\ x-w=0)$ and $(z-y=0,\ x+w=0)$, which are the reciprocals of the lines 12′ and 13′ respectively. The nodal curve is made up of these two lines and of the line $y=0$, $w=0$ (reciprocal of edge), as will presently appear; so that we have $b'=3$.

112. The equations of the cuspidal curve are

$$\begin{aligned} L^2 - 12w^2M &= 0, \\ LM + \ \ 9w^2N &= 0, \\ 4M^2 + \ \ 3LN &= 0. \end{aligned}$$

Attending to the two equations

$$\begin{aligned} L^2 - 12w^2M &= y^4 + 8y^2xw + 16x^2w^2 + 24yzw^2 + 48w^4 = 0, \\ LM + 9w^2N &= y^3z + 2y^2w^2 + 4xyzw + (8 - 72 =) - 64xw^3 + 18z^2w^2 = 0, \end{aligned}$$

these surfaces are each of the order 4, and the order of their intersection is $=16$. But the two surfaces contain in common the line $(y=0,\ w=0)$ 7 times; in fact on the first surface this is a cusp-nodal line $4xw + y^2 + Ay^{\frac{5}{2}} = 0$; and on the second surface it is a nodal line $w(4xy + 18zw) = 0$; the sheet $w=0$ is more accurately $4xw + y^2 + By^3 \ldots = 0$; whence in the intersection with the first surface the line counts 5 times in respect of the first sheet and 2 times in respect of the second sheet; together $(5+2=)$ 7 times, and the residual curve is of the order $(16-7=)$ 9.

113. I say that the cuspidal curve is made up of this curve of the 9th order, and of the line $y=0$, $w=0$ (reciprocal of the edge) once; so that $c'=10$. In fact, considering the line in question $y=0$, $w=0$ in relation to the surface, the equation of the surface (attending only to the lowest terms in y, w) may be written

$$-xz^2(y^2+4xw)^2 + w(-y^3z^2) + w^2(-36xyz^3) + \&\text{c.} = 0,$$

giving in the vicinity of the line

$$4xw + y^2 = Ay^{\frac{5}{2}},$$

and then

$$-xz^2A^2 + \frac{z^3}{x}\left(\tfrac{1}{4} - \tfrac{36}{16}\right) = 0,$$

that is, $A^2 = -2\,\dfrac{z}{x^2}$ or $4xw + y^2 = \sqrt{-2}\,.\,\dfrac{z^{\frac{1}{2}}}{x}\,y^{\frac{5}{2}}$; wherefore the line is a cusp-nodal line, counting once as a nodal and once as a cuspidal line; and so giving the foregoing results $b'=3$, $c'=10$.

114. I revert to the equation which exhibits the nodal lines $(x-w=0,\ y+z=0)$, $(x+w=0,\ y-z=0)$ for the purpose of showing that they have respectively no pinch-points; that is, that in regard to each of them we have $j'=0$. In fact for the first

of these lines, neglecting the terms which contain $x-w$, $y+z$ conjointly in an order above the second, the equation may be written

$$\begin{array}{lll} & 64w^3(x+w)^2 & (x-w)^2 \\ + & 8w^2(x+w)^2(x+3w)(z+y)^2 & \\ + & 8w^2(z-y)^2(x-3w)(x-w)^2 & \\ - & 32w^2(z-y)\ (x+\ w)\,x & (x-w)(z+y) \\ + & \tfrac{1}{2}w\ (z-y)^2(11x^2-27w^2)(z+y)^2 & \\ - & w\ (z-y)^3(3x-7w) & (x-w)(z+y) \\ + & \tfrac{1}{4}w\ (z-y)^4 & (x-w)^2 \\ + & y^3z\ (z-y) & (x-w)(z+y) \\ - & y^3x\ (z-y) & (z+y)^2=0, \end{array}$$

viz. this is

$$(A,\ B,\ C \between x-w,\ z+y)^2=0,$$

where, collecting the terms and reducing the values by means of the equations $x-w=0$, $z+y=0$, or say by writing $x=w$, $-y=z$, we have

$$\begin{array}{rlrl} A= & 64w^3(x+w)^2 & = & 256w^5 \\ & +\ 8w^2(z-y)^2(x-3w) & - & 64w^3z^2 \\ & +\ \tfrac{1}{4}w^2(z-y)^4 & + & 4wz^4 \\ & & = & 4w(z^2-8w^2)^2, \\ B= & -32(z-y)(x+w)\,xw^2 & =- & 128w^4z \\ & -w(z-y)^3(3x-7w) & + & 32w^2z^3 \\ & +y^3z(z-y) & - & 2z^5 \\ & & =- & 2z(z^2-8w^2)^2, \\ C= & 8w^2(x+w^2)(x+3w) & = & 128w^5 \\ & +\ \tfrac{1}{2}w(z-y)^2(11x^2-27w^2) & - & 32w^3z^2 \\ & -\ xy^3(z-y) & + & 2wz^4 \\ & -\ xy^3(z-y) & + & 2wz^4 \\ & & = & 2w(z^2-8w^2)^2. \end{array}$$

Hence the condition $4AC-B^2=0$ of a pinch-point is $(z^2-8w^2)^5=0$, so that the pinch-points (if any) would be at the points $x-w=0$, $y+z=0$, $z^2-8w^2=0$; or say at $x, y, z, w = 1, -2\sqrt{2}, 2\sqrt{2}, 1$. But these values give $L, M, N = 12, 12, -16$; values which satisfy the equations $L^2-12w^2M=0$, $LM+9w^2N=0$, $4M^2+3LN=0$, and as the points in question are obviously not on the line $y=0$, $w=0$, they lie on the ninthic component of the cuspidal curve, being in fact points β', and not pinch-points.

The line $y=0$, $w=0$ *quà* nodal line would have every point a pinch-point, but being part of the cuspidal curve, no point thereof is regarded as a pinch-point; that is, in regard to this line also we have $j'=0$. And therefore for the entire nodal curve $j'=0$.

115. The cuspidal ninthic curve is a unicursal curve, the equations of which can be very readily obtained by considering it as the reciprocal of the spinode torse; we in fact have

$$x : y : z : w = ZW + 2XY : 2YZ + X^2 : WX + Y^2 - 3Z^2 : ZX,$$

or substituting for X, Y, Z, W their values ($= 16\theta^2$, $4\theta + 16\theta^5$, $16\theta^3$, $-5 - 8\theta^4 - 16\theta^8$) and omitting a common factor $16\theta^2$, we find for the cuspidal curve

$$x : y : z : w = 3\theta + 24\theta^5 - 16\theta^9 : 24\theta^2 + 32\theta^6 : -4 - 48\theta^4 : 16\theta^3$$

(values which verify the equation $Xx + Yy + Zz + Ww = 0$); the spinode curve being thus of the order $= 9$ as mentioned.

For $\theta = \infty$ we have the singular point ($y = 0$, $z = 0$, $w = 0$) (reciprocal of torsal biplane), and in the vicinity thereof $x : y : z : w = 1 : -2\theta^{-3} : 3\theta^{-5} : -\theta^{-6}$, therefore

$$\left(\frac{y}{x}\right)^2 = -4\frac{w}{x}, \quad \left(\frac{y}{x}\right)^5 = -\tfrac{32}{27}\left(\frac{z}{x}\right)^3.$$

For $\theta = 0$ we have the singular point $x = 0$, $y = 0$, $w = 0$ (reciprocal of the other biplane), and in the vicinity thereof $x : y : z : w = -\tfrac{3}{4}\theta : -6\theta^2 : 1 : -4\theta^3$, therefore

$$\frac{y}{z} = -\tfrac{32}{3}\left(\frac{x}{z}\right)^2, \quad \frac{w}{z} = \tfrac{256}{27}\left(\frac{x}{z}\right)^3.$$

116. The section of the surface by the plane $z = 0$ is an interesting curve. Writing $z = 0$ in the equation of the surface, I find that the resulting equation may be written

$$(64w^3,\ 144xw^2,\ w^3 + 76x^2w + xy^2)(w^2 + 27x^2,\ y^2 - 32xw)^2 = 0,$$

where observe that

$$64w^3(w^3 + 76x^2w + xy^2) - (72xw^2)^2 = 64w^3[w(w^2 + 27x^2) + x(y^2 - 32xw)];$$

so that the curve has the four cusps $w^2 + 27x^2 = 0$, $y^2 - 32xw = 0$; the plane $z = 0$ intersects the cuspidal ninthic curve in the point ($y = 0$, $z = 0$, $w = 0$) counting 5 times, and in the last-mentioned four points: in fact, writing in the equations of the ninthic curve $z = 0$, that is $1 + 12\theta^4 = 0$, we find $x, y, w = \tfrac{8}{9}\theta, \tfrac{64}{3}\theta^2, 16\theta^3$, and thence $w^2 + 27x^2 = \tfrac{64}{3}\theta^2(1 + 12\theta^4) = 0$, $y^2 - 32xw = 0$.

The curve has also nodes at the points ($y = 0$, $x + w = 0$; $y = 0$, $x - w = 0$), viz. these are the intersections of the plane $z = 0$ with the nodal lines ($y - z = 0$, $x + w = 0$) and ($y + z = 0$, $x - w = 0$), and it has at the point ($y = 0$, $w = 0$) (intersection of its plane with the cusp-nodal line $y = 0$, $w = 0$, and quintic intersection with the cuspidal ninthic) a singular point $= 2$ cusps $+ 7$ nodes; hence the curve has cusps $= (4 + 2 =) 6$; nodes $(2 + 7 =) 9$; or 2 nodes $+ 3$ cusps $= 36$; class $= 6$, as it should be.

Section VIII $= 12 - 3C_2$.

Article Nos. 117 to 125. Equation $Y^3 + Y(X+Z+W) + 4aXZW = 0$.

117. The diagram of the lines and planes is

		Lines.												
Planes.		6	5	4	3	2	1	$\bar{9}$	$\bar{8}$	$\bar{7}$	9	8	7	
VIII $=12-3C_2$.		$\overline{12}$ $\overline{27}$			6 × 2 = 12				3 × 1 = 3			3 × 4 = 12		
7	3 × 2 = 6									• •			• •	Planes each touching along an axis and containing the corresponding transversal.
8									• •			• •		
9								• •			• •			
12	3 × 2 = 6					•	•			• •				Biradial planes each containing two rays of the same node.
34				•	•				• •					
56		•	•					• •						
13	6 × 4 = 24				• •		• •					•		Planes each containing an axis, and two rays through the terminal nodes respectively.
24				• •		• •						•		
16		• •					• •				•			
25			• •			• •					•			
46		• •		• •									•	
35			• •		• •								•	
789	1 × 8 = 8										• •	• •	• •	Plane through the three axes.
$\overline{789}$	1 × 1 = 1							•	•	•				Plane through the three transversals.
$\bar{1}4$	45													
		Rays, 1, 2 through node 89. 3, 4 " 97. 5, 6 " 78.						Transversals, meeting the three axes respectively.			Axes, each joining two nodes.			

118. Take m_1, m_2 as the roots of the equation $(m-1)^2 = 4am$, so that $m_1 + m_2 = 2 + 4a$, $m_1 m_2 = 1$, then the planes are

$$
\begin{array}{ll}
X = 0, & [\ 7] \\
Y = 0, & [\ 8] \\
Z = 0, & [\ 9] \\
Y + Z + X = 0, & [12] \\
Y + X + W = 0, & [\ 34] \\
Y + Z + W = 0, & [\ 56] \\
Y = (m_1 - 1) X, & [\ 13] \\
Y = (m_2 - 1) X, & [\ 24] \\
Y = (m_2 - 1) Z, & [\ 16] \\
Y = (m_1 - 1) Z, & [\ 25] \\
Y = (m_1 - 1) W, & [\ 46] \\
Y = (m_2 - 1) W, & [\ 35] \\
Y = 0, & [789] \\
Y + X + Z + W = 0, & [789]
\end{array}
$$

119. And the lines are

a	b	c	f	g	h	equations may be written
0	0	0	0	0	1	(7) $X = 0,\ Y = 0$
0	0	0	1	0	0	(8) $Z = 0,\ Y = 0$
0	1	0	0	0	0	(9) $W = 0,\ Y = 0$
1	1	1	0	0	0	$(\bar{7})$ $Y + Z + X = 0,\ W = 0$
0	0	1	-1	1	0	(8) $Y + X + W = 0,\ Z = 0$
1	0	0	0	-1	1	(9) $Y + Z + W = 0,\ X = 0$
0	0	0	$\frac{1}{m_1 - 1}$	1	$\frac{1}{m_2 - 1}$	(1) $Y = (m_1 - 1) X = (m_2 - 1) Z$
0	0	0	$\frac{1}{m_2 - 1}$	1	$\frac{1}{m_1 - 1}$	(2) $Y = (m_2 - 1) X = (m_1 - 1) Z$
-1	$\frac{1}{m_1 - 1}$	0	0	0	$\frac{1}{m_2 - 1}$	(3) $Y = (m_2 - 1) W = (m_1 - 1) X$
-1	$\frac{1}{m_2 - 1}$	0	0	0	$\frac{1}{m_1 - -1}$	(4) $Y = (m_1 - 1) W = (m_2 - 1) X$
0	$-\frac{1}{m_1 - 1}$	1	$\frac{1}{m_2 - 1}$	0	0	(5) $Y = (m_1 - 1) Z = (m_2 - 1) W$
0	$-\frac{1}{m_2 - 1}$	1	$\frac{1}{m_1 - 1}$	0	0	(6) $Y = (m_2 - 1) Z = (m_1 - 1) W$

120. The three transversals are each facultative; $\rho' = b' = 3$; $t' = 0$.

121. Hessian surface. The equation is

$$4aXZW(3Y + X + Z + W) + Y^2(X^2 + Z^2 + W^2 - 2XZ - 2XW - 2ZW) = 0.$$

The complete intersection with the cubic surface is made up of the lines ($Y=0$, $X=0$), ($Y=0$, $Z=0$), ($Y=0$, $W=0$) (the axes) each twice, and of a sextic curve which is the spinode curve; $\sigma' = 6$.

The spinode curve is a complete intersection 2×3; the equations may in fact be written

$$\begin{gathered} Y^3 + \ Y^2(X + Z + W) + 4aXZW = 0, \\ 3Y^2 + 4Y\ (X + Z + W) + 4\,(XZ + XW + ZW) = 0; \end{gathered}$$

the nodes D, C, A are nodes (double points) of the curve, the tangents at each node being the nodal rays.

Each of the transversals is a single tangent of the spinode curve; in fact for the transversal $Y + Z + X = 0$, $W = 0$, these equations of course satisfy the equation of the cubic surface; and substituting in the equation of the Hessian, we have $Y^2(X - Z)^2 = 0$. But $Y + Z + X = 0$, $W = 0$, $Y = 0$ is a point on the axis $W = 0$, $Y = 0$, not belonging to the spinode curve; we have only the point of contact $Y + X + Z = 0$, $W = 0$, $X - Z = 0$. Hence $\beta' = 3$.

Reciprocal Surface.

122. The equation is found by means of the binary cubic,

$$aT(T - yU)^2 + (T - xU)(T - zU)(T - wU),$$

viz. writing for shortness

$$\begin{aligned} \beta &= x + z + w, \\ \gamma &= xz + xw + zw, \\ \delta &= xzw; \end{aligned}$$

this is a binary cubic $(*\between T,\ U)^3$, the coefficients whereof are

$$3\,(a + 1),\ -2ay - \beta,\ ay^2 + \gamma,\ -3\delta,$$

and the equation is hence found to be

$$\begin{aligned} &4a^3y^3(y^3 - \beta y^2 + \gamma y - \delta) \\ &+ a^2\{(12\gamma - \beta^2)\,y^4 - (8\beta\gamma + 36\delta)\,y^3 + (30\beta\delta + 8\gamma^2)\,y^2 -- 36\gamma\delta y + 27\delta^2\} \\ &+ 2a\,\{(6\gamma^2 - \beta^2\gamma - 9\beta\delta)\,y^2 + (12\beta^2\delta - 2\beta\gamma^2 - 18\gamma\delta)\,y + 2\gamma^3 + 27\delta^2 - 9\beta\gamma\delta\} \\ &- (\beta^2\gamma^2 + 18\beta\gamma\delta - 4\beta^3\delta - 4\gamma^3 - 27\delta^2) = 0; \end{aligned}$$

or substituting for β, γ, δ in the first and last lines their values

$$(= x + z + w,\ xz + xw + zw,\ xzw),$$

this is

$$\begin{aligned}
&4a^3y^3(y-x)(y-z)(y-w)\\
&+a^2\{(12\gamma-\beta^2)y^4-(8\beta\gamma+36\delta)y^3+(30\beta\delta+8\gamma^2)y^2-36\gamma\delta y+27\delta^2\}\\
&+2a\{(6\gamma^2-\beta^2\gamma-9\beta\delta)y^2+(12\beta^2\delta-2\beta\gamma^2-18\gamma\delta)y+2\gamma^3+27\delta^2-9\beta\gamma\delta\}\\
&-(x-z)^2(x-w)^2(z-w)^2=0.
\end{aligned}$$

123. The nodal curve is made up of the lines $(y = x = z)$, $(y = x = w)$, $(y = z = w)$, reciprocals of the three transversals.

To show this I remark that, writing

$$\begin{aligned}
\beta' &= (x-y)+(z-y)+(w-y),\\
\gamma' &= (x-y)(z-y)+(x-y)(w-y)+(z-y)(w-y),\\
\delta' &= (x-y)(z-y)(w-y),
\end{aligned}$$

the equation of the surface may be written

$$\begin{aligned}
&4a^3y^3(y-x)(y-z)(y-w)\\
&+a^2\{y^2(12\beta'\delta'-\gamma'^2)+x\,.\,18\gamma'\delta'+27\delta'^2\}\\
&+2a\{y(-6\beta'^2\delta'+2\beta'\gamma'^2+9\gamma'\delta')+2\gamma'^3+27\delta'^2-9\beta'\gamma'\delta'\}\\
&-(x-z)^2(x-w)^2(z-w)^2=0,
\end{aligned}$$

whence observing that γ' is of the order 1 and δ' of the order 2 in $(x-y)$, $(z-y)$ conjointly, each term of the equation is at least of the second order in $(x-y)$, $(z-y)$ conjointly; or we have $y = x = z$, a nodal line; and similarly the other two lines are nodal lines.

124. The foregoing transformed equation is most readily obtained by reverting to the cubic in T, U, viz. writing $p = x - y$, $r = z - y$, $s = w - y$, and therefore $x = y + p$, $z = y + r$, $w = y + s$, the cubic function (putting therein $T = V + yU$) becomes

$$a(V+yU)V^2+(V-pU)(V-rU)(V-sU);$$

writing $\beta', \gamma', \delta' = p + r + s,\ pr + ps + rs,\ prs$, the coefficients are $(3(a+1),\ ay-\beta',\ \gamma',\ -3\delta')$, and the equation of the surface is thus obtained in the form

$$\begin{aligned}
&27(a+1)^2\delta'^2\\
&+18(a+1)(ay-\beta')\gamma'\delta'\\
&+\ \ 4(a+1)\gamma'^3\\
&-\ \ 4(ay-\beta')^3\delta'\\
&-\ \ \ (ay-\beta')^2\gamma'^2=0,
\end{aligned}$$

which, arranging in powers of a, and reversing the sign, is the foregoing transformed result.

125. The cuspidal curve is given by the equations

$$\left\|\begin{matrix} 3(a+1), & -2ay-\beta, & ay^2+\gamma \\ -2ay-\beta, & ay^2+\gamma, & -\delta \end{matrix}\right\| = 0,$$

or say by the equations

$$3(a+1)(ay^2+\gamma)-(2ay+\beta)^2=0,$$

that is

$$a(a-3)y^2+4a\beta y-3(a+1)\gamma=0,$$

and

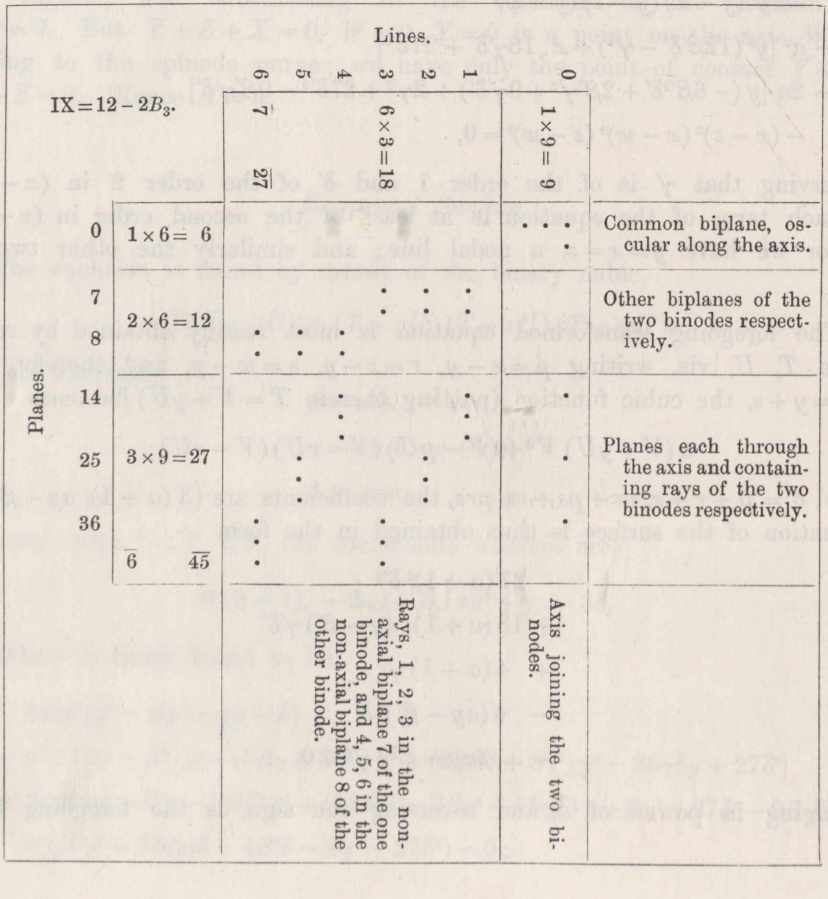
$$-3(a+1)\delta+(2ay+\beta)(ay^2+\gamma)=0,$$

consequently $c'=6$. It is to be added that the cuspidal curve is a complete intersection, 2×3.

Section $\text{IX}=12-2B_3$.

Article Nos. 126 to 136. Equation $WXZ+(a, b, c, d\between X, Y)^3=0$.

126. The diagram of the lines and planes is



127. Writing $(a, b, c, d \between X, Y)^3 = -d\,(f_1X - Y)\,(f_2X - Y)\,(f_3X - Y)$, the planes are

$$\begin{array}{ll} X = 0, & [\ 0] \\ Z = 0, & [\ 7] \\ W = 0, & [\ 8] \\ f_1X - Y = 0, & [14] \\ f_2X - Y = 0, & [25] \\ f_3X - Y = 0, & [36]\,; \end{array}$$

and the lines are

$$\begin{array}{lll} X = 0, & Y = 0, & (0) \\ f_1X - Y = 0, & Z = 0, & (1) \\ f_2X - Y = 0, & Z = 0, & (2) \\ f_3X - Y = 0, & Z = 0, & (3) \\ f_1X - Y = 0, & W = 0, & (4) \\ f_2X - Y = 0, & W = 0, & (5) \\ f_3X - Y = 0, & W = 0, & (6). \end{array}$$

128. There is no facultative line; $\rho' = b' = 0$, $t' = 0$; and hence also $\beta' = 0$.

129. Hessian surface. The equation is

$$X\,\{ZW\,(cX + dY) - 3X\,(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2\} = 0,$$

so that the Hessian breaks up into the plane $X = 0$ (axial or common biplane) and into a cubic surface.

The complete intersection of the Hessian with the cubic surface is made up of the line $X = 0$, $Y = 0$ (the axis) four times; and of a system of four conics, which is the spinode curve; $c' = 8$.

In fact combining the equations

$$WXZ + (a,\ b,\ c,\ d \between X,\ Y)^3 = 0$$

and

$$ZW\,(cX + dY) - 3X\,(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2 = 0,$$

these intersect in the axis once, and in a curve of the eighth order which breaks up into four conics; for we can from the two equations deduce

$$(a,\ b,\ c,\ d \between X,\ Y)^3\,(cX + dY) + 3X^2\,(ac - b^2,\ ad - bc,\ bd - c^2 \between X,\ Y)^2 = 0,$$

that is

$$(4ac - 3b^2,\ ad,\ bd,\ cd,\ d^2 \between X,\ Y)^4 = 0,$$

a system of four planes each intersecting the cubic $XZW + (a,\ b,\ c,\ d \between X,\ Y)^3 = 0$ in the axis and a conic; whence, as above, spinode curve is four conics.

It is easy to see that the tangent planes along any conic on the surface pass through a point, and form therefore a quadric cone; hence in particular the spinode torse is made up of the quadric cones which touch the surface along the four conics respectively.

Reciprocal Surface.

130. The equation is obtained by means of the binary cubic

$$X(xX+yY)^2+4zw\,(a,\ b,\ c,\ d\,\between X,\ Y)^3,$$

viz. calling this $(*\between X,\ Y)^3$ the coefficients are

$$(3x^2+12azw,\ 2xy+12bzw,\ y^2+12czw,\ 12dzw).$$

The equation is found to be

$$\begin{aligned}&432\,(a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2)\,z^3w^3\\ &+216\,[(ad^2-3bcd+2c^3)\,x^2+(-2acd+4b^2d-2bc^2)\,xy+(-abd+2ac^2-b^2c)\,y^2]\,z^2w^2\\ &+\quad 9\,[3d^2x^4-12cdx^3y+(10bd+8c^2)\,x^2y^2-(4ad+8bc)\,xy^3+(4ac-b^2)\,y^4]\,zw\\ &-\quad y^3\,(dx^3-3cx^2y+3bxy^2-ay^3)=0.\end{aligned}$$

The section by the plane $w=0$ (reciprocal of $B_3=D$) is the line $w=0$, $y=0$ (reciprocal of edge) three times, and the lines $w=0$, $dx^3-3cx^2y+3bxy^2-ay^3=0$ (reciprocals of the biplanar rays). And similarly for the section by the plane $z=0$ (reciprocal of $B_3=C$).

The section by the plane $y=0$ is made up of the lines $(y=0,\ z=0)$, $(y=0,\ w=0)$ each once, and of two conics, $y=0$,

$$\begin{aligned}&16\,(a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2)\,z^2w^2\\ &+\ \ 8\,(ad^2-3bcd+2c^3)\,x^2zw\\ &+\quad d^2x^4=0.\end{aligned}$$

131. There is not any nodal curve; $b'=0$.

132. Cuspidal curve. The equations may be written

$$\left\|\begin{matrix}3x^2+12azw, & 2xy+12bzw, & y^2+12czw\\ 2xy+12bzw, & y^2+12czw, & 12dzw\end{matrix}\right\|=0.$$

Forming the equations

$$\begin{aligned}&(bd-c^2)\,.\,144z^2w^2+2\,(dxy-cy^2)\,.\,12zw\ -y^4=0,\\ &(ad-bc)\,.\,144z^2w^2+(3dx^2-2cxy-by^2)\,.\,12zw\ -2xy^3=0,\end{aligned}$$

these are two quartic surfaces having in common the lines $(y=0,\ w=0)$, $(y=0,\ z=0)$: attending to the line $(y=0,\ z=0)$, this is on the second surface an oscular line, $z=\dfrac{1}{18dxw}y^3$; on the first surface it is a nodal line, the one tangent plane being $6\,(bd-c^2)\,w\,.\,z+dxz\,.\,y=0$, the other tangent plane being $z=0$, but the line being in regard to this sheet an oscular line, $z=\dfrac{1}{24dxw}y^3$. Hence in the intersection of the two surfaces the line counts $(1+3=)$ 4 times; similarly the line $y=0$, $w=0$ counts $(1+3=)$ four times; and there is a residual intersection of the order $(16-4-4=)$ 8, which is the cuspidal curve; $c'=8$.

133. The cuspidal curve is a system of four conics; in fact from the preceding equations written in the forms

$$(bd - c^2,\ 2\,(dxy - cy^2),\ -y^4 \between 12zw,\ 1)^2 = 0,$$

$$(ad - bc,\ 3dx^2 - 2cxy - by^2,\ -2xy^3 \between 12zw,\ 1)^2 = 0,$$

eliminating zw, we obtain

$$\left(\begin{array}{l} 3\,(bd - c^2), \\ 2\,(-ad^2 - 3bcd + 4c^3), \\ 6\,(acd + b^2d - 2bc^2), \\ 6\,(bc - ad)\,b, \\ a^2d - b^3, \end{array}\right) \between x,\ y)^4 = 0,$$

which shows that the cuspidal curve lies in four planes, and it hence consists of four conics; these are of course the reciprocals of the quadric cones which touch the cubic surface along the four conics which make up the spinode curve.

134. The equation of the surface, attending only to the terms of the second order in y, z, w, is $27d^2x^4zw = 0$; it thus appears that the point $y = 0$, $z = 0$, $w = 0$ (reciprocal of the plane $X = 0$) (which is oscular along the axis joining the two binodes, or BB-axis) is a binode on the reciprocal surface, the biplanes being $z = 0$, $w = 0$, viz. these are the planes reciprocal to the binodes ($X = 0$, $Y = 0$, $W = 0$) and ($X = 0$, $Y = 0$, $Z = 0$) of the cubic surface; we have thus $B' = 1$.

It is proper to remark that the binode $y = 0$, $z = 0$, $w = 0$ is not on the cuspidal curve, as its being so would probably imply a higher singularity.

135. A simple case, presenting the same singularities as the general one, is when $a = d$, $b = c = 0$: to diminish the numerical coefficients assume $a = d = \frac{1}{12}$, the cubic surface is thus $12XZW + X^3 + Y^3 = 0$, and the equation of the reciprocal surface, multiplying it by 4, becomes

$$\begin{aligned} & z^3w^3 \\ & + 6x^2z^2w^2 \\ & + (9x^4 - 12x^3y)\,zw \\ & - 4y^3\,(x^3 - y^3) = 0, \end{aligned}$$

viz. this is the surface

$$\begin{aligned} & 4y^6 \\ & - 4y^3x\,(x^2 + 3zw) \\ & + zw\,(3x^2 + zw)^2 = 0 \end{aligned}$$

considered in the Memoir "On the Theory of Reciprocal Surfaces." The cuspidal curve is, as there shown, composed of the four conics $y = 0$, $3x^2 + zw = 0$ and $y^3 - 2x^3 = 0$, $x^2 - zw = 0$; and it is there shown that the two points ($x = 0$, $y = 0$, $z = 0$), ($x = 0$, $y = 0$, $w = 0$), each reckoned eight times, are to be considered as off-points of the reciprocal surface.

136. The like investigation applies to the general surface, and we have thus $\theta'=16$; the points in question are still the points $(x=0, y=0, z=0)$, $(x=0, y=0, w=0)$; viz. these are the points of intersection of the surface by the line $(x=0, y=0)$, which points are also the common points of intersection of the four conics which compose the cuspidal curve, that is, they are quadruple points on the cuspidal curve; it does not appear that the points are on this account, viz. *quà* quadruple points of the cuspidal curve, off-points of the surface, nor does this even show that the points should be reckoned each eight times. As already remarked, the singularity requires a more complete investigation.

Section $X=12-B_4-C_2$.

Article Nos. 137 to 143. Equation $WXZ+(X+Z)(Y^2-X^2)=0$.

137. The diagram of the lines and planes is

		Lines.					
$X=12-B_4-C_2$.		12	1′ 2′	1 2	3	0	
		1×1= 1	2×2= 4	2×4= 8	1×6= 6	1×8= 8	
		$\overline{7}$ $\overline{27}$					
Planes. 0	1×12=12				• •	• • • •	Biplane touching along axis, and containing edge.
3	1×12=12			• • • • • •	• •		Other biplane.
11′ 22′	2× 8=16		• • • • • • • •	• • • •		• •	Planes each through the axis and containing a biplanar ray and a cnicnodal ray.
3′	1× 3= 3	• •			• •		Plane touching along the edge and containing the mere line.
1′2′	1× 2= 2	• •	• •				Biradial plane through the two cnicnodal rays.
	$\overline{6}$ $\overline{45}$						
		Mere line, being a transversal.	Cnicnodal rays.	Biplanar rays in the non axial biplane.	Edge of binode, being a transversal.	Axis, through the two nodes.	

138. The planes are and the lines are

$X=0,$	[0]	$X=0,\ Y=0,$	(0)
$Z=0,$	[3]	$X=0,\ Z=0,$	(3)
$X-Y=0,$	[11′]	$X-Y=0,\ Z=0,$	(1)
$X+Y=0,$	[22′]	$X+Y=0,\ Z=0,$	(2)
$W=0,$	[3′]	$X-Y=0,\ W=0,$	(1′)
$X+Z=0,$	[1′2′]	$X+Y=0,\ W=0,$	(2′)
		$X+Z=0,\ W=0,$	(12).

139. The facultative lines are the edge counting twice, and the mere line;

$$\rho=b'=3\,;\ t'=1.$$

140. Hessian surface. The equation is

$$X(X+Z)(ZW+3X^2-XZ)+Y^2(X-Z)^2=0.$$

The complete intersection with the surface consists of the line ($X=0$, $Y=0$), the axis, four times; the line ($X=0$, $Z=0$), the edge, twice; and a sextic curve, which is the spinode curve; $c'=6$.

Writing the equations of the surface and the Hessian in the form

$$X(ZW+Y^2)-X^3+Z(Y^2-X^2)=0,$$

$$X(X+Z)(ZW+Y^2)+(Z-3X)\{-X^3+Z(Y^2-X^2)\}=0,$$

we see that the equations of the spinode curve may be written

$$ZW+Y^2=0,$$

$$-X^3+Z(Y^2-X^2)=0,$$

viz. the curve is a complete intersection, 2×3.

There is at B_4 a triple point $\frac{Y}{W}=-\left(\frac{Z}{W}\right)^2$, $\frac{X}{W}=-\left(\frac{Z}{W}\right)^{\frac{4}{3}}$; and at C_2 a double point, the tangents coinciding with the nodal rays $W=0$, $Y^2-X^2=0$.

The edge and the mere line are each of them single tangents of the spinode curve. But the edge counting twice in the nodal curve, its contact with the spinode curve will also count twice, that is, we have $\beta'=2.1+1,=3$.

Reciprocal Surface.

141. The equation is obtained by means of the binary cubic

$$4w^2X(X+Z)^2+4wZ(X+Z)(xX+zZ)+y^2XZ^2\,;$$

or calling this $(*\between X,Z)^3$, the coefficients are

$$(12w^2,\ 8w^2+4wx,\ 4w^2+4wx+4wz+y^2,\ 12wz).$$

136. The like investigation applies to the general surface, and we have thus $\theta' = 16$; the points in question are still the points $(x=0, y=0, z=0)$, $(x=0, y=0, w=0)$; viz. these are the points of intersection of the surface by the line $(x=0, y=0)$, which points are also the common points of intersection of the four conics which compose the cuspidal curve, that is, they are quadruple points on the cuspidal curve; it does not appear that the points are on this account, viz. *quà* quadruple points of the cuspidal curve, off-points of the surface, nor does this even show that the points should be reckoned each eight times. As already remarked, the singularity requires a more complete investigation.

Section $X = 12 - B_4 - C_2$.

Article Nos. 137 to 143. Equation $WXZ + (X+Z)(Y^2 - X^2) = 0$.

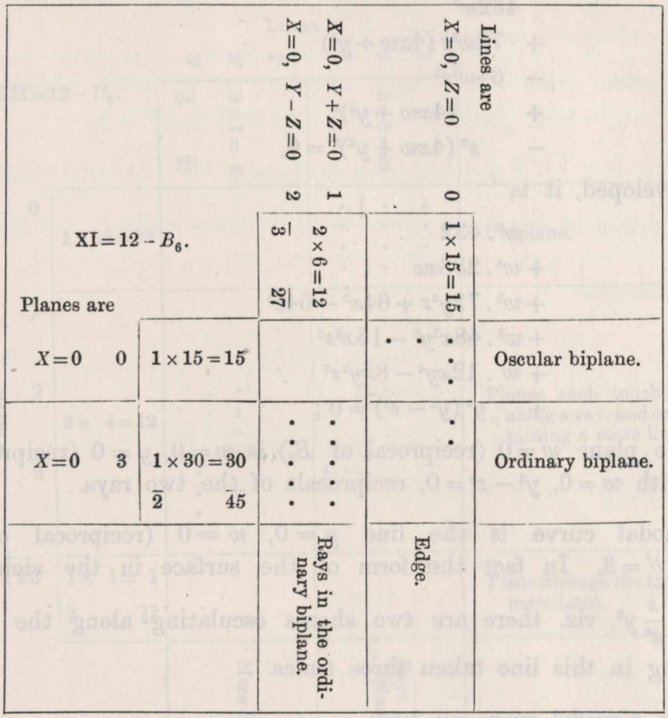
137. The diagram of the lines and planes is

			Lines.							
$X = 12 - B_4 - C_2$.			0	3	1	2	1′	2′	12	
			1×8=8	1×6=6	2×4=8		2×2=4		1×1=1	
									$\overline{7}$ $\overline{27}$	
Planes.	0	1×12=12	• • • •	• •						Biplane touching along axis, and containing edge.
	3	1×12=12		• •	• • •	• • •				Other biplane.
	11′	2× 8=16	•		• •		• • • •			Planes each through the axis and containing a biplanar ray and a cnicnodal ray.
	22′		•			• •		• • • •		
	3′	1× 3= 3		• •					• • •	Plane touching along the edge and containing the mere line.
	1′2′	1× 2= 2					•	•	• •	Biradial plane through the two cnicnodal rays.
	$\overline{6}$	$\overline{45}$								
			Axis, through the two nodes.	Edge of binode, being a transversal.	Biplanar rays in the non-axial biplane.		Cnicnodal rays.		Mere line, being a transversal.	

Section $\mathrm{XI} = 12 - B_6$.

Article Nos. 144 to 149. Equation $WXZ + Y^2Z + X^3 - Z^3 = 0$.

144. The diagram of the lines and planes is



where the equations of the lines and planes are shown in the margins of the diagram.

145. The edge is a facultative line counting three times; this will appear from the discussion of the reciprocal surface. Therefore $\rho' = b' = 3$; $t' = 1$.

146. Hessian surface. This is

$$Z(WXZ + Y^2Z - 3X^3 - 3Z^3) = 0,$$

breaking up into $Z = 0$, the oscular biplane, and into a cubic surface (itself a surface $\mathrm{XI} = 12 - B_6$). The complete intersection with the cubic surface is made up of the line $X = 0$, $Z = 0$ (the edge) six times, and of a residual sextic ($= 3$ conics), which is the spinode curve; $c' = 6$.

The equations of the sextic are in fact $XZ + Y^2 = 0$, $X^3 + Z^3 = 0$, so that this consists of three conics, each in a plane passing through the edge.

The edge touches each of the three conics at the point $X = 0$, $Z = 0$, $Y = 0$; but it must be reckoned as a single tangent of the spinode curve, and then counting it three times, $\beta' = 3$.

and thence the equation is found to be

$$\begin{aligned}
&16w^4\left[y^2-(x-z)^2\right]\\
&+16w^3\left[(2x-5z)\,y^2-2\,(x-2z)\,(x-z)^2\right]\\
&+\ \ 8w^2\left[y^4+(x^2-xz+6z^2)\,y^2-2x^2\,(x-z)^2\right]\\
&+\ \ 4w\left[(2x+3z)\,y^4-2x^2\,(x+z)\,y^2\right]\\
&+\qquad y^4\,(y^2-x^2)=0,
\end{aligned}$$

where the section by the plane $w=0$ (reciprocal of binode) is $y^4(y^2-x^2)=0$, viz. this is the line $w=0$, $y=0$ (reciprocal of the edge) four times, and the lines $w=0$, $y^2-x^2=0$ (reciprocals of the biplanar rays).

The section by the plane $z=0$ is found to be $(y^2-x^2)(y^2+4xw+4w^2)^2=0$, viz. this is the two lines $z=0$, $y^2-x^2=0$ (reciprocals of the nodal rays), and the conic $z=0$, $y^2+4xw+4w^2=0$ (reciprocal of the nodal cone $WX+Y^2-X^2=0$) twice.

142. Nodal curve. The equation shows that the line $y=0$, $x-z=0$ (reciprocal of the line $W=0$, $X+Z=0$) is a nodal line on the surface.

It also shows that the line $y=0$, $w=0$ (reciprocal of the edge) is a tacnodal line (= 2 nodal lines) on the surface; in fact attending only to the lowest terms in y, w, we have

$$-x^2\left[16\,(x-z)^2\,w^2+8\,(x+z)\,wy^2+y^4\right]=0,$$

that is,

$$4\,(x-z)\,w+\frac{\sqrt{x}\pm\sqrt{z}}{\sqrt{x}\mp\sqrt{z}}\,y^2=0,$$

two values, $w=Ay^2$, $w=By^2$, which indicates a tacnodal line.

The nodal curve is thus made up of the line $y=0$, $x-z=0$ once, and the line $y=0$, $w=0$ twice; $b'=3$.

143. Cuspidal curve. The equations

$$\left\|\begin{matrix}12w^2, & 8w^2+4wx, & 4w^2+4wx+4wz+y^2\\ 8w^2+4wx, & 4w^2+4wx+4wz+y^2, & 12wz\end{matrix}\right\|=0$$

give

$$\begin{aligned}
&(4w+2x)^2-3\,(4w^2+4wx+4wz+y^2)=0,\\
&-36w^2z+(2w+x)\,(4w^2+4wx+4wz+y^2)=0,
\end{aligned}$$

or, as these are more simply written,

$$\begin{aligned}
&4w^2+\ 4wx-12wz+4x^2-3y^2=0,\\
&8w^3+12w^2x-28w^2z+w\,(4x^2+4xz+2y^2)+xy^2=0,
\end{aligned}$$

so that the cuspidal curve is a complete intersection 2×3; $c'=6$.

Section XI = 12 − B_6.

Article Nos. 144 to 149. Equation $WXZ + Y^2Z + X^3 - Z^3 = 0$.

144. The diagram of the lines and planes is

			$X=0,\ Y-Z=0$	$X=0,\ Y+Z=0$	$X=0,\ Z=0$	Lines are
			2	1	0	
XI = 12 − B_6.			$\overline{3}$ $\overline{27}$	$2 \times 6 = 12$	$1 \times 15 = 15$	
Planes are						
$X=0$	0	$1 \times 15 = 15$			· · · ·	Oscular biplane.
$X=0$	3	$1 \times 30 = 30$ $\overline{2}$ $\overline{45}$	· · · · ·	· · · · ·	· ·	Ordinary biplane.
			Rays in the ordinary biplane.		Edge.	

where the equations of the lines and planes are shown in the margins of the diagram.

145. The edge is a facultative line counting three times; this will appear from the discussion of the reciprocal surface. Therefore $\rho' = b' = 3$; $t' = 1$.

146. Hessian surface. This is

$$Z(WXZ + Y^2Z - 3X^3 - 3Z^3) = 0,$$

breaking up into $Z = 0$, the oscular biplane, and into a cubic surface (itself a surface XI = 12 − B_6). The complete intersection with the cubic surface is made up of the line $X = 0$, $Z = 0$ (the edge) six times, and of a residual sextic (= 3 conics), which is the spinode curve; $c' = 6$.

The equations of the sextic are in fact $XZ + Y^2 = 0$, $X^3 + Z^3 = 0$, so that this consists of three conics, each in a plane passing through the edge.

The edge touches each of the three conics at the point $X = 0$, $Z = 0$, $Y = 0$; but it must be reckoned as a single tangent of the spinode curve, and then counting three times, $\beta' = 3$.

Reciprocal Surface.

147. The equation is obtained by means of the binary cubic

$$(12w^2,\ 4zw,\ y^2+4xw,\ -12w^2 \between Z,\ X)^3,$$

viz. it is

$$\begin{aligned}
&432w^6\\
+\ &72w^3z\,(4xw+y^2)\\
-\ &64w^3z^3\\
+\ &\qquad(4xw+y^2)^3\\
-\ &\ z^2\,(4xw+y^2)^3=0,
\end{aligned}$$

or, completely developed, it is

$$\begin{aligned}
&w^6\,.\,432\\
+\,&w^4\,.\,288xz\\
+\,&w^3\,.\,72y^2z+64x^3-64z^3\\
+\,&w^2\,.\,48x^2y^2-16x^2z^2\\
+\,&w\ .\,12xy^4-8xy^2z^2\\
+\,&\ \ y^4\,(y^2-z^2)=0\,;
\end{aligned}$$

the section by the plane $w=0$ (reciprocal of B_6) is $w=0$, $y=0$ (reciprocal of edge) four times, together with $w=0$, $y^2-z^2=0$, reciprocals of the two rays.

148. The nodal curve is the line $y=0$, $w=0$ (reciprocal of edge counting as three lines); $b'=3$. In fact the form of the surface in the vicinity is given by $w=-\dfrac{1}{4x}y^2\pm\tfrac{1}{4}\sqrt{\dfrac{z}{x^5}}\,y^3$, viz. there are two sheets osculating along the line in question, that is intersecting in this line taken three times.

149. For the cuspidal curve we have

$$\left\|\begin{matrix} 12w^2, & 4zw, & y^2+4xw\\ 4zw, & y^2+4xw, & -12w^2\end{matrix}\right\|=0,$$

giving

$$\begin{aligned}
12xw+3y^2\quad-4z^2&=0,\\
36w^3\ +4wxz+y^2z&=0\,;
\end{aligned}$$

or multiplying the first by $3z$ and subtracting the second we have $108w^3+4z^3=0$. Hence the equations are

$$\begin{aligned}
z^3+27w^3&=0,\\
12xw+3y^2-4z^2&=0,
\end{aligned}$$

viz. the cuspidal curve is made up of three conics lying in planes through the line $z=0$, $w=0$.

The curve may be put in evidence by writing the equation of the surface in the form

$$(3y^2+5z^2+12xw,\ 24z,\ 16 \between 3y^2-4z^2+12xw,\ z^3+27w^3)^2=0,$$

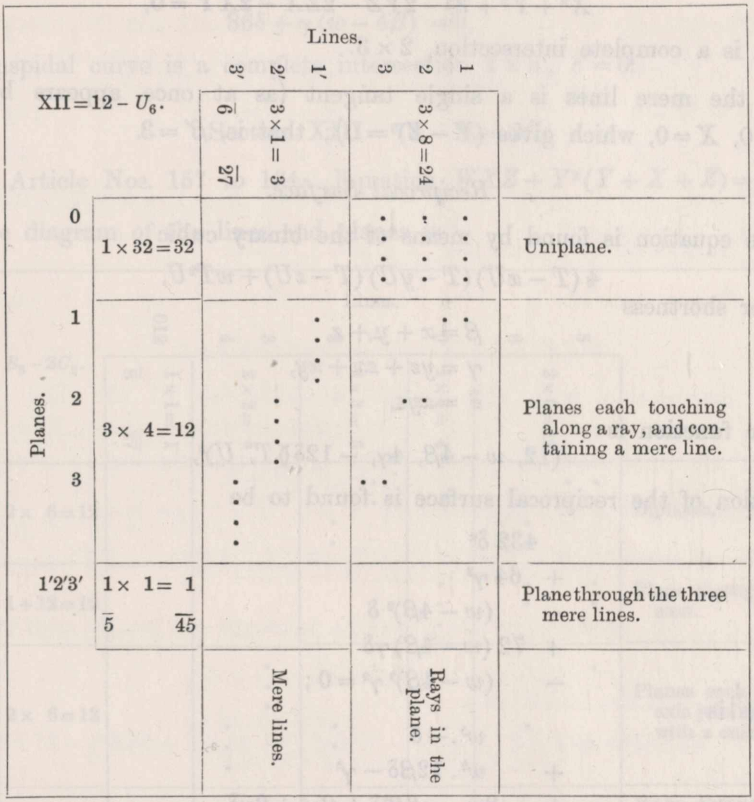
where

$$16\,(3y^2+5z^2+12xw)-144z^2=16\,(3y^2-4z^2+12xw).$$

Section XII $= 12 - U_6$.

Article Nos. 150 to 156. Equation $W(X+Y+Z)^2 + XYZ = 0$.

150. The diagram of the lines and planes is



151. The planes are

$$\begin{aligned} X+Y+Z&=0, &&[0]\\ X&=0, &&[1]\\ Y&=0, &&[2]\\ Z&=0, &&[3]\\ W&=0, &&[1'2'3'] \end{aligned}$$

The lines are

$$\begin{aligned} &X=0,\ Y+Z=0, &&(1)\\ &Y=0,\ Z+X=0, &&(2)\\ &Z=0,\ X+Y=0, &&(3)\\ &X=0,\ W=0, &&(1')\\ &Y=0,\ W=0, &&(2')\\ &Z=0,\ W=0, &&(3'). \end{aligned}$$

152. The three mere lines are each facultative: $\rho' = b' = 3$; $t' = 1$.

153. Hessian surface. The equation is

$$(X+Y+Z)^2(X^2+Y^2+Z^2-2YZ-2ZX-2XY)=0,$$

viz. the surface consists of the uniplane $X+Y+Z=0$ twice, and of a quadric cone having its vertex at U_6, and touching each of the planes $X=0$, $Y=0$, $Z=0$.

The complete intersection with the cubic surface is made up of the rays each twice and of a residual sextic, which is the spinode curve; $\sigma' = 6$.

The equations of the spinode curve are

$$W(X+Y+Z)^2 + XYZ = 0,$$
$$X^2+Y^2+Z^2-2YZ-2ZX-2XY=0,$$

viz. the curve is a complete intersection, 2 × 3.

Each of the mere lines is a single tangent (as at once appears by writing for instance $W=0$, $X=0$, which gives $(Y-Z)^2=0$); that is, $\beta'=3$.

Reciprocal Surface.

154. The equation is found by means of the binary cubic

$$4(T-xU)(T-yU)(T-zU)+wT^2U,$$

viz. writing for shortness

$$\begin{aligned}\beta &= x+y+z,\\ \gamma &= yz+zx+xy,\\ \delta &= xyz,\end{aligned}$$

then the cubic function is

$$(12,\ w-4\beta,\ 4\gamma,\ -12\delta \between T,\ U)^3,$$

and the equation of the reciprocal surface is found to be

$$\begin{aligned}&432\,\delta^2\\ +\ &64\,\gamma^3\\ -\ &(w-4\beta)^3\,\delta\\ +\ &72\,(w-4\beta)\,\gamma\delta\\ -\ &(w-4\beta)^2\,\gamma^2=0\,;\end{aligned}$$

expanding, this is

$$\begin{aligned}&w^3\,.\,-\delta\\ +\ &w^2\,.\,12\beta\delta-\gamma^2\\ +\ &8w\,.\,-6\beta^2\delta+\beta\gamma^2+9\gamma\delta\\ +\ &16\,(4\beta^3\delta-\beta^2\gamma^2-18\beta\gamma\delta+4\gamma^3+27\delta^2)=0\,;\end{aligned}$$

or substituting for β, γ, δ in the first and last lines, this is

$$\begin{aligned}&w^3\,.\,-xyz\\ +\ &w^2\,.\,(12\beta\delta-\gamma^2)\\ +\ &8w\,.\,-6\beta^2\delta+\beta\gamma^2+9\gamma\delta\\ +\ &16\,(y-z)^2(z-x)^2(x-y)^2=0\end{aligned}$$

(where $\beta, \gamma, \delta = x+y+z,\ yz+zx+xy,\ xyz$). The section by the plane $w=0$ (reciprocal of the unode) is made up of the lines $w=0$, $y-z=0$; $w=0$, $z-x=0$; $w=0$, $x-y=0$ (reciprocals of the rays) each twice.

155. The nodal curve is at once seen to consist of the lines $(y=0, z=0)$, $(z=0, x=0)$, $(x=0, y=0)$, reciprocals of the facultative lines; in fact, in regard to (y, z) conjointly γ is of the order 1, and δ is of the order 2; hence every term of the equation is of the order 2 in y, z; and the like as to the other two lines: $b'=3$ as above.

156. For the cuspidal curve we have

$$\left\| \begin{array}{ccc} 12\ , & w-4\beta, & 4\gamma \\ w-4\beta, & 4\gamma\ \ , & -12\delta \end{array} \right\| = 0,$$

or say

$$48\gamma - \ \ (w-4\beta)^2 = 0,$$
$$36\delta + \gamma\,(w-4\beta) = 0,$$

whence the cuspidal curve is a complete intersection 2×3; $c'=6$.

Section XIII $= 12 - B_3 - 2C_2$.

Article Nos. 157 to 164. Equation $WXZ + Y^2(Y+X+Z) = 0$.

157. The diagram of the lines and planes is

	XIII $=12-B_3-2C_2$.		Lines. 012	4	3	2	1	0	6	5	
			$\overline{8}$ $1\times1=1$ $\overline{27}$	$2\times2=4$		$2\times3=6$		$1\times4=4$	$2\times6=12$		
Planes.	1	$2\times\ 6=12$					• •			• • •	Biplanes.
	2					• •			• • •		
	056	$1+12=12$						• • •	• •	• •	Plane through the three axes.
	5	$2\times\ 6=12$			• • •		• •			•	Planes each through an axis joining the binode with a cnicnode.
	6			• • •		• •			•		
	34	$1\times\ 4=\ 4$		• •	• •			•			Plane through the axis joining the two cnicnodes.
	12	$1\times\ 3=\ 3$	• • •			•	•				Planes through the biplanar rays.
	0	$1\times\ 2=\ 2$ $\overline{8}$ $\overline{45}$	• •					• •			Plane touching along the axis which joins the two cnicnodes.
			Transversal.	Cnicnodal rays, one through each cnicnode.		Biplanar rays, one in each biplane, and being each a transversal.		Axis joining the two cnicnodes.	Axes, each joining the binode with a cnicnode.		

158.

The planes are		The lines are	
$X=0,$	[1]	$X=0,\ Y=0,$	(5)
$Z=0,$	[2]	$Z=0,\ Y=0,$	(6)
$Y=0,$	[056]	$Y=0,\ W=0,$	(0)
$Y+X=0,$	[5]	$X=0,\ Y+Z=0,$	(1)
$Y+Z\ =0,$	[6]	$Z=0,\ Y+X=0,$	(2)
$Y-W=0,$	[34]	$W=Y=-Z,$	(3)
$X+Y+Z=0,$	[12]	$W=Y=-X,$	(4)
$W=0,$	[0]	$W=0;\ X+Y+Z=0,$	(012)

159. The transversal is facultative; $\rho'=b'=1,\ t'=0.$

160. The Hessian surface is

$$WXZ(3Y+X+Z)+Y^2(Z-X)^2=0.$$

The complete intersection with the surface is made up of the line $Y=0,\ X=0$ (CB-axis) three times; the line $Y=0,\ Z=0$ (CB-axis) three times; line $Y=0,\ W=0$ (CC-axis) twice, and of a residual quartic, which is the spinode curve; $\sigma'=4$.

161. Representing the two equations by $U=0,\ H=0$, we have

$$(3Y+X+Z)\,U-H=Y^2(3Y^2+4YX+4YZ+4XZ),\ =MY^2 \text{ suppose},$$

and

$$27(X+Z)\,U+9H=9WXZ(3Y+4X+4Z)+36Y^2(X^2+XZ+Z^2)+27Y^3(X+Z);$$

but

$$(-9(X+Z)\,Y+16XZ)\,M=$$
$$64X^2Z^2+28YXZ(X+Z)-Y^2(36X^2+28XZ+36Z^2)-27Y^3(X+Z),$$

whence

$$27(X+Z)\,U+9H+(-9XY-9ZY+16XZ)\,M$$
$$=ZX\,\{12Y^2+28Y\overline{X+Z}+64XZ+9W(3Y+4X+4Z)\};$$

or, as this may also be written,

$$\begin{array}{ll} 27Y^2(X+Z)\,U & +9Y^2H \\ +(-9YX-9YZ+16XZ)(3Y+X+Z)\,U & +(9Y\overline{X+Z}-16XZ)\,H, \end{array}$$

that is,

$$\{-9Y(X+Z)^2+48YXZ+16XZ(X+Z)\}\,U+\{9Y^2+9Y\overline{X+Z}-16XZ\}\,H$$
$$=Y^2ZX\,\{12Y^2+28Y(Z+X)+64XZ+9W(3Y+4X+4Z)\}=0;$$

and we thus obtain the equation of the residual quartic, or spinode curve, in the form

$$3Y^2+\ \ 4Y(X+Z)+\ \ 4XZ=0,$$
$$12Y^2+28Y(X+Z)+64XZ+9W(3Y+4X+4Z)=0.$$

The spinode curve is thus a complete intersection, 2×2; and since the first surface is a cone having its vertex on the second surface, we see moreover that the spinode curve is a nodal quadriquadric. Instead of the last equation we may write more simply

$$4Y(X+Z)+16XZ+3W(3Y+4X+4Z)=0.$$

The equations of the transversal are $W=0$, $X+Y+Z=0$, and substituting in the equations of the spinode curve we obtain from each equation $(X-Z)^2=0$, that is, the transversal is a single tangent of the spinode curve; $\beta'=1$.

Reciprocal Surface.

162. The equation of the cubic is derived from that belonging to $\text{VI}=12-B_3-C_2$ by writing therein $a=b=0$, $c=\frac{1}{3}$, $d=1$. Making this change in the formulæ for the reciprocal surface of the case just referred to, we have

$$\begin{aligned} L &= y^2+4(x+z)w, \\ M &= 2x(y+2w), \\ N &= -4x^2, \\ P &= 16x^2(y+w-x-z); \end{aligned}$$

and substituting in the equation

$$L^2P+8zM^3-9zLMN-27z^2wN^2=0,$$

the equation divides by x^2; or throwing this out, the equation is

$$\begin{aligned} &(y^2+4xw+4zw)^2(y+w-x-z) \\ &-8xz(y+2w)^3 \\ &+9xz(y^2+4xw+4zw)(y+2w) \\ &-27x^2z^2w=0; \end{aligned}$$

reducing, this is

$$\begin{aligned} &w^3.16(x-z)^2 \\ &+w^2\left\{\begin{array}{l} \quad y^2(x+z) \\ +2y(x^2-4xz+z^2) \\ +(x+z)(2x-z)(-x+2z) \end{array}\right\} \\ &+w\left\{\begin{array}{l} \quad y^4 \\ +8y^3(x+z) \\ -2y^2(4x^2+23xz+4z^2) \\ +36xyz(x+z) \\ -27x^2z^2 \end{array}\right\} \\ &+y^3(y-x)(y-z)=0. \end{aligned}$$

The section by the plane $w=0$ (reciprocal of B_3) is $w=0$, $y=0$ (the edge) three times; and $w=0$, $y-x=0$; $w=0$, $y-z=0$ (reciprocals of the CB-axes).

163. Nodal curve. This is the line $y=x=z$; wherefore $b'=1$. To put the line in evidence, write for a moment $x=y+\alpha$, $z=y+\gamma$, then the equation is readily converted into

$$\begin{aligned}
&w^3 . 16(\alpha-\gamma)^2 \\
&+ w^2 \begin{Bmatrix} -y(\alpha^2-4\alpha\gamma+\gamma^2) \\ +(\alpha+\gamma)(2\alpha-\gamma)(-\alpha+2\gamma) \end{Bmatrix} \\
&+ w \begin{Bmatrix} y^2(\alpha^2-10\alpha\gamma+\gamma^2) \\ -18y\alpha\gamma(\alpha+\gamma) \\ -27\alpha^2\gamma^2 \end{Bmatrix} \\
&+ \ y^3\alpha\gamma = 0,
\end{aligned}$$

which, each term being at least of the second order in α, γ $(=x-y,\ z-y)$, exhibits the nodal line in question.

164. Cuspidal curve. Multiplying by 27, the equation may be written

$$\begin{aligned}(7y-3x-3z-5w,\ -y+6w,\ -w)(y^2+16yw-12xw-12zw+16w^2,& \\ -20y^2+24yx+24yz-27xz-8yw+16w^2)^2=0,&\end{aligned}$$

where

$$4w(7y-3x-3z-5w)+(-y+6w)^2=y^2+16yw-12(x+z)w+16w^2;$$

and we have thus in evidence the cuspidal curve,

$$\begin{aligned} y^2+16yw-12(x+z)w+16w^2=0, \\ -20y^2+24y(x+z)-27xz-8yw+16w^2=0, \end{aligned}$$

which is a complete intersection, 2×2, or quadriquadric curve; $c'=4$.

Section XIV $= 12 - B_5 - C_2$.

Article Nos. 165 to 171. Equation $WXZ + Y^2Z + YX^2 = 0$.

165. The diagram of the lines and planes is

			$W=0,\ Y=0$ 3 $\overline{4}$ $1\times 2=2$ $\overline{27}$	$Z=0,\ Y=0$ 2 $1\times 5=5$	$X=0,\ Z=0$ 1 $1\times 10=10$	$X=0,\ Y=0$ 0 $1\times 10=10$	Lines are
$\mathrm{XIV}=12-B_5-C_2$.							
Planes are							
$Z=0$	12	$1\times 15=15$		• • •	• • • •		Torsal biplane.
$X=0$	01	$1\times 20=20$			• •	• • • • •	Ordinary biplane.
$Y=0$	023	$1\times 10=10$ $\overline{3}$ $\overline{45}$	• • • • •	• •		•	Plane through axis and the two rays.
			Cnicnodal ray.	Biplanar ray in torsal biplane.	Edge.	Axis.	

where the equations of the planes and lines are shown in the margins.

166. The edge is a facultative line, as will appear from the discussion of the reciprocal surface: $\rho' = b' = 1$; $t' = 0$.

167. Hessian surface. The equation is

$$WXZ^2 + Y^2Z^2 - 3X^2YZ + X^4 = 0.$$

The complete intersection with the surface is made up of the line $X=0$, $Y=0$ (the axis) five times, the line $X=0$, $Z=0$ (the edge) four times, and a skew cubic, the equations of which may be written

$$\left\| \begin{matrix} X, & Y, & W \\ 4Z, & X, & -5Y \end{matrix} \right\| = 0.$$

In fact from the equations $U=0$, $H=0$ we deduce $H - ZU = X^2(X^2 + 4YZ) = 0$; and if in $U=0$ we write $X^2 = 4YZ$, it becomes $Z(XW + 5Y^2) = 0$; and then in $5U=0$, writing $5Y^2 = -XW$, we have

$$5WXZ + Z(-XW) + 5X^2Y = X(5XY + 4ZW) = 0.$$

168. I say that the spinode curve is made up of the edge $X=0$, $Z=0$ once, and of the cubic curve; and therefore $\sigma' = 4$.

In fact in the reciprocal surface the cuspidal curve is made up of the skew cubic, and of a line the reciprocal of the axis, being a cusp-nodal line, and so counting once as part of the cuspidal curve: the pencil of planes through the line is thus part of the cuspidal torse; and reverting to the original cubic surface, we have the axis as part of the spinode curve: I assume that it counts *once*.

The edge is a single tangent of the spinode curve; $\beta' = 1$.

Reciprocal Surface.

169. The equation is obtained by means of the binary cubic

$$4wZ^2(Xx+Zz)+X(YZ-wX)^2,$$

or, as this may be written,

$$(3w^2,\ -2yw,\ y^2+4xw,\ 12zw \between X,\ Z)^3.$$

The equation is in the first instance obtained in the form

$$\begin{aligned} & 108w^6z^2 \\ - \ & 32w^4y^3z \\ + \ & 36w^4yz\,(y^2+4xw) \\ + \ & \quad w^2 \ \ (y^2+4xw)^3 \\ - \ & \quad w^2y^2\,(y^2+4xw)^2 = 0\,; \end{aligned}$$

but the last two terms being together $= 4w^3x\,(y^2+4xw)^2$, the whole divides by $4w^3$, and it then becomes

$$\begin{aligned} & 27w^3z^2 \\ - \ & 8wy^3z \\ + \ & 9wyz\,(y^2+4xw) \\ + \ & \qquad x\,(y^2+4xw)^2 = 0\,; \end{aligned}$$

or, expanding, it is

$$\begin{aligned} & w^3\,.\,27z^2 \\ + \ & w^2\,.\,36xyz+16x^3 \\ + \ & w\ .\quad y^3z+\ 8x^2y^2 \\ + \ & \qquad\quad xy^4 = 0. \end{aligned}$$

The section by the plane $w=0$ (reciprocal of B_5) is $w=0$, $y=0$ (reciprocal of edge) four times, together with $w=0$, $x=0$ (reciprocal of biplanar ray).

The section by the plane $z=0$ (reciprocal of C_2) is $x\,(y^2+4xw)^2=0$, viz. this is $z=0$, $y^2+4xw=0$ (reciprocal of nodal cone) twice, together with $z=0$, $x=0$ (reciprocal of nodal ray).

170. Nodal curve. This is the line $w=0$, $y=0$, reciprocal of edge. The equation in the vicinity is $y = -\frac{1}{4x}\,w \pm \sqrt{-\frac{z}{8x^4}\,w^{\frac{5}{2}}}$, showing that the line is a cusp-nodal line counting once in the nodal and once in the cuspidal curve: wherefore $b' = 1$.

171. Cuspidal curve. The equation of the surface may be written

$$(x,\ -y,\ 3w \between 12xw - y^2,\ 9zw + 8xy)^2 = 0,$$

where $4x \,.\, 3w - y^2 = 12xw - y^2$. This exhibits the cuspidal curve $12xw - y^2 = 0$, $9zw + 8xy = 0$, breaking up into the line $w = 0$, $y = 0$ (reciprocal of edge) and a skew cubic; the line is really part of the cuspidal curve, or $c' = 4$.

The equations of the cuspidal cubic may be written in the more complete form

$$\left\| \begin{matrix} 12x, & y, & z \\ y, & w, & -8x \end{matrix} \right\| = 0.$$

Section XV = 12 − U_7.

Article Nos. 172 to 176. Equation $WX^2 + XZ^2 + Y^2Z = 0$.

172. The diagram of the lines and planes is

			$Z=0,\ W=0$	$X=0,\ Z=0$	$X=0,\ Y=0$	Lines are
			3	2	1	
XV=12 − U_7.			$1\times 1=1$	$1\times 10=10$	$1\times 16=16$	
			$\overline{3}$ $\overline{27}$			
Planes are						
$X=0$	21	$1\times 40=40$		· · · ·	· · · · · ·	Uniplane.
$Z=0$	23	$1\times 5=5$	· · · · ·	· ·		Plane touching along the single ray.
	$\overline{2}$	$\overline{45}$				
			Mere line.	Single ray.	Torsal ray.	

where the equations of the lines and planes are shown in the margins.

173. The mere line is facultative: $\rho' = b' = 1$; $t' = 0$.

174. The Hessian surface is

$$X^2(XZ - Y^2) = 0,$$

viz. this is the uniplane $X = 0$ twice, and a quadric cone having its vertex at U_7.

The complete intersection with the surface is made up of $X=0$, $Y=0$ (torsal ray) six times; $X=0$, $Z=0$ (single ray) twice; and of a residual quartic, which is the spinode curve; $\sigma'=4$.

The equations of the spinode curve are $XZ-Y^2=0$, $XW+2Z^2=0$; the first surface is a cone having its vertex on the second surface; and the curve is thus a nodal quadriquadric.

The mere line is a single tangent of the spinode curve; $\beta'=1$.

Reciprocal Surface.

175. The equation is obtained by means of the binary cubic

$$(-3y^2,\ 2yz,\ 4xw,\ 6yw \between X,\ Y)^3,$$

viz. throwing out the factor y, the equation is

$$w^2(-64x^3)+w(-16x^2z^2+72xy^2z+27y^4)+16y^2z^3=0.$$

The section by the plane $w=0$ (reciprocal of U_7) is $w=0$, $z=0$ (reciprocal of torsal ray) three times, and $w=0$, $y=0$ (reciprocal of single ray) twice.

Nodal curve. This is the line $x=0$, $y=0$, reciprocal of the mere line: $b'=1$.

Cuspidal curve. The equation of the surface may be written

$$(64x,\ -16z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0,$$

where

$$4\,.\,64x(-3w)-256z^2=-256(z^2+3xw).$$

This exhibits the cuspidal curve $z^2+3xw=0$, $9y^2+4zx=0$, where the surfaces are each of them cones; the vertex of the second cone is on the first cone, and the two cones have at this point a common tangent plane; the curve is thus a cuspidal quadriquadric.

176. {The equation

$$(64x,\ \ -16z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0$$

resembles that of a quintic torse, viz. the equation of a quintic torse is

$$(\ \ x,\ -\ \ 4z,\ \ \ 8w \between z^2-2wx,\ \ y^2-2zx)^2=0,$$

which equation, writing $9y$ for y, $-2x$ for x, and $\tfrac{3}{4}w$ for w, becomes

$$(-2x,\ -\ \ 4z,\ \ \ 6w \between z^2+3xw,\ 9y^2+4zx)^2=0,$$

or, what is the same thing,

$$(\ \ x,\ \ \ \ 2z,\ -3w \between z^2+3xw,\ 9y^2+4zx)^2=0;$$

and developing, this is

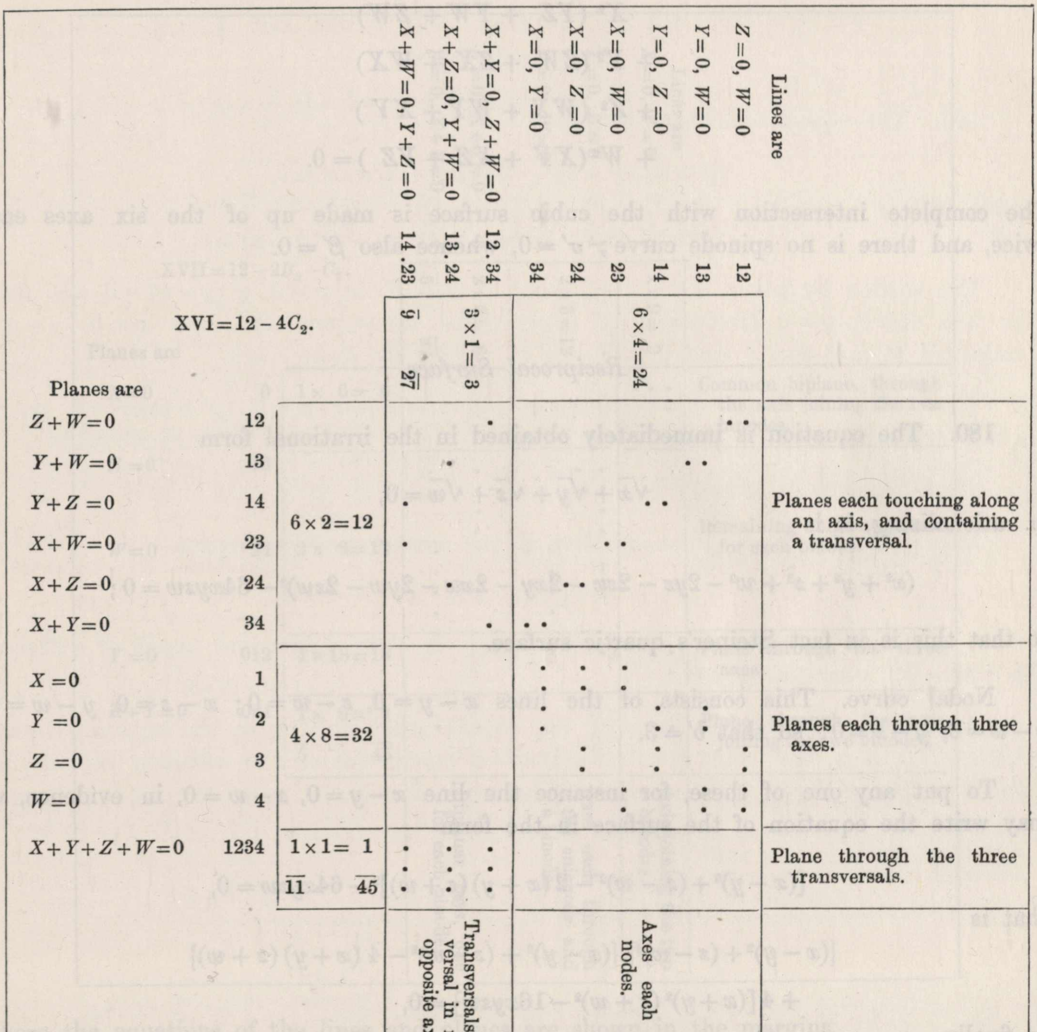
$$\begin{aligned} & x^3\,.\,w^2 \\ &+x^2\,.-2z^2w \\ &+x\ \,.-18y^2zw+z^4 \\ &\qquad\ -27y^4w+2y^2z^3=0, \end{aligned}$$

which, however, differs from the equation of the reciprocal surface, not only in the numerical coefficients, but by the presence of a term xz^4.}

Section $\mathrm{XVI} = 12 - 4C_2$.

Article No. 177 to 180. Equation $W(XY + XZ + YZ) + XYZ = 0$.

177. The diagram of the lines and planes is



where the equations of the lines and planes are shown in the margins.

178. The transversals are each facultative: $\rho' = b' = 3$; $t' = 1$.

179. Hessian surface. The equation is

$$4XYZW - (X+Y+Z+W)(WXY+WXZ+WYZ+XYZ) = 0,$$

or, what is the same thing,

$$\begin{aligned} &X^2(YZ+YW+ZW) \\ &+Y^2(ZW+ZX+WX) \\ &+Z^2(WX+WY+XY) \\ &+W^2(XY+XZ+YZ) = 0. \end{aligned}$$

The complete intersection with the cubic surface is made up of the six axes each twice, and there is no spinode curve; $\sigma' = 0$, whence also $\beta' = 0$.

Reciprocal Surface.

180. The equation is immediately obtained in the irrational form

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 0,$$

or rationalizing, it is

$$(x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw)^2 - 64xyzw = 0;$$

so that this is in fact Steiner's quartic surface.

Nodal curve. This consists of the lines $x-y=0$, $z-w=0$; $x-z=0$, $y-w=0$; $x-w=0$, $y-z=0$; so that $b'=3$.

To put any one of these, for instance the line $x-y=0$, $z-w=0$, in evidence, we may write the equation of the surface in the form

$$[(x-y)^2+(z-w)^2-2(x+y)(z+w)]^2 - 64xyzw = 0,$$

that is

$$\begin{aligned} &\{(x-y)^2+(z-w)^2\}\{(x-y)^2+(z-w)^2-4(x+y)(z+w)\} \\ &\qquad + 4[(x+y)^2(z+w)^2 - 16xyzw] = 0, \end{aligned}$$

or finally

$$\begin{aligned} &\{(x-y)^2+(z-w)^2\}\{(x-y)^2+(z-w)^2-4(x+y)(z+w)\} \\ &\qquad + 4\{(x-y)^2(z-w)^2 + 4xy(z-w)^2 + 4zw(x-y)^2\} = 0, \end{aligned}$$

where each term is at least of the second order in $x-y$, $z-w$.

There is no cuspidal curve; $c'=0$.

Section XVII $=12-2B_3-C_2$.

Article Nos. 181 to 185. Equation $WXZ+XY^2+Y^3=0$.

181. The diagram of the lines and planes is

XVII $=12-2B_3-C_2$.

Planes are			Lines are $W=0,\ X+Y=0$ 4 $Z=0,\ X+Y=0$ 3	$Y=0,\ W=0$ 2 $Y=0,\ Z=0$ 1	$X=0,\ Y=0$ 0	
			$2\times3=6$ $\overline{5}$ $\overline{27}$	$2\times6=12$	$1\times9=9$	
$X=0$	0	$1\times\ 6=\ 6$			• • • •	Common biplane, through the axis joining the two binodes.
$Z=0$	13		• • • •	• • • • •		
$W=0$	24	$2\times\ 6=12$	• • • •	• • • • •		Remaining biplanes, one for each binode.
$Y=0$	012	$1\times18=18$		• •	• •	Plane through the three axes.
$X+Y=0$	034	$1\times\ 9=\ 9$ $\overline{5}$ $\overline{45}$	• •		•	Plane through the axis joining the two binodes.
			Biplanar rays, one for each binode.	Axes each through the cnicnode and a binode.	Axis joining the two binodes.	

where the equations of the lines and planes are shown in the margins.

182. There is no facultative line; $b'=\rho'=0,\ t'=0$.

183. The Hessian surface is

$$X(WXZ+3YZW+XY^2)=0,$$

viz. this breaks up into $X=0$ (the common biplane), and into a cubic surface.

The complete intersection with the cubic surface is made up of $X=0$, $Y=0$ (BB-axis) four times, of $Y=0$, $Z=0$ and $Y=0$, $W=0$ (CB-axes) each three times; and of a residual conic, which is the spinode curve; $\sigma'=2$. The equations of the spinode curve are $Y^2-3ZW=0$, $4X+3Y=0$; viz. it lies in a plane passing through the BB-axis; since there is no facultative line, $\beta'=0$.

Reciprocal Surface.

184. The equation is found to be

$$(y^2+4zw)^2-xy^3-36xyzw+27x^2zw=0,$$

or say this is

$$16z^2w^2+(8y^2-36xy+27x^2)\,zw+y^3\,(y-x)=0.$$

The section by plane $w=0$ (reciprocal of $B_3=D$) is $w=0$, $y^3\,(y-x)=0$, viz. this is the line $w=0$, $y=0$ (reciprocal of edge) three times, and the line $w=0$, $y-x=0$ (reciprocal of ray) once; and the like as to section by plane $z=0$.

The section by plane $x=0$ (reciprocal of $C_2=A$) is $x=0$, $(y^2+4zw)^2=0$, viz. this is the conic (reciprocal of nodal cone) twice.

There is no nodal curve; $b'=0$.

185. Cuspidal curve. The equation of the surface may be written

$$(1,\ -y,\ 3zw\,)\!(\,y^2-12zw,\ 9x-8y)^2=0,$$

where $4\,.\,1\,.\,3zw-y^2=-(y^2-12zw)$; and there is thus a cuspidal conic $y^2-12zw=0$, $9x-8y=0$: wherefore $c'=2$.

Attending only to the terms of the second order in y, z, w, the equation becomes $x^2zw=0$; that is, the point $y=0$, $z=0$, $w=0$ (reciprocal of the common biplane) is a binode of the surface; or there is the singularity $B'=1$.

Section XVIII $= 12 - B_4 - 2C_2$.

Article Nos. 186 to 189. Equation $WXZ + Y^2(X+Z) = 0$.

186. The diagram of the lines and planes are

			$W=0,\ X+Z=0$	$X=0,\ Z=0$	$Y=0,\ W=0$	$Y=0,\ Z=0$	$Y=0,\ X=0$	Lines are
			4	3	0	2	1	
XVIII $=12-B_4-2C_2$.			$1\times1=1$ $\overline{5}$ $\overline{27}$	$1\times6=6$	$1\times4=4$	$2\times8=16$		
Planes are								
$Y=0$	0	$1\times16=16$			· · · ·	· ·	· ·	Plane through the three axes.
$X=0$	01	$2\times12=24$		· ·			· · ·	Biplanes.
$Z=0$	02			· ·		· · ·		
$X+Z=0$	34	$1\times\ 3=\ 3$	· · ·	· ·				Plane touching along edge and containing the mere line.
$W=0$	04	$1\times\ 2=\ 2$ $\overline{5}$ $\overline{45}$	· ·		· ·			Plane touching along axis through the two cnicnodes and containing the mere line.
			Mere line, being also a transversal.	Edge of the binode.	Axis through the two cnicnodes.	Axes, each through the binode and a cnicnode.		

where the equations of the lines and planes are shown in the margins.

187. The mere line is facultative; the edge is also facultative counting twice (this will appear from the discussion of the reciprocal surface): $b' = \rho' = 3$, $t' = 1$.

188. The Hessian surface is

$$(X+Z)\,WXZ+(X-Z)^2\,Y^2=0.$$

The complete intersection with the cubic surface is $Y=0$, $Z=0$ and $Y=0$, $X=0$ (the CB-axes) each four times; $Y=0$, $W=0$ (BB-axis) twice; and $X=0$, $Z=0$ (the edge) twice. There is no spinode curve, $\sigma'=0$; wherefore also $\beta'=0$.

Reciprocal Surface.

189. The equation is obtained from the binary quadric $4w\,(X+Z)(Xx+Zz)+y^2XZ$, or say

$$(8wx,\ 4w\,(x+z)+y^2,\ 8wz \between X,\ Z)^2.$$

The equation is thus

$$(y^2+4wx+4wz)^2-64w^2xz=0,$$

or in an irrational form

$$iy+2\,\sqrt{wx}+2\,\sqrt{wz}=0.$$

The section by the plane $w=0$ (reciprocal of B_4) is $w=0$, $y=0$ (reciprocal of edge) four times.

The section by the plane $z=0$ (reciprocal of $C_2=C$) is $z=0$, $y^2+4wx=0$ (reciprocal of nodal cone) twice; and similarly for the section by $x=0$ (reciprocal of $C_2=A$).

Nodal curve. Writing the equation in the form

$$y^4+8wy^2\,(z+x)+16w^2\,(x-z)^2=0,$$

we have a nodal line $y=0$, $x-z=0$, reciprocal of the mere line:

and writing the equation in the form

$$w=-\frac{1}{4\,(\sqrt{x}\pm\sqrt{z})^2}\,y^2,$$

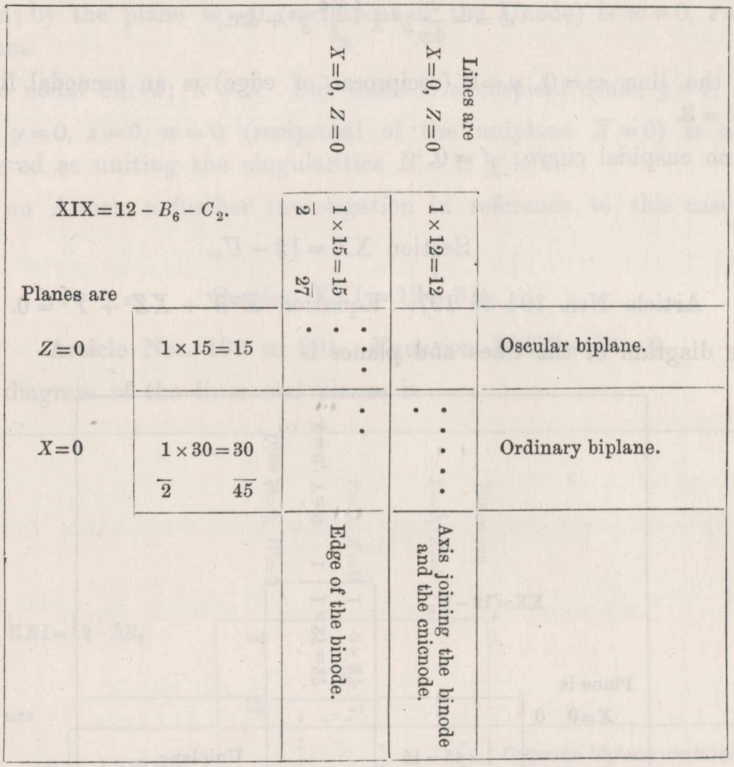
we have $y=0$, $w=0$ (reciprocal of edge), a tacnodal line counting as two lines; $b'=3$.

There is no cuspidal curve; $c'=0$.

Section XIX $= 12 - B_6 - C_2$.

Article Nos. 190 to 193. Equation $WXZ + Y^2Z + X^3 = 0$.

190. The diagram of the lines and planes is



where the equations of the lines and planes are shown in the margins.

191. The axis is a facultative line counting three times (as will appear from the reciprocal surface); $\rho' = b' = 3$, $t' = 1$.

192. The Hessian surface is

$$Z(WXZ + Y^2Z - 3X^3) = 0,$$

viz. this is the oscular biplane $Z = 0$ and a cubic surface.

The complete intersection with the cubic surface is made up of $X = 0$, $Z = 0$ (the edge) six times, and $X = 0$, $Y = 0$ (the axis) six times. There is no spinode curve, $\sigma' = 0$; whence also $\beta' = 0$.

Reciprocal Surface.

193. The equation is at once found to be

$$64zw^3 + (y^2 + 4xw)^2 = 0.$$

The section by the plane $w=0$ (reciprocal of B_6) is $w=0$, $y=0$ (reciprocal of edge) four times. The section by the plane $z=0$ (reciprocal of C_2) is $z=0$, $y^2+4xw=0$ (reciprocal of nodal cone) twice.

Nodal curve. The equation gives

$$w=-\frac{1}{4x}y^2 \pm \frac{i\sqrt{z}}{x^{\frac{5}{2}}}y^3+\&c.,$$

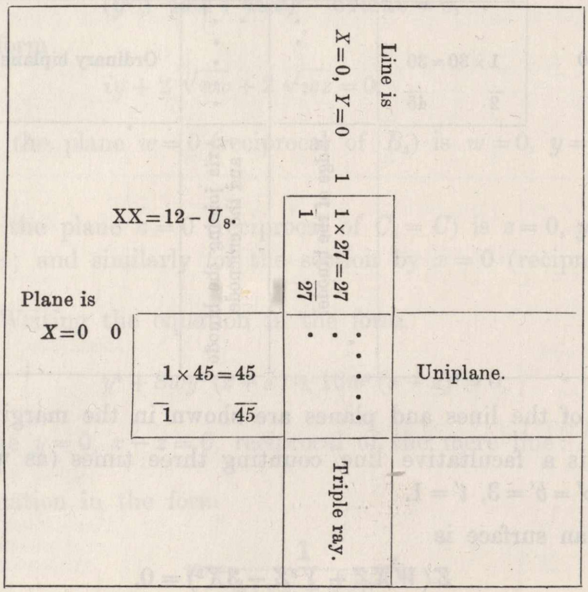
showing that the line $w=0$, $y=0$ (reciprocal of edge) is an oscnodal line counting as three lines; $b'=3$.

There is no cuspidal curve; $c'=0$.

Section XX $=12-U_8$.

Article Nos. 194 to 197. Equation $X^2W+XZ^2+Y^3=0$.

194. The diagram of the lines and planes is



where the equations of the line and plane are shown in the margins.

195. There is no facultative line; $b'=\rho'=0$, $t'=0$.

196. The Hessian surface is $X^3Y=0$, viz. this is the uniplane $X=0$, three times, and the plane $Y=0$ through the ray. The complete intersection with the cubic surface is made up of $X=0$, $Y=0$ (the ray) ten times, and of a residual conic, which is the spinode curve; $\sigma'=2$.

The equations of the spinode conic are $Y=0$, $XW+Z^2=0$, viz. the plane of the conic passes through the ray. Since there is no facultative line, $\beta'=0$.

Reciprocal Surface.

197. The equation is at once found to be

$$27(z^2+4xw)^2-64w^3y=0.$$

The section by the plane $w=0$ (reciprocal of the Unode) is $w=0$, $z=0$ (reciprocal of ray) four times.

There is no nodal curve; $b'=0$. But there is a cuspidal conic, $y=0$, $z^2+4xw=0$.

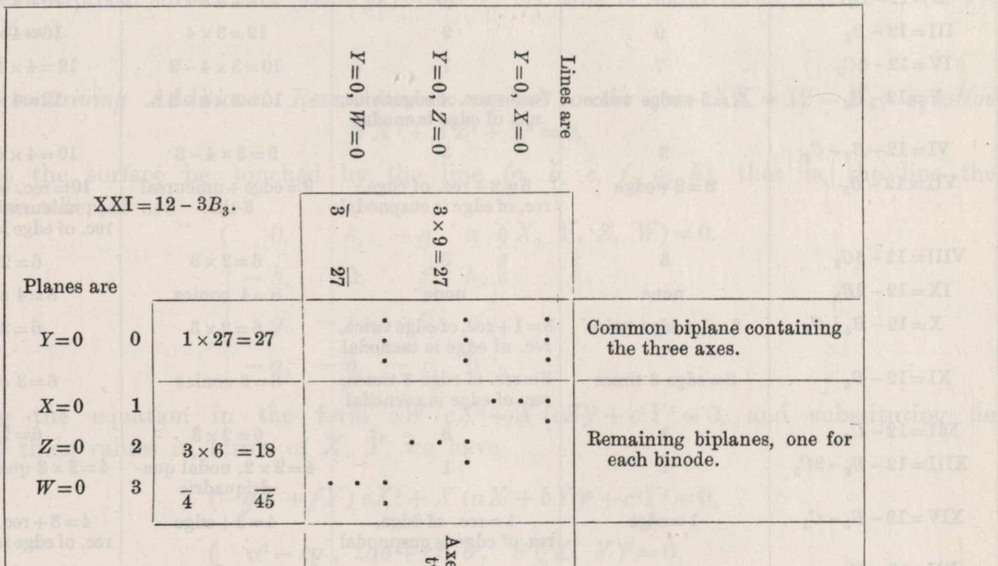
The point $y=0$, $z=0$, $w=0$ (reciprocal of the uniplane $X=0$) is a point which must be considered as uniting the singularities $B'=1$, $\chi'=2$.

I give in an Annex a further investigation in reference to this case of the cubic surface.

Section XXI $=12-3B_3$.

Article Nos. 198 to 201. Equation $WXZ+Y^3=0$.

198. The diagram of the lines and planes is



where the equations of the lines and planes are shown in the margins.

199. There is no facultative line; $\rho'=b'=0$, $t'=0$.

200. The Hessian surface is $XYZW=0$, the common biplane and the other biplanes each once. The complete intersection with the surface consists of the axes each four times; there is no spinode curve, $\sigma'=0$; whence also $\beta'=0$.

Reciprocal Surface.

201. This is $27xzw - y^3 = 0$, viz. it is a cubic surface of the form $\mathrm{XXI} = 12 - 3B_3$.

There is no nodal curve, $b' = 0$, and no cuspidal curve, $c' = 0$. Moreover $B' = 3$.

Article No. 202. *Synopsis for the foregoing sections.*

202. I annex the following synopsis, for the several cases, of the facultative lines (or node-couple curve) and of the spinode curve of the cubic surface; also of the nodal curve and the cuspidal curve of the reciprocal surface. It is to be observed that in designating a curve, for instance, as $18 = 4 \times 5 - 2$, this means that it is a curve of the order 18, the partial intersection of a quartic surface and a quintic surface, but without any explanation of the nature of the common curve 2 which causes the reduction, viz. without explaining whether this is a conic or a pair of lines, and so in other cases; this may be seen by reference to the proper section of the Memoir.

	Facultative lines.	Nodal curve.	Spinode curve.	Cuspidal curve.
$\mathrm{I} = 12$	27	27	$12 = 3 \times 4$	$24 = 6 \times 4$
$\mathrm{II} = 12 - C_2$	15	15	$12 = 3 \times 4$	$18 = 4 \times 5 - 2$
$\mathrm{III} = 12 - B_3$	9	9	$12 = 3 \times 4$	$16 = 4 \times 5 - 4$
$\mathrm{IV} = 12 - 2C_2$	7	7	$10 = 3 \times 4 - 2$	$12 = 4 \times 4 - 2 - 2$
$\mathrm{V} = 12 - B_4$	7 = 5 + edge twice	7 = 5 + rec. of edge twice, rec. of edge tacnodal	$10 = 3 \times 4 - 2$	$12 = 4 \times 4 - 4$
$\mathrm{VI} = 12 - B_3 - C_2$	3	3	$9 = 3 \times 4 - 3$	$10 = 4 \times 4 - 4 - 2$
$\mathrm{VII} = 12 - B_5$	3 = 2 + edge	3 = 2 + rec. of edge, rec. of edge is cuspnodal	9 = edge + unicursal 8-thic	10 = rec. of edge + unicursal 9-thic, rec. of edge is cuspnodal
$\mathrm{VIII} = 12 - 3C_2$	3	3	$6 = 2 \times 3$	$6 = 2 \times 3$
$\mathrm{IX} = 12 - 2B_3$	none	none	8 = 4 conics	8 = 4 conics
$\mathrm{X} = 12 - B_4 - C_2$	3 = 1 + edge twice	3 = 1 + rec. of edge twice, rec. of edge is tacnodal	$6 = 2 \times 3$	$6 = 2 \times 3$
$\mathrm{XI} = 12 - B_6$	3 = edge 3 times	3 = rec. of edge 3 times, rec. of edge is oscnodal	6 = 3 conics	6 = 3 conics
$\mathrm{XII} = 12 - U_6$	3	3	$6 = 2 \times 3$	$6 = 2 \times 3$
$\mathrm{XIII} = 12 - B_3 - 2C_2$	1	1	$4 = 2 \times 2$, nodal quadriquadric	$4 = 2 \times 2$ quadriquadric
$\mathrm{XIV} = 12 - B_5 - C_2$	1 = edge	1 = rec. of edge, rec. of edge is cuspnodal	4 = 3 + edge	4 = 3 + rec. of edge, rec. of edge is cuspnodal
$\mathrm{XV} = 12 - U_7$	1	1	$4 = 2 \times 2$, nodal quadriquadric	$4 = 2 \times 2$ cuspidal quadriquadric
$\mathrm{XVI} = 12 - 4C_2$	3	3	none	none
$\mathrm{XVII} = 12 - 2B_3 - C_2$	none	none	2 = conic	2 = conic
$\mathrm{XVIII} = 12 - B_4 - 2C_2$	3 = 1 + edge twice	1 + rec. of edge twice, rec. of edge tacnodal	none	no
$\mathrm{XIX} = 12 - B_6 - C_2$	3 = axis 3 times	3 = rec. of axis 3 times, rec. of axis oscnodal	none	none
$\mathrm{XX} = 12 - U_8$	none	none	2 = conic	2 = conic
$\mathrm{XXI} = 12 - 3B_3$	none	none	none	none

I pass now to the two cases of cubic scrolls.

Article No. 203. Section XXII $= S(1, 1)$. Equation $X^2W + Y^2Z = 0$.

203. As this is a scroll there is here no question of the 27 lines and 45 planes; there is a nodal line $X = 0$, $Y = 0$, $(b = 1)$ and a single directrix line, $Z = 0$, $W = 0$.

The Hessian surface is $X^2Y^2 = 0$; the complete intersection with the cubic surface is made up of $X = 0$, $Y = 0$ (the nodal line) eight times, and of the lines $X = 0$, $Z = 0$, and $Y = 0$, $W = 0$, each twice.

The reciprocal surface is $x^2z - y^2w = 0$; viz. this is a like scroll, XXII $= S(1, 1)$; $c' = 0$, $b' = 1$.

Article No. 204. Section XXIII $= S(\overline{1, 1})$. Equation $X(XW + YZ) + Y^3 = 0$.

204. This is also a scroll; there is a nodal line $X = 0$, $Y = 0$, and a single directrix line united therewith.

The Hessian surface is $X^4 = 0$; the complete intersection with the cubic surface is $X = 0$, $Y = 0$ (the nodal line) twelve times.

The reciprocal surface is $w(xw + yz) - z^3 = 0$; viz. this is a like scroll, XXIII $= S(\overline{1, 1})$; $c' = 0$, $b' = 1$.

Annex containing Additional Researches in regard to the case XX $= 12 - U_8$; *equation* $WX^2 + XZ^2 + Y^3 = 0$.

Let the surface be touched by the line (a, b, c, f, g, h), that is, the line the equations whereof are

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array}\right| \between X, Y, Z, W) = 0.$$

Writing the equation in the form $cW \,.\, cX^2 + X(cZ)^2 + c^2Y^3 = 0$, and substituting for cW, cZ their values in terms of X, Y, we have

$$(-gX + fY)\,cX^2 + X(aX + bY)^2 + c^2Y^3 = 0,$$

that is

$$(\ a^2 - cg\,,\ 2ab + cf,\ b^2,\ \ c^2 \between X, Y)^3 = 0,$$

or say

$$\big(3(a^2 - cg),\ 2ab + cf,\ b^2,\ 3c^2 \between X,\ Y)^3 = 0,$$

viz. the condition of contact is obtained by equating to zero the discriminant of the cubic function. We have thus

$$\begin{aligned} & 27c^4(a^2 - cg)^2 \\ + \ & 4b^6(a^2 - cg) \\ + \ & 4c^2(2ab + cf)^3 \\ - \ & b^4(2ab + cf)^2 \\ - \ & 18b^2c^2(a^2 - cg)(2ab + cf) = 0, \end{aligned}$$

viz. this is

$$\begin{aligned}&+27\,a^4c^2\\&-\ 4\,a^3b^3\\&+30\,a^2b^2cf\\&-54\,a^2c^3g\\&+36\,ab^3cg\\&+24\,abc^2f^2\\&+\ 4\,b^5h\\&-\ 1\,b^4f^2\\&+18\,b^2c^2fg\\&+27\,c^4g^2\\&+\ 4\,c^3f^3=0,\end{aligned}$$

which is the condition in order that the line (a, b, c, f, g, h) may touch the surface $X^2W+XZ^2+Y^3=0$; and if we unite thereto the conditions that the line shall pass through a given point $(\alpha, \beta, \gamma, \delta)$, we have in effect the equation of the circumscribed cone, vertex $(\alpha, \beta, \gamma, \delta)$.

Writing (f, g, h, a, b, c) in place of (a, b, c, f, g, h), we obtain

$$\begin{aligned}&27\,f^4h^2\\&-\ 4\,f^3g^3\\&+30\,f^2g^2ha\\&-54\,f^2h^3b\\&+36\,fg^3hb\\&+24\,fgh^2a^2\\&+\ 4\,g^5c\\&-\ 1\,g^4a^2\\&+18\,g^2h^2ab\\&+27\,h^4f^2\\&+\ 4\,h^3a^3=0\end{aligned}$$

as the condition that the line (a, b, c, f, g, h) shall touch the reciprocal surface

$$27\,(4\,xw+z^2)^2+64\,y^3w=0\,;$$

and if we consider a, b, c, f, g, h as standing for

$$\gamma y-\beta z,\ \alpha z-\gamma x,\ \beta x-\alpha y,\ \delta x-\alpha w,\ \delta y-\beta w,\ \delta z-\gamma w,$$

values which satisfy the relation

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array} \right| \Big)\!\Big(\alpha, \beta, \gamma, \delta\Big) = 0,$$

then the equation in (a, b, c, f, g, h) is that of the circumscribed cone, vertex $(\alpha, \beta, \gamma, \delta)$; the order being (as it should be) $a' = 6$.

The cuspidal conic is $y = 0$, $4xw + z^2 = 0$, and we at once obtain $a^2 - 4cg = 0$ as the condition that the line (a, b, c, f, g, h) shall pass through the cuspidal cone. Hence attributing to (a, b, c, f, g, h) the foregoing values, we have

$$a^2 - 4cg = 0$$

for the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the cuspidal conic; this is of course a quadric cone, $c' = 2$. I proceed to determine the intersections of the two cones.

Representing by $\Theta = 0$ the foregoing equation of the circumscribed cone, and putting for shortness

$$X = 27h^2 (f^2 - bh) - 2g^2 (2fg + ah),$$

I find that we have identically

$$\Theta - (f^2 - bh) X + (g^4 - 4ah^3 - 8fgh^2)(a^2 - 4cg) - (32fg^2h + 16agh^2)(af + bg + ch) = 0:$$

whence in virtue of the relation $af + bg + ch = 0$, we see that the equations $\Theta = 0$, $a^2 - 4cg = 0$, are equivalent to

$$(f^2 - bh) X = 0, \quad a^2 - 4cg = 0,$$

or the twelve lines of intersection break up into the two systems

$$f^2 - bh = 0, \quad a^2 - 4cg = 0,$$

and

$$(X =) \quad 27h^2 (f^2 - bh) - 2g^2 (2fg + ah) = 0, \quad a^2 - 4cg = 0.$$

To determine the lines in question, observe that we have

$$\left(\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array} \right| \Big)\!\Big(\alpha, \beta, \gamma, \delta\Big) = 0;$$

and we can by the first three of these express a, b, c linearly in terms of f, g, h; the equations $f^2 - bh = 0$, $a^2 - 4cg = 0$, $27h^2(f^2 - bh) - 2g^2(2fg + ah) = 0$ become thus homogeneous equations in (f, g, h); the equations may in fact be written

$$\begin{aligned}\delta^2 (a^2 - 4cg) &= (\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf = 0,\\ \delta\ (f^2 - bh\,) &= \delta f^2 - \alpha h^2 + \gamma hf = 0,\\ \delta X \qquad &= 27h^2 (\delta f^2 - \alpha h^2 + \gamma hf) + 2g^2 (\beta h^2 - \gamma gh - 2\delta fg) = 0,\end{aligned}$$

viz. interpreting (f, g, h) as coordinates *in plano*, the first equation represents a conic, the second a pair of lines, and the third a quartic.

We have identically

$$\begin{aligned}&\{2\beta\delta f - (\gamma^2 + 4\alpha\delta)\, g + \beta\gamma h\}^2 - 4\beta^2\delta\, (\delta f^2 - \alpha h^2 + \gamma hf)\\ &\qquad = (\gamma^2 + 4\alpha\delta)\, \{(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf\}\,;\end{aligned}$$

and it thus appears that the two conics touch at the points given by the equations

$$\delta f^2 - \alpha h^2 + \gamma hf = 0,$$

$$2\beta\delta f - (\gamma^2 + 4\alpha\delta)\, g + \beta\gamma h = 0:$$

we have moreover

$$\begin{aligned}-(\gamma^2 + 4\alpha\delta)(\beta h^2 - \gamma gh - 2\delta fg) &= 4\beta\delta\, (\delta f^2 - \alpha h^2 + \gamma hf)\\ &\quad + (-2\delta f - \gamma h)\, [2\beta\delta f - (\gamma^2 + 4\alpha\delta)\, g + \beta\gamma h],\end{aligned}$$

hence at the last-mentioned two points $-\beta h^2 + \gamma gh + 2\delta fg$ is $= 0$; and the quartic $X = 0$ thus passes through these two points.

The conic $(a^2 - cg) = 0$ and the quartic $X = 0$ intersect besides (as is evident) in the point $g = 0$, $h = 0$ reckoning as two points, since it is a node of the quartic; and they must consequently intersect in four more points: to obtain these in the most simple manner, write for a moment

$$\Omega = -(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2,$$

then we have identically

$$\begin{aligned}16\beta^2\delta g^2\, (\delta f^2 - \alpha h^2 + \gamma hf\,) - \Omega^2 &= -\,[(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2] + 4\beta^2g^2\, (\gamma h + 2\delta f\,)^2,\\ = -\,\{(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 &- 2\beta\gamma gh - 2\beta\delta fg\}\, \{(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 + 2\beta\gamma gh + 4\beta\delta fg\},\\ = -\,\delta\, (a^2 - 4cg)\, &\{(\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 + 2\beta\gamma gh + 4\beta\delta fg\}\,;\end{aligned}$$

and moreover

$$2\beta\, (\beta h^2 - 2\delta fg - \gamma gh) - \Omega = (\gamma^2 + 4\alpha\delta)\, g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta fg = \delta\, (a^2 - 4cg).$$

Hence when $a^2 - 4cg = 0$, we have

$$\delta f^2 - \alpha h^2 + \gamma hf = \frac{\Omega^2}{16\beta^2\delta g^2},\ \beta h^2 - 2\delta fg - \gamma gh = \frac{\Omega}{2\beta}\,;$$

and substituting these values in the equation $X = 0$, it becomes

$$27h^2 \cdot \frac{\Omega^2}{16\beta^2\delta} + 2g^2 \cdot \frac{\Omega}{2\beta} = 0,$$

viz. multiplying by $16\beta^2\delta$, and omitting the factor Ω, this is

$$27h^2\Omega + 16\beta\delta g^4 = 0,$$

or finally

$$16\beta\delta g^4 - 27(\gamma^2 + 4\alpha\delta) g^2h^2 + 27\beta^2h^4 = 0,$$

a pencil of four lines, each passing through the point $g = 0$, $h = 0$, and therefore intersecting the conic

$$(\gamma^2 + 4\alpha\delta) g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf = 0$$

at that point and at one other point; and we have thus four points of intersection, which are the required four points.

Recapitulating, the conic $a^2 - 4cg = 0$ meets the sextic $(f^2 - bh) X = 0$ in the two points

$$\begin{cases} \delta f^2 - \alpha h^2 + \gamma hf = 0, \\ 2\beta\delta f - (\gamma^2 + 4\alpha\delta) g + \beta\gamma h = 0 \end{cases}$$

each three times, in the point $g = 0$, $h = 0$ twice, and in the four points

$$\begin{cases} 16\beta\delta g^4 - 27(\gamma^2 + 4\alpha\delta) g^2h^2 + 27\beta^2h^4 = 0, \\ (\gamma^2 + 4\alpha\delta) g^2 + \beta^2h^2 - 2\beta\gamma gh - 4\beta\delta hf = 0 \end{cases}$$

each once. Or reverting to the proper significations of (a, b, c, f, g, h), instead of points we have 2 lines each three times, a line twice, and 4 lines each once; the line $g = 0$, $h = 0$, that is, $g = 0$, $h = 0$, $a = 0$, being, it will be observed, the line $\frac{y}{\beta} = \frac{z}{\gamma} = \frac{w}{\delta}$ drawn from $(\alpha, \beta, \gamma, \delta)$ to the point $y = 0$, $z = 0$, $w = 0$, which is the reciprocal of the uniplane $X = 0$: the twelve lines are the $a'c'$ lines of intersection of the circumscribed cone a' with the cuspidal cone c', viz. $a'c' = [a'c'] + 3\sigma' + \chi'$; $[a'c'] = 4$ referring to the last-mentioned four lines; $\sigma' = 2$ to the two lines; and $\chi' = 2$ to the line $g = 0$, $h = 0$, $a = 0$, which it thus appears must in the present case be reckoned twice.