399.

ON THE CUBICAL DIVERGENT PARABOLAS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. IX. (1868), pp. 185—189.]

NEWTON reckons five forms, viz. these are the *simplex*, the *complex*, the *crunodal*, the *acnodal*, and the *cuspidal*, but as noticed by Murdoch, the simplex has three different forms, the *ampullate*, the *neutral*, and the *campaniform*. We have thus the 8 forms at once distinguishable by the eye.

Plücker has in all 13 species, the division into species being established or completed geometrically by reference to the asymptotic cuspidal curve (or asymptotic semi-cubical parabola), and analytically as follows, viz. writing the equation in the form

$$y^2 = x^3 - 3cx + 2d,$$

the different species are

$$\begin{array}{lll}
\text{simplex,} & \left.\begin{array}{l} y^2 = x^3 - 3cx + 2d \\ y^2 = x^3 - 3cx - 2d \end{array}\right\} c^3 < d^2, & \begin{array}{l}\text{ampullate,} \\ \text{campaniform,}\end{array} \\
\quad ,, & y^2 = x^3 + 2d, & \text{neutral,} \\
\quad ,, & y^2 = x^3 - 2d, & \text{campaniform,} \\
\quad ,, & y^2 = x^3 + 3cx + 2d, & \quad ,, \\
\quad ,, & y^2 = x^3 + 3cx, & \quad ,, \\
\quad ,, & y^2 = x^3 + 3cx - 2d, & \quad ,, \\
\text{complex,} & \left.\begin{array}{l} y^2 = x^3 - 3cx + 2d \\ y^2 = x^3 - 3cx - 2d \end{array}\right\} c^3 > d^2, & \\
\quad ,, & y^2 = x^3 - 3cx, & \\
\text{acnodal,} & y^2 = x^3 - 3cx - 2c\sqrt{(c)}, & \\
\text{crunodal,} & y^2 = x^3 - 3cx + 2c\sqrt{(c)}, & \\
\text{cuspidal,} & y^2 = x^3\,; &
\end{array}$$

but of the simplex species, there are five which are to the eye campaniform, and the three complex species have with each other a close resemblance in form.

I remark as regards the simplex forms, that the tangents at the two inflexions meet in a point R on the axis, and that the ampullate, the neutral, and campaniform forms are distinguished from each other according to the position of R, viz. for the ampullate form, R lies within the curve, for the campaniform form R lies without the curve, and for the neutral form, R is at infinity. It is to be observed, as regards the complex forms, that here R always lies without the curve, between the infinite branch and the oval.

The further division of the simplex and complex forms so as to obtain the $7+3$ species of Plücker, may be effected by considering in conjunction with the point R a certain other point I on the axis; it is to be remarked that excluding the inflexion at infinity the cubical divergent parabola has in all eight inflexions, two real and six imaginary, viz. the inflexions lie by pairs on four ordinates, or if x be the abscissa corresponding to an inflexion, x is determined by a quartic equation; this equation has always two real and two imaginary roots, each of the imaginary roots gives a pair of imaginary inflexions; one of the real roots gives a positive value for y^2 and therefore two real inflexions, the tangents at these meet in the above-mentioned point R on the axis; the other real root gives a negative value for y^2 and therefore two imaginary inflexions, but the tangents at these meet in a real point on the axis, and this I call the point I. It is clear that for each of the four pairs of inflexions the tangents at the two inflexions meet at a point on the axis, so that if X be the abscissa of such point, then X is determined by a quartic equation; two of the roots of this equation are imaginary, the other two roots are real, and correspond to the points R and I respectively.

The equation of the curve being as above

$$y^2 = x^3 - 3cx + 2d,$$

then the coordinate x belonging to a pair of inflexions is found by the equation

$$x^4 - 6cx^2 + 8dx - 3c^2 = 0,$$

or what is the same thing,

$$(1,\ 0,\ -c,\ 2d,\ -3c^2 \between x,\ 1)^4 = 0,$$

(the invariant I is $=0$, and hence the discriminant, $=-27J^2$, is negative, or the roots are two real, two imaginary, as already mentioned): the corresponding value of X is easily found to be

$$X = \frac{x^3 + 3cx - 4d}{3(x^2 - c)},$$

and we thence obtain

$$3cX^4 - 4dX^3 - 6c^2X^2 + 12cdX - (c^3 + 4d^2) = 0,$$

or what is the same thing,

$$(3c,\ -d,\ -c^2,\ 3cd,\ -c^3 - 4d^2 \between X,\ 1)^4 = 0,$$

for the equation in X; the quadrinvariant I is $=0$, and hence the discriminant, $=-27J^2$, is negative; that is, the roots are two real, two imaginary, as already mentioned.

Considering the simplex forms, first, if $c=0$, then for the curve

$$y^2 = x^3 + 2d,$$

it appears that R lies at infinity, I within the curve; and for the curve

$$y^2 = x^3 - 2d,$$

that R lies without the curve, I at infinity.

It further appears that when $d=0$, or for the curve,

$$y^2 = x^3 + 3cx,$$

R, I lie equidistant from the vertex, R without, I within the curve.

Hence in the curve

$$y^2 = x^3 + 3cx + 2d,$$

since, when $d=0$, the points R, I are equidistant from the vertex, and for $c=0$, the point R is at infinity, it is easy to infer by continuity that the points R, I lie R without, I within the curve, I being nearer to the vertex.

And similarly in the curve

$$y^2 = x^3 + 3cx - 2d,$$

that the points R, I lie R without, I within the curve, R being nearer to the vertex.

Again, in the curve

$$y^2 = x^3 - 3cx + 2d,$$

since, in the curve $y^2 = x^3 + 3cx + 2d$, R is without, I within the curve, and as c becomes $=0$, R passes off to infinity, it appears that c having changed its sign, or for the curve now in question, R having passed through infinity, will be situate within the curve; that is, R, I lie each of them within the curve.

And similarly for the curve

$$y^2 = x^3 - 3cx - 2d,$$

it appears that R, I lie each without the curve.

Hence, finally, for the simplex forms, we have the 7 species of Plücker, viz.

$$y^2 = x^3 - 3cx + 2d,\ c^3 < d^2,$$

simplex ampullate, R, I within the curve;

$$y^2 = x^3 - 3cx - 2d,\ c^3 < d^2,$$

simplex campaniform, R, I without the curve;

$$y^2 = x^3 + 2d,$$

simplex neutral, I within the curve, R at infinity;

$$y^2 = x^3 - 2d,$$

simplex campaniform quasi-neutral, R without the curve, I at infinity;

$$y^2 = x^3 + 3cx + 2d,$$

simplex campaniform, R without and further from, I within and nearer to the curve;

$$y^2 = x^3 + 3cx,$$

simplex campaniform equidistant, viz. R and I are equidistant from the curve, R without and I within;

$$y^2 = x^3 + 3cx - 2d,$$

simplex campaniform, R without and nearer to, I within and further from the curve.

Passing to the complex forms, suppose for a moment that α is the diameter of the oval and β the distance of the oval from the vertex of the infinite branch; the equation of the curve then is $y^2 = x(x-\alpha)(x-\alpha-\beta)$, or changing the origin so as to make the term in x^2 to vanish, this is

$$y^2 = (x + \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta)(x - \tfrac{1}{3}\alpha + \tfrac{1}{3}\beta)(x - \tfrac{1}{3}\alpha - \tfrac{2}{3}\beta),$$

or, what is the same thing,

$$y^2 = x^3 - \tfrac{2}{3}(\alpha^2 + \alpha\beta + \beta^2)x - \tfrac{1}{27}(\alpha - \beta)(2\alpha + \beta)(\alpha + 2\beta),$$

or comparing this with $y^2 = x^3 - 3cx + 2d$, d is $= +$, 0 or $-$, as $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$, or say as the oval is smaller, mean, or larger; viz. the magnitude of the oval is estimated by the relation which the diameter thereof bears to the distance of the oval from the infinite branch. In the case $d = 0$, or for the curve $y^2 = x^3 - 3cx$ it appears (as for the corresponding simplex form $y^2 = x^3 + 3cx$) that the points R, I are equidistant from the point $x = 0$, which is in the present case the middle vertex, or vertex of the oval which vertex is nearest to the infinite branch. As the oval diminishes, so that the curve becomes ultimately acnodal, I remaining within the oval ultimately coincides with the acnode; and as the oval increases so that the curve becomes ultimately crunodal, R remaining between the oval and the infinite branch, ultimately coincides with the crunode; and it hence easily appears by continuity that for a smaller oval I is nearer to, R further from the middle vertex; while for a larger oval, I is further from, R nearer to the middle vertex. Hence for the complex forms the species are

$$y^2 = x^3 - 3cx + 2d,$$

smaller oval, I nearer to, R further from the middle vertex;

$$y^2 = x^3 - 3cx,$$

mean oval, R and I equidistant from the middle vertex;

$$y^2 = x^3 - 3cx - 2d,$$

larger oval, I further from, R nearer to the middle vertex: and the division into species is thus completed.

Cambridge, June 16, 1865.