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REPRODUCTION OF EULER'S MEMOIR OF 1758 ON THE ROTATION OF A SOLID BODY.

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EULER'S Memoir "Du mouvement de rotation des corps solides autour d'un axe variable," *Mém. de Berlin*, 1758, pp. 154—193 (printed in 1765), seems to have been written subsequently to the memoir with a similar title in the Berlin *Memoirs* for 1760, and to the "Theoria Motus Corporum Solidorum &c.," *Rostock*, 1765, and there are contained in the first-mentioned memoir some very interesting results which appear to have escaped the notice of later writers on the subject; viz. Euler succeeds in integrating the equations of motion *without the assistance furnished by the consideration of the invariable plane.* In reproducing these results I make the following alterations in Euler's notation, viz. instead of x, y, z I write p, q, r; instead of Ma^2, Mb^2, Mc^2 (where M is the mass) I write A, B, C, these quantities denoting the principal moments, and in some equations where the omission or insertion of the factor M is really immaterial I write A, B, C in the place of a^2, b^2, c^2; moreover instead of Euler's A, B, C $\left(\text{which denote respectively } \frac{b^2-c^2}{a^2}, \frac{c^2-a^2}{b^2}, \frac{a^2-b^2}{c^2}\right)$ I write L, M, N; but in other respects Euler's notation is preserved. The equations of motion are

$$Adp + (C-B)\,qrdt = 0,$$
$$Bdq + (A-C)\,rpdt = 0,$$
$$Cdr + (B-A)\,pqdt = 0\,;$$

so that putting for shortness

$$L = \frac{B-C}{A}, \quad M = \frac{C-A}{B}, \quad N = \frac{A-B}{C},$$

and introducing the auxiliary quantity u such that $du = pqrdt$, we have

$$p^2 = \mathfrak{A} + 2Lu,$$
$$q^2 = \mathfrak{B} + 2Mu,$$
$$r^2 = \mathfrak{C} + 2Nu,$$

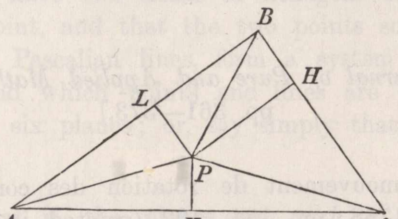
where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are constants of integration, and thence

$$t = \int \frac{du}{\sqrt{\{(\mathfrak{A} + 2Lu)(\mathfrak{B} + 2Mu)\}(\mathfrak{C} + 2Nu)}},$$

where the integral may without loss of generality be taken from $u = 0$; u, and consequently p, q, r, are thus given functions of t; and it is moreover clear that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are the initial values of p^2, q^2, r^2. We have also if ω be the angular velocity round the instantaneous axis

$$\omega^2 = \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + 2(L + M + N)u.$$

Euler then assumes that the position in space of the principal axes is geometrically determined as follows, viz. (treating the axes as points on a sphere) it is assumed that the distances from a fixed point P of the sphere are respectively l, m, n, and that



the inclinations of these distances to a fixed arc PQ are respectively λ, μ, ν. We have then the geometrical relations

$$\cos^2 l + \cos^2 m + \cos^2 n = 1;$$

$$\sin(\mu - \nu) = \frac{\cos l}{\sin m \sin n}, \quad \cos(\mu - \nu) = -\frac{\cos m \cos n}{\sin m \sin n},$$

$$\sin(\nu - \lambda) = \frac{\cos m}{\sin n \sin l}, \quad \cos(\nu - \lambda) = -\frac{\cos n \cos l}{\sin n \sin l},$$

$$\sin(\lambda - \mu) = \frac{\cos n}{\sin l \sin m}, \quad \cos(\lambda - \mu) = \frac{\cos l \cos m}{\sin l \sin m};$$

whence also

$$\sin \mu = \frac{-\cos \lambda \cos n - \sin \lambda \cos l \cos m}{\sin l \sin m},$$

$$\cos \mu = \frac{\sin \lambda \cos n - \cos \lambda \cos l \cos m}{\sin l \sin m},$$

$$\sin \nu = \frac{\cos \lambda \cos m + \sin \lambda \cos l \cos n}{\sin l \sin n},$$

$$\cos \nu = \frac{-\sin \lambda \cos m - \cos \lambda \cos l \cos n}{\sin l \sin n}.$$

The geometrical equations connecting the resolved angular velocities p, q, r with the differentials of $l, m, n, \lambda, \mu, \nu$ are

$$\begin{aligned} dl\ \sin l &= dt\,(q\cos n - r\cos m), & d\lambda \sin^2 l &= -\,dt\,(q\cos m + r\cos n),\\ dm \sin m &= dt\,(r\cos l - p\cos n), & d\mu \sin^2 m &= -\,dt\,(r\cos n + p\cos l),\\ dn\ \sin n &= dt\,(p\cos m - q\cos l), & d\nu\ \sin^2 n &= -\,dt\,(p\cos l + q\cos m). \end{aligned}$$

Multiplying the equations of motion respectively by $\cos l$, $\cos m$, $\cos n$, and adding, we obtain an equation which is reducible to the form

$$d\,(Ap\cos l + Bq\cos m + Cr\cos n) = 0,$$

whence integrating

$$Ap\cos l + Bq\cos m + Cr\cos n = \mathfrak{D},$$

$\mathfrak{D}$ being a constant of integration. One other integral equation is necessary for the determination of the angles l, m, n. The expressions for dl, dm, dn give at once

$$p\,dl\sin l + q\,dm\sin m + r\,dn\sin n = 0.$$

Instead of the arcs l, m, n, Euler introduces a new variable v, such that

$$v = p\cos l + q\cos m + r\cos n;$$

by means of the last preceding equation, we find

$$dv = dp\cos l + dq\cos m + dr\cos n,$$

and then, substituting for dp, dq, dr, their values,

$$dv = \left(\frac{L\cos l}{p} + \frac{M\cos m}{q} + \frac{N\cos n}{r}\right)du,$$

from which the relation between v and u is to be determined. We have

$$\begin{aligned} \cos^2 l + \cos^2 m + \cos^2 n &= 1,\\ Ap\cos l + Bq\cos m + Cr\cos n &= \mathfrak{D},\\ p\cos l + q\cos m + r\cos n &= v, \end{aligned}$$

which give $\cos l$, $\cos m$, $\cos n$ in terms of u, v; the resulting formulæ contain the radical

$$\sqrt{\left\{\begin{aligned}&(L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2) - \mathfrak{D}^2(x^2+y^2+z^2)\\ &\qquad + 2\mathfrak{D}v\,(Ap^2 + Bq^2 + Cr^2) - v^2(A^2p^2 + B^2q^2 + C^2r^2)\end{aligned}\right\}},$$

which for shortness is represented by $\sqrt{\{(\cdot\cdot)\}}$. We then have

$$\cos l = \frac{\mathfrak{D}p\,(NCq^2 - MBr^2) + BCpv\,(Mr^2 - Nq^2) + LAqr\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

$$\cos m = \frac{\mathfrak{D}q\,(LAr^2 - NCp^2) + CAqv\,(Np^2 - Lr^2) + MBrp\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

$$\cos n = \frac{\mathfrak{D}r\,(MBp^2 - LAq^2) + ABrv\,(Lq^2 - Mp^2) + NCpq\sqrt{\{(\cdot\cdot)\}}}{L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2},$$

and substituting these values in the differential equation

$$\frac{dv}{du} = \frac{L\cos l}{p} + \frac{M\cos m}{q} + \frac{N\cos n}{r},$$

the equation to be integrated becomes

$$\frac{dv}{du}(L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2) = LMN\mathfrak{D}(Ap^2 + Bq^2 + Cr^2) - LMNv(A^2p^2 + B^2q^2 + C^2r^2)$$
$$+ \frac{1}{pqr}(L^2Aq^2r^2 + M^2Br^2p^2 + N^2Cp^2q^2).$$

Now substituting for p, q, r their values, we have

$$L^2A^2q^2r^2 + M^2B^2r^2p^2 + N^2C^2p^2q^2 = L^2A^2\mathfrak{B}\mathfrak{C} + M^2B^2\mathfrak{C}\mathfrak{A} + N^2C^2\mathfrak{A}\mathfrak{B} - 2LMNu(\mathfrak{A}A^2 + \mathfrak{B}B^2 + \mathfrak{C}C^2),$$

$$L^2Aq^2r^2 + M^2Br^2p^2 + N^2Cp^2q^2 = L^2A\mathfrak{B}\mathfrak{C} + M^2B\mathfrak{C}\mathfrak{A} + N^2C\mathfrak{A}\mathfrak{B} - 2LMNu(\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C),$$

$$p^2 + q^2 + r^2 = \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + 2(L + M + N)u,$$

$$Ap^2 + Bq^2 + Cr^2 = \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C,$$

$$A^2p^2 + B^2q^2 + C^2r^2 = \mathfrak{A}A^2 + \mathfrak{B}B^2 + \mathfrak{C}C^2:$$

and writing for shortness

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = E,$$

$$\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C = F,$$

$$\mathfrak{A}A^2 + \mathfrak{B}B^2 + \mathfrak{C}C^2 = G,$$

$$L^2A\mathfrak{B}\mathfrak{C} + M^2B\mathfrak{C}\mathfrak{A} + N^2C\mathfrak{A}\mathfrak{B} = H,$$

$$L^2A^2\mathfrak{B}\mathfrak{C} + M^2B^2\mathfrak{C}\mathfrak{A} + N^2C^2\mathfrak{A}\mathfrak{B} = K,$$

where $K = EG - F^2$, substituting these values and observing that

$$L + M + N = -LMN,$$

the radical of the formula becomes

$$\surd\{(\cdot\cdot)\} = \surd(K - 2LMNGu + 2\mathfrak{D}^2LMNu - \mathfrak{D}^2E + 2\mathfrak{D}Fv - Gv^2),$$

and the differential equation becomes

$$\frac{dv}{du}(K - 2LMNGu) = LMN\mathfrak{D}F - LMNGv + \frac{1}{pqr}(H - 2LMNFu)\surd\{(\cdot\cdot)\},$$

which can be reduced to the form

$$\frac{Kdv - LMNF\mathfrak{D}du - 2LMNGudv + LMNGvdu}{\surd\{K - \mathfrak{D}^2E + 2LMN(\mathfrak{D}^2 - G)u + 2\mathfrak{D}Fv - Gv^2\}} = \frac{Hdu - 2LMNFudu}{\surd\{(2Lu + \mathfrak{A})(2Mu + \mathfrak{B})(2Nu + \mathfrak{C})\}}.$$

Euler remarks that as the right-hand side of the equation contains only the variable u, the solution will be effected if we can find a function of u, a multiplier of the left-hand side; he had elsewhere explained the method of finding such multipliers, and applying it to the equation in hand, the multiplier of the left-hand side, and therefore of the equation itself, is found to be $\frac{1}{K-2LMNGu}$, or what is the same thing $\frac{\sqrt{(G)}}{K-2LMNGu}$.

Multiplying by this quantity, the right-hand side may for shortness be represented by dU, so that

$$dU = \frac{(H-2LMNFu)\sqrt{(G)}\,du}{(K-2LMNGu)\sqrt{\{(2Lu+\mathfrak{A})(2Mu+\mathfrak{B})(2Nu+\mathfrak{C})\}}},$$

and U may be considered as a given function of u, or what is the same thing of t.

As regards the left-hand side, attending to the equation $K=EG-F^2$, the radical multiplied into $\sqrt{(G)}$ may be presented under the form

$$\sqrt{[\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}]};$$

and consequently the left-hand side becomes

$$\frac{(K-2LMNGu)\,Gdv+LMNG\,(Gv-\mathfrak{D}F)\,du}{(K-LMNGu)\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}}},$$

which putting for the moment $K-2LMNGu=p^2$, $Gv-\mathfrak{D}F=q$, $G-\mathfrak{D}^2=f^2$, becomes $\frac{pdq-qdp}{p\sqrt{(f^2p^2-q^2)}}$, the integral of which is $\sin^{-1}\frac{q}{fp}$; hence restoring the values of p, q, f, the integral is

$$\sin^{-1}\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{(K-2LMNGu)}}.$$

Hence considering the constant of integration as included in U, or writing

$$U=\mathfrak{C}+\int\frac{(H-2LMNFu)\sqrt{(G)}\,du}{(K-2LMNGu)\sqrt{\{(2Lu+\mathfrak{A})(2Mu+\mathfrak{B})(2Nu+\mathfrak{C})\}}},$$

we have for the required integral of the differential equation

$$\sin^{-1}\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=U,$$

whence also

$$\frac{Gv-\mathfrak{D}F}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=\sin U,$$

and

$$\frac{\sqrt{[\{(G-\mathfrak{D}^2)(K-2LMNGu)-(Gv-\mathfrak{D}F)^2\}]}}{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}=\cos U,$$

so that the value of the original radical is

$$\sqrt{\{(\cdot\cdot)\}} = \frac{\sqrt{(G-\mathfrak{D}^2)}\sqrt{\{(K-2LMNGu)\}}}{\sqrt{(G)}}\cos U.$$

Substituting in the expressions for the cosines of the arcs l, m, n, these values of v and the radical; the formulæ after some reductions become

$$\cos l = \frac{\mathfrak{D}Ap}{G} + \frac{BCp(M\mathfrak{C}-N\mathfrak{B})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{LAqr\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

$$\cos m = \frac{\mathfrak{D}Bq}{G} + \frac{CAq(N\mathfrak{A}-L\mathfrak{C})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{MBrp\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

$$\cos n = \frac{\mathfrak{D}Cr}{G} + \frac{ABr(L\mathfrak{B}-M\mathfrak{A})\sqrt{(G-\mathfrak{D}^2)}}{G\sqrt{(K-2LMNGu)}}\sin U + \frac{NCpq\sqrt{(G-\mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K-2LMNGu)}}\cos U,$$

where for shortness p, q, r are retained in place of their values $\sqrt{(2Lu+\mathfrak{A})}$, $\sqrt{(2Mu+\mathfrak{B})}$, $\sqrt{(2Nu+\mathfrak{C})}$.

The values of l, m, n being known, that of λ could be determined by the differential equation

$$d\lambda = -\frac{dt\,(q\cos m + z\cos n)}{\sin^2 l},$$

and then the values of μ, ν would be determined without any further integration; but it is better to consider, in the place of any one of the principal axes in particular, the instantaneous axis, which is a line inclined to these at angles α, β, γ, the cosines of which are $\frac{p}{\omega}$, $\frac{q}{\omega}$, $\frac{r}{\omega}$ (if as before $\omega^2 = p^2+q^2+r^2$). Considering the instantaneous axis as a point of the sphere, let j denote the distance OP from the fixed point P, and ϕ the inclination OPQ of this distance to the fixed arc PQ. We have

$$\cos j = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n,$$

$$\sin j\cos\phi = \cos\alpha\sin l\cos\lambda + \cos\beta\sin m\cos\mu + \cos\gamma\sin n\cos\nu,$$

$$\sin j\sin\phi = \cos\alpha\sin l\sin\lambda + \cos\beta\sin m\sin\mu + \cos\gamma\sin n\sin\nu,$$

$$\cos(\lambda-\phi) = \frac{\cos\alpha - \cos l\cos j}{\sin l\sin j}, \quad \sin(\lambda-\phi) = \frac{\cos\gamma\cos m - \cos\beta\cos n}{\sin l\sin j},$$

$$\cos(\mu-\phi) = \frac{\cos\beta - \cos m\cos j}{\sin m\sin j}, \quad \cos(\mu-\phi) = \frac{\cos\alpha\cos n - \cos\gamma\cos l}{\sin m\sin j},$$

$$\cos(\nu-\phi) = \frac{\cos\gamma - \cos n\cos j}{\sin n\sin j}, \quad \cos(\nu-\phi) = \frac{\cos\beta\cos l - \cos\alpha\cos m}{\sin n\sin j},$$

so that λ, μ, ν are determined in terms of j and ϕ. These expressions give

$$d\phi = \frac{1}{\gamma^2 \sin^2 j}\{\cos l\,(qdr - rdq) + \cos m\,(rdp - pdr) + \cos n\,(pdq - qdp)\},$$

which is reducible to Euler's equation

$$d\phi = dt\,\frac{p\,(M\mathfrak{C} - N\mathfrak{B})\cos l + q\,(N\mathfrak{A} - L\mathfrak{C})\cos m + r\,(L\mathfrak{B} - M\mathfrak{A})\cos n}{E - 2LMNu - v^2},$$

and thence, substituting for $\cos l$, $\cos m$, $\cos n$ their values, and observing that

$$Ap^2(M\mathfrak{C} - N\mathfrak{B}) + Bq^2(N\mathfrak{A} - L\mathfrak{C}) + Cr^2(L\mathfrak{B} - M\mathfrak{A}) = -(H - 2LMNFu),$$

$$BCp^2(M\mathfrak{C} - N\mathfrak{B})^2 + CAq^2(N\mathfrak{A} - L\mathfrak{C})^2 + ABr^2(L\mathfrak{B} - M\mathfrak{A})^2 = F(H - 2LMNFu),$$

$$LA\,(M\mathfrak{C} - N\mathfrak{B}) + MB\,(N\mathfrak{A} - L\mathfrak{C}) + NC\,(L\mathfrak{B} - M\mathfrak{A}) = LMNF,$$

the equation becomes

$$d\phi\,(E - 2LMNu - v^2) \div dt = \frac{-\mathfrak{D}\,(H - 2LMFu)}{G} + \frac{F\,(H - 2LMNFu)\sqrt{(G - \mathfrak{D}^2)}}{G\sqrt{(K - 2LMNGu)}}\sin U$$
$$+ \frac{LMNFpqr\sqrt{(G - \mathfrak{D}^2)}}{\sqrt{(G)}\sqrt{(K - 2LMNGu)}}\cos U,$$

where it is to be remarked that

$$G^2(E - 2LMNu - v^2)$$
$$= (G - \mathfrak{D}^2)F^2 + G\,(K - 2LMNGu) - (G - \mathfrak{D}^2)(K - 2LMNGu)\sin^2 U$$
$$- 2\mathfrak{D}F\sqrt{(G - \mathfrak{D}^2)}\sqrt{\{(K - 2LMNGu)\}}\sin U.$$

Now

$$dU = \frac{dt\,(H - 2LMNFu)\sqrt{(G)}}{K - 2LMNGu},\quad du = pqrdt,$$

the differential $d\phi$ can be expressed as a fraction, the numerator whereof is

$$-\mathfrak{D}dU\,(K - 2LMNGu)\sqrt{(G)} + FdU\sqrt{\{G\,(G - \mathfrak{D}^2)(K - 2LMNGu)\}}\sin U$$
$$+ \frac{LMNFGdu\sqrt{\{G\,(G - \mathfrak{D}^2)\}}}{\sqrt{(K - 2LMNGu)}}\cos U,$$

and the denominator

$$(G - \mathfrak{D}^2)F^2 + G\,(K - 2LMNGu) - 2\mathfrak{D}F\sqrt{(G - \mathfrak{D}^2)(K - 2LMNGu)}\sin U$$
$$- (G - \mathfrak{D}^2)(K - 2LMNGu)\sin^2 U.$$

To simplify, write

$$\sqrt{(K - 2LMNGu)} = s,\quad \sqrt{(G - \mathfrak{D}^2)} = h,$$

the numerator is

$$-\mathfrak{D}s^2\,dU\sqrt{(G)}+Fhs\,dU\sqrt{(G)}\sin U-Fhs\sqrt{(G)}\cos U,$$

and the denominator

$$h^2F^2+Gs^2-2\mathfrak{D}Fhs\sin U-h^2s^2\sin^2 U,$$

which, observing that $h^2=G-\mathfrak{D}^2$, is equal to

$$(Fh-\mathfrak{D}s\sin U)^2+Gs^2\cos^2 U,$$

and we have

$$d\phi=\frac{-\mathfrak{D}s^2dU+Fhs\sin U dU-Fhds\cos U}{(Fh-\mathfrak{D}s\sin U)^2+Gs^2\cos^2 U}\sqrt{(G)}$$

the integral of which is

$$\phi+\mathfrak{F}=\tan^{-1}\frac{Fh-\mathfrak{D}s\sin U}{s\cos U\sqrt{(G)}},$$

where $\mathfrak{F}$ is the constant of integration, or substituting for h, s their values, the equation is

$$\tan(\phi+\mathfrak{F})=\frac{F\sqrt{(G-\mathfrak{D}^2)}-\mathfrak{D}\sin U\sqrt{(K-2LMNGu)}}{\cos U\sqrt{\{G(K-2LMNGu)\}}}.$$

It may be added that

$$\omega\cos j=v=\frac{1}{G}\left[\mathfrak{D}F+\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U\right],$$

and therefore

$$\cos j=\frac{\mathfrak{D}F+\sqrt{\{(G-\mathfrak{D}^2)(K-2LMNGu)\}}\sin U}{G\sqrt{(E-2LMNu)}}.$$

Euler remarks that the complexity of the solution owing to the circumstance that the fixed point P is left arbitrary; and that the formulæ may be simplified by taking this point so that $G-\mathfrak{D}^2=0$, and he gives the far more simple formulæ corresponding to this assumption; this is in fact taking the point P in the *direction of the normal to the invariable plane,* and the resulting formulæ are identical with the ordinary formulæ for the solution of the problem. The term *invariable plane* is not used by Euler, and seems to have first occurred in Lagrange's "Essai sur le problème de trois corps," *Prix de l'Acad. de Berlin*, t. IX., 1772.

To prove the before-mentioned equation for $d\phi$; starting from the equations

$$\cos j=\cos\alpha\cos l+\cos\beta\cos m+\cos\gamma\cos n=\frac{v}{\omega},$$

$$\sin j\cos\phi=\cos\alpha\sin l\cos\lambda+\cos\beta\sin m\cos\mu+\cos\gamma\sin n\cos\nu$$

$$\sin j\sin\phi=\cos\alpha\sin l\cos\lambda+\cos\beta\sin m\sin\mu+\sin\gamma\sin n\sin\nu,$$

we have

$$\cos j\,dj\cos\phi-\sin j\sin\phi\,d\phi$$

$$=-\sin\alpha\,d\alpha\sin l\cos\lambda-\text{\&c.}+\cos\alpha\cos\lambda\cos l\,dl+\text{\&c.}-\cos\alpha\sin l\sin\lambda\,d\lambda+\text{\&c.},$$

the second term is

$$\frac{p}{\omega}\cos\lambda\cot\ l\ (q\cos n - r\cos m)$$

$$+\frac{q}{\omega}\cos\mu\cot m\,(r\cos l - p\cos n\,)$$

$$+\frac{r}{\omega}\cos\nu\cot\ n\ (p\cos m - q\cos l),$$

and the third term is

$$+\frac{p}{\omega}\sin\lambda\,\mathrm{cosec}\ l\ (q\cos m + r\cos n\,)$$

$$+\frac{q}{\omega}\sin\mu\,\mathrm{cosec}\,m\,(r\cos n + p\cos l)$$

$$+\frac{r}{\omega}\sin\nu\,\mathrm{cosec}\,n\ (p\cos\ l + q\cos m).$$

Hence the second and third terms together are

$$=\frac{pq}{\omega}\left(\cos\lambda\,\frac{\cos l\cos n}{\sin l} - \cos\mu\,\frac{\cos m\cos n}{\sin m} + \sin\lambda\,\frac{\cos m}{\sin l} + \sin\mu\,\frac{\cos l}{\sin m}\right) + \&c.,$$

$$=\frac{pq}{\omega}\begin{cases}-\cos\lambda\sin n\cos(\nu-\lambda) + \sin\lambda\sin n\sin(\nu-\lambda)\\ +\cos\mu\sin n\cos(\mu-\nu) + \sin\mu\sin n\sin(\mu-\nu)\end{cases} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\begin{cases}-\cos\lambda\cos(\nu-\lambda) + \sin\lambda\sin(\nu-\lambda)\\ +\cos\mu\cos(\mu-\nu) + \sin\mu\sin(\mu-\nu)\end{cases} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\begin{cases}-\cos\{\lambda+(\nu-\lambda)\}\\ +\cos\{\mu-(\mu-\nu)\}\end{cases} + \&c.,$$

$$=\frac{pq}{\omega}\sin n\,(-\cos\nu + \cos\nu) + \&c., = 0\,;$$

we have therefore

$$\cos j\,dj\cos\phi - \sin j\sin\phi\,d\phi$$

$$= -\sin\alpha\,d\alpha\sin l\cos\lambda - \sin\beta\,d\beta\sin m\cos\mu - \sin\gamma\,d\gamma\sin n\cos\nu,$$

$$= d\,\frac{p}{\omega}\,.\sin l\cos\lambda + d\,\frac{q}{\omega}\,.\sin m\cos\mu + d\,\frac{r}{\omega}\,.\sin n\sin\nu$$

$$= +\frac{1}{\omega}\ (\sin l\cos\lambda\,dp + \sin m\cos\mu\,dq + \sin n\cos\nu\,dr)$$

$$-\frac{d\omega}{\omega^2}\,(\sin l\cos\lambda\,p + \sin m\cos\mu\,q + \sin n\cos\nu\,r)$$

$$= -\cot j\cos\phi\,d\frac{v}{\omega} - \sin j\sin\phi\,d\phi.$$

Hence therefore

$$\sin j \sin \phi d\phi = -\cot j \cos \phi d \frac{v}{\omega}$$

$$-\frac{1}{\omega}(\sin l \cos \lambda \, dp + \sin m \cos \mu \, dq + \sin n \cos \nu \, dr)$$

$$+\frac{d\omega}{\omega^2}(\sin l \cos \lambda \, p + \sin m \cos \mu \, q + \sin n \cos \nu \, r)$$

$$= -\cot j \cos \phi \frac{1}{\omega}(\cos l \, dp + \cos m \, dq + \cos n \, dr)$$

$$+\cot j \cos \phi \frac{d\omega}{\omega^2}(p \cos l + q \cos m + r \cos n)$$

$$-\frac{1}{\omega}(\sin l \cos \lambda \, dp + \sin m \cos \mu \, dq + \sin n \cos \nu \, dr)$$

$$+\frac{d\omega}{\omega^2}(\sin l \cos \lambda \, . \, p + \sin m \cos \mu \, . \, q + \sin n \cos \nu \, . \, r)$$

$$= \frac{1}{\omega}\{(-\cot j \cos \phi \cos l - \sin l \cos \lambda)\, dp + \&c.\}$$

$$+\frac{d\omega}{\omega^2}\{(\ \ \cot j \cos \phi \cos l + \sin l \cos \lambda)\, p \ \ + \&c.\}.$$

But we have

$$\cos(\lambda - \phi) = \frac{\cos \alpha - \cos l \cos j}{\sin l \sin j},$$

$$= \frac{\cos \alpha}{\sin l \sin j} - \cot l \cot j,$$

$$\sin(\lambda - \phi) = \frac{\cos \gamma \cos m - \cos \beta \cos n}{\sin l \sin j}$$

and thence

$$\cos \lambda = \cos\{(\lambda - \phi) + \phi\} = \frac{\cos \phi (\cos \alpha - \cos l \cos j) - \sin \phi (\cos \gamma \cos m - \cos \beta \cos n)}{\sin l \sin j},$$

whence also

$$\cot j \cos \phi \cos l + \sin l \cos \lambda$$

$$= \frac{1}{\sin j}\{\cos \phi \cos l + \cos \phi (\cos \alpha - \cos l \cos j) - \sin \phi (\cos \gamma \cos m - \cos \beta \cos n)\},$$

$$= \frac{1}{\sin j}\{\cos \alpha \cos \phi - \sin \phi (\cos \gamma \cos m - \cos \beta \cos n)\},$$

$$= \frac{1}{\omega \sin j}\{p \cos \phi - \sin \phi (r \cos m - q \cos n)\}.$$

Hence the expression for $\sin j \sin\phi\, d\phi$ is

$$= -\frac{1}{\omega^2 \sin j}\,[\{p\cos\phi - \sin\phi\,(r\cos m - q\cos n)\}\,dp + \ldots]$$

$$+\frac{d\omega}{\omega^3 \sin j}\,[\{p\cos\phi - \sin\phi\,(r\cos m - q\cos n)\}\;p + \ldots]$$

$$= -\frac{1}{\omega^2 \sin j}\,[\omega\, d\omega\cos\phi - \sin\phi\,\{(r\cos m - q\cos n)\,dp + \ldots\}]$$

$$+\frac{d\gamma}{\omega^3 \sin j}\,\omega^2\cos\phi \qquad = \sin j\sin\phi\, d\phi,$$

or finally

$$\sin j \sin\phi\, d\phi = \frac{1}{\omega^2}\frac{\sin\phi}{\sin j}\,[(r\cos m - q\cos n)\,dp + \&\text{c.}],$$

that is

$$d\phi = \frac{1}{\omega^2\sin^2 j}\left\{\begin{array}{l}(r\cos m - q\cos n)\,dp\\ +(p\cos n - r\cos l)\,dq\\ +(q\cos l - p\cos m)\,dr\end{array}\right\},$$

which is the required expression for $d\phi$.

Recapitulating, A, B, C, p, q, r denote as usual,

$$L = \frac{B-C}{A},\; M = \frac{C-A}{B},\; N = \frac{A-B}{C},\; du = pqrdt,$$

$$p = \surd(\mathfrak{A} + 2Lu),$$

$$q = \surd(\mathfrak{B} + 2Mu),$$

$$r = \surd(\mathfrak{C} + 2Nu);$$

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = E,$$

$$\mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C = F,$$

$$\mathfrak{A}A^2 + \mathfrak{B}B^2 + \mathfrak{C}C^2 = G;$$

$$L^2A\mathfrak{B}\mathfrak{C} + M^2B\mathfrak{C}\mathfrak{A} + N^2C\mathfrak{A}\mathfrak{B} = H,$$

$$L^2A^2\mathfrak{B}\mathfrak{C} + M^2B^2\mathfrak{C}\mathfrak{A} + N^2C^2\mathfrak{A}\mathfrak{B} = K;$$

so that

$$K = EG - F^2,$$

$$U = \mathfrak{C} + \int\frac{(H - 2LMNFu)\,du\,\surd(G)}{(K - 2LMNGu)\,\surd\{(\mathfrak{A}+2Lu)(\mathfrak{B}+2Mu)(\mathfrak{C}+2Nu)\}},$$

$$\cos l = \frac{\mathfrak{D}Ap}{G} + \frac{BCp\,(B\mathfrak{C} - C\mathfrak{B})\sqrt{(G - \mathfrak{D}^2)}}{G\sqrt{(K - 2LMNGu)}} \sin U + \frac{LAqr\sqrt{(G - \mathfrak{D}^2)}}{\sqrt{\{G\,(K - 2LMNGu)\}}} \cos U,$$

$$\cos m = \frac{\mathfrak{D}Bq}{G} + \frac{CAq\,(C\mathfrak{A} - A\mathfrak{C})\sqrt{(G - \mathfrak{D}^2)}}{G\sqrt{(K - 2LMNGu)}} \sin U + \frac{MBrp\sqrt{(G - \mathfrak{D}^2)}}{\sqrt{\{G\,(K - 2LMNGu)\}}} \cos U,$$

$$\cos n = \frac{\mathfrak{D}Cr}{G} + \frac{ABr\,(A\mathfrak{B} - B\mathfrak{A})\sqrt{(G - \mathfrak{D}^2)}}{G\sqrt{(K - 2LMNGu)}} \sin U + \frac{NCpq\sqrt{(G - \mathfrak{D}^2)}}{\sqrt{\{G\,(K - 2LMNGu)\}}} \cos U,$$

$$\omega^2 = E - 2LMNu,$$

$$\cos j = \frac{\mathfrak{D}F + \sqrt{\{(G - \mathfrak{D}^2)(K - 2LMNGu)\}} \sin U}{G\sqrt{(E - 2LMNu)}},$$

$$v = p\cos l + q\cos m + r\cos n$$

$$= \frac{1}{G}\left[\mathfrak{D}F + \sqrt{\{(G - \mathfrak{D}^2)(K - 2LMNGu)\}} \sin U\right],$$

$$\tan(\phi + \mathfrak{F}) = \frac{F\sqrt{(G - \mathfrak{D}^2)} - \mathfrak{D}\sin U\sqrt{(K - 2LMNGu)}}{\cos U\sqrt{\{G\,(K - 2LMNGu)\}}}.$$

[The angles which determine the position of the body are thus expressed in terms of u, which is given as a function of t by the foregoing equation $du = pqrdt$, where p, q, r denote given functions of u.]