403.

ON PASCAL'S THEOREM.

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I CONSIDER the following question: to find a point such that its polar plane in regard to a given system of three planes is the same as its polar plane in regard to another given system of three planes.

The equations of any six planes whatever may be taken to be $X=0$, $Y=0$, $Z=0$, $U=0$, $V=0$, $W=0$, where

$$X+Y+Z+U+V+W=0,$$

$$aX+bY+cZ+fU+gV+hW=0,$$

and so also any quantities X, Y, Z, U, V, W satisfying these relations may be regarded as the coordinates of a point in space; we pass to the ordinary system of quadriplanar coordinates by merely substituting for V, W their values as linear functions of X, Y, Z, U.

This being so, the equations of the given systems of three planes may be taken to be

$$XYZ=0,\quad UVW=0,$$

and if we take for the coordinates of the required point (x, y, z, u, v, w), where

$$x+y+z+u+v+w=0,$$

$$ax+by+cz+fu+gv+hw=0,$$

then the equations of the two polar planes are

$$\frac{X}{x}+\frac{Y}{y}+\frac{Z}{z}=0,\quad \frac{U}{u}+\frac{V}{v}+\frac{W}{w}=0,$$

respectively, and we have to find (x, y, z, u, v, w), such that these two equations may represent the same plane, or that the two equations may in virtue of the linear relations between (X, Y, Z, U, V, W) be the same equation.

The ordinary process by indeterminate multipliers gives

$$\frac{1}{x}+\lambda+\mu a=0,$$

$$\frac{1}{y}+\lambda+\mu b=0,$$

$$\frac{1}{z}+\lambda+\mu c=0,$$

$$\frac{k}{u}+\lambda+\mu f=0,$$

$$\frac{k}{v}+\lambda+\mu g=0,$$

$$\frac{k}{w}+\lambda+\mu h=0,$$

and we have the before-mentioned linear relations between (x, y, z, u, v, w); these last are satisfied by the values

$$(x, y, z, u, v, w)=\left(\frac{1}{a-\theta}, \frac{1}{b-\theta}, \frac{1}{c-\theta}, -\frac{1}{f-\theta}, -\frac{1}{g-\theta}, -\frac{1}{h-\theta}\right),$$

if only

$$\frac{1}{a-\theta}+\frac{1}{b-\theta}+\frac{1}{c-\theta}-\frac{1}{f-\theta}-\frac{1}{g-\theta}-\frac{1}{h-\theta}=0;$$

in fact, θ satisfying this equation, the relation

$$x+y+z+u+v+w=0$$

is obviously satisfied; and observing that we have

$$ax=\frac{a}{a-\theta}=1+\frac{\theta}{a-\theta}, \ldots, fu=\frac{-f}{f-\theta}=-1-\frac{\theta}{f-\theta}, \ldots,$$

we have

$$ax+by+cz+fu+gv+hw$$

$$=\Sigma\left(1+\frac{\theta}{a-\theta}\right)-\Sigma\left(1+\frac{\theta}{f-\theta}\right),$$

$$=\theta\left(\Sigma\frac{1}{a-\theta}-\Sigma\frac{1}{f-\theta}\right), =0,$$

so that the relation $ax+by+cz+fu+gv+hw=0$ is also satisfied. Substituting the foregoing values of (x, y, z, u, v, w) the six equations containing k, λ, μ, will be all of them satisfied if only

$$\mu=-1, \ \lambda=\theta, \ k=-1.$$

The coordinates of the required point thus are

$$\left(\frac{1}{a-\theta}, \frac{1}{b-\theta}, \frac{1}{c-\theta}, -\frac{1}{f-\theta}, -\frac{1}{g-\theta}, -\frac{1}{h-\theta}\right),$$

where

$$\frac{1}{a-\theta}+\frac{1}{b-\theta}+\frac{1}{c-\theta}-\frac{1}{f-\theta}-\frac{1}{g-\theta}-\frac{1}{h-\theta}=0;$$

and, the equation in θ being of the fourth order, there are thus four points, say the points O_1, O_2, O_3, O_4, which have each of them the property in question.

It will be convenient to designate the planes $X=0$, $Y=0$, $Z=0$, $U=0$, $V=0$, $W=0$ as the planes a, b, c, f, g, h respectively; the line of intersection of the planes $X=0$, $Y=0$ will then be the line ab, and the point of intersection of the planes $X=0$, $Y=0$, $Z=0$ the point abc; and so in other cases.

I say that from any one of the points O it is possible to draw

a line meeting the lines $af.bg.ch$ (1),

„ „ $ag.bh.cf$ (2),

„ „ $ah.bf.cg$ (3),

„ „ $af.bh.cg$ (4),

„ „ $ag.bf.ch$ (5),

„ „ $ah.bg.cf$ (6),

and consequently, that the four points O are the four common points of the six hyperboloids passing through these triads of lines respectively.

In fact, considering θ as determined by the foregoing quartic equation, and writing for shortness

$$(a-\theta)X=A, \quad (f-\theta)U=F,$$
$$(b-\theta)Y=B, \quad (g-\theta)V=G,$$
$$(c-\theta)Z=C, \quad (h-\theta)W=H,$$

so that

$$A+B+C+F+G+H=0,$$

the equations $A+F=0$, $B+G=0$, $C+H=0$, are equivalent to two equations only, and it is at once seen, that these are in fact the equations of a line through the point O meeting the three lines af, bg, ch respectively.

The equation $A+F=0$, is in fact satisfied by the values $X : U = \frac{1}{a-\theta} : -\frac{1}{f-\theta}$, and by $X=0$, $U=0$; it is consequently the equation of the plane through O and the line af; similarly, $B+G=0$ is the equation of the plane through O and the line

bg; and $C+H=0$ is the equation of the plane through O and the line ch; and the three equations being equivalent to two equations only, the planes have a common line which is the line in question.

The equations of the six lines thus are:

$$\begin{array}{llll} (1) & A+F=0, & B+G=0, & C+H=0, \\ (2) & A+G=0, & B+H=0, & C+F=0, \\ (3) & A+H=0, & B+F=0, & C+G=0, \\ (4) & A+F=0, & B+H=0, & C+G=0, \\ (5) & A+G=0, & B+F=0, & C+H=0, \\ (6) & A+H=0, & B+G=0, & C+F=0. \end{array}$$

It is further to be noticed, that if in any one of these systems, for instance in the system $A+F=0$, $B+G=0$, $C+H=0$, we consider θ as an arbitrary quantity, then the equations are those of any line whatever cutting the lines af, bg, ch; and hence eliminating θ, we have the equation of the hyperboloid through the three lines af, bg, ch; the equations of the six hyperboloids are thus found to be

$$(1) \quad \frac{ax+fu}{x+u} = \frac{by+gv}{y+v} = \frac{cz+hw}{z+w},$$

$$(2) \quad \frac{ax+gv}{x+v} = \frac{by+hw}{y+w} = \frac{cz+fu}{z+u},$$

$$(3) \quad \frac{ax+hw}{x+w} = \frac{by+fu}{y+u} = \frac{cz+gv}{z+v},$$

$$(4) \quad \frac{ax+fu}{x+u} = \frac{by+hw}{y+w} = \frac{cz+gv}{z+v},$$

$$(5) \quad \frac{ax+gv}{x+v} = \frac{by+fu}{y+u} = \frac{cz+hw}{z+w},$$

$$(6) \quad \frac{ax+hw}{x+w} = \frac{by+gv}{y+v} = \frac{cz+fu}{z+u},$$

respectively; the equations in the same line being of course equivalent to a single equation.

For each one of the six lines we have

$$(A, B, C)=(-F, -G, -H)$$

in some order or other, and it is thus seen that the six lines lie on a cone of the second order, the equation whereof is

$$A^2+B^2+C^2-F^2-G^2-H^2=0.$$

Consider now the six planes a, b, c, f, g, h, and taking in the first instance an arbitrary point of projection, and a plane of projection which is also arbitrary—the line of intersection ab of the planes a and b will be projected into a line ab, and the point of intersection of the planes a, b, c into a point abc; and so in other cases. We have thus a plane figure, consisting of the fifteen lines ab, ac, ... gh, and of the twenty points abc, abf, ... fgh; and which is such, that on each of the lines there lie four of the points, and through each of the points there pass three of the lines, viz. the points abc, abf, abg, abh lie on the line ab; and the lines bc, ca, ab meet in the point abc, and so in other cases. If now the point of projection instead of being arbitrary, be one of the above-mentioned four points O, then the projections of the lines af, bg, ch meet in a point, and the like for each of the six triads of lines; that is in the plane figure we have six points 1, 2, 3, 4, 5, 6, each of them the intersection of three lines as shown in the diagram,

$$1 = af \,.\, bg \,.\, ch,$$

$$2 = ag \,.\, bh \,.\, cf,$$

$$3 = ah \,.\, bf \,.\, cg,$$

$$4 = af \,.\, bh \,.\, cg,$$

$$5 = ag \,.\, bf \,.\, ch,$$

$$6 = ah \,.\, bg \,.\, cf,$$

and these six points lie in a conic. It is clear that the lines af, ag, ah; bf, bg, bh; cf, cg, ch are the lines 14, 25, 36; 35, 16, 24; 26, 34, 15 respectively.

Conversely, starting from the points 1, 2, 3, 4, 5, 6 on a conic, and denoting the lines 14, 25, 36; 35, 16, 24; 26, 34, 15 (being, it may be noticed, the sides and diagonals of the hexagon 162435) in the manner just referred to, then it is possible to complete the figure of the fifteen lines ab, ac, ... gh and of the twenty points abc, abf, ... fgh, such that each line contains upon it four points, and that through each point there pass three lines, in the manner already mentioned.

Of the fifteen lines, nine, viz. the lines af, ag, ah; bf, bg, bh; cf, cg, ch are, as has been seen, lines through two of the six points 1, 2, 3, 4, 5, 6; the remaining lines are bc, ca, ab; gh, hf, fg. These are Pascalian lines,

bc	of the hexagon	162435,
ca	„	152634,
ab	„	142536,
gh	„	152436,
hf	„	142635,
bg	„	162534,

which appears thus, viz.

$$\text{line } bc \text{ contains points } bcf\,,\quad bcg\,,\quad bch\,,$$
$$= bf\,.\,cf,\quad bg\,.\,cg,\quad bh\,.\,ch,$$
$$= 35\,.\,26,\quad 16\,.\,34,\quad 24\,.\,15\,;$$

that is, bc is the Pascalian line of the hexagon 162435; and the like for the rest of the six lines.

The twenty points abc, $abf, \ldots fgh$ are as follows, viz. omitting the two points abc, fgh, the remaining eighteen points are the Pascalian points (the intersections of pairs of lines each through two of the points 1, 2, 3, 4, 5, 6) which lie on the Pascalian lines bc, ca, ab, gh, hf, fg respectively; the point abc is the intersection of the Pascalian lines bc, ca, ab, and the point fgh is the intersection of the Pascalian lines gh, hf, fg, the points in question being two of the points P (Steiner's twenty points, each the intersection of three Pascalian lines).

We thus see that we have two triads of hexagons such that the Pascalian lines of each triad meet in a point, and that the two points so obtained, together with the eighteen points on the six Pascalian lines, form a system of twenty points lying four together on fifteen lines, and which points and lines are the projections of the points and lines of intersection of six planes; or, say simply that the figure is the projection of the figure of six planes.

It is to be added, that if the planes are a, b, c, f, g, h, then the point of projection is any one of the four points which have the same polar plane in regard to the system of the planes a, b, c, and in regard to the system of the planes f, g, h. The consideration of the solid figure affords a demonstration of the existence as well of the six Pascalian lines as of the two points each the intersection of three of these lines.