398.

ON A CERTAIN SEXTIC DEVELOPABLE, AND SEXTIC SURFACE CONNECTED THEREWITH.

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I PROPOSE to consider [first] the sextic developable derived from a quartic equation, viz. taking this to be $(a, b, c, d, e \between t, 1)^4 = 0$, where (a, b, c, d, e) are any linear functions of the coordinates (x, y, z, w), the equation of the developable in question is

$$(ae - 4bd + 3c^2)^3 - 27(ace - ad^2 - b^2e + 2bcd - c^3)^2 = 0.$$

I have already, in the paper "On a Special Sextic Developable," *Quarterly Journal of Mathematics*, vol. VII. (1866), pp. 105—113, [373], considered a particular case of this surface, viz. that in which c was $= 0$, the geometrical peculiarity of which is that the cuspidal edge is there an excubo-quartic curve (of a special form, having two stationary tangents), whereas in the general case here considered it is a sextic curve. There was analytically the convenience that the linear functions being only the four functions a, b, d, e, these could be themselves taken as coordinates, whereas in the present case we have the five linear functions a, b, c, d, e.

The developable

$$(ae - 4bd + 3c^2)^3 - 27(ace - ad^2 - b^2e + 2bcd - c^3)^2 = 0$$

is a sextic developable having for its cuspidal curve the sextic curve

$$ae - 4bd + 3c^2 = 0,$$

$$ace - ad^2 - b^2e + 2bcd - c^3 = 0,$$

(say $I = 0$, $J = 0$, as usual), and having besides a nodal curve the equations of which may be written

$$\begin{matrix} 6(ac - b^2) & : & 3(ad - bc) & : & ae + 2bd - 3c^2 & : & 3(be - cd) & : & 6(ce - d^2) & : & 9J \\ = a & : & b & : & c & : & d & : & e & : & I, \end{matrix}$$

viz. these equations are really equivalent to two equations, and they represent a curve of the fourth order which is an excubo-quartic. We may in fact find the equations of the nodal curve by assuming $(a, b, c, d, e \between t, 1)^4$ to be a perfect square, say to avoid fractions that it is $= 3(\alpha t^2 + 2\beta t + \gamma)^2$, then we have

$$a : b : c : d : e = 3\alpha^2 : 3\alpha\beta : \alpha\gamma + 2\beta^2 : 3\beta\gamma : 3\gamma^2,$$

which equations as involving the two arbitrary parameters $\alpha : \beta : \gamma$, give two equations between (a, b, c, d, e), and we may at once by means of them verify the above-mentioned equations of the nodal curve. It also hereby appears that the nodal curve is as stated an excubo-quartic curve; viz. we have between a, b, c, d, e a single linear relation, that is a quadric relation between α, β, γ, and this equation may be satisfied identically by taking for α, β, γ properly determined quadric functions of a variable parameter θ; whence a, b, c, d, e are proportional to quartic functions of the variable parameter θ, or the curve is an excubo-quartic.

The equations of the nodal curve may be presented under a somewhat different form; viz. the cubi-covariant of $(a, b, c, d, e \between t, 1)^4 = 0$ being

$$\left\{\begin{array}{l} -\ a^2d + 3abc - 2b^3 \\ -\ a^2e - 2abd + 9ac^2 - 6b^2c \\ -\ 5abe + 15acd - 10b^2d \\ +\ 10ad^2 - 10b^2e \\ +\ 5ade + 10bd^2 - 15bce \\ +\ ae^2 + 2bde - 9c^2e + 6cd^2 \\ +\ be^2 - 3cde + 2d^3 \end{array}\right\} \between (t, 1)^6 = 0,$$

say this function, multiplied by 6 to avoid fractions, is

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g} \between t, 1)^6,$$

that is

$$\mathrm{a} = 6(-a^2d + 3abc - 2b^3),$$

$$\mathrm{b} = 1(-a^2e - 2abd + 9ac^2 - 6b^2c),$$

$$\mathrm{c} = 2(-abe + 3acd - 2b^2d),$$

$$\mathrm{d} = 3(+ad^2 - b^2e),$$

$$\mathrm{e} = 2(+ade + 2bd^2 - 3bce),$$

$$\mathrm{f} = 1(+ae^2 + 2bde - 9c^2e + 6cd^2),$$

$$\mathrm{g} = 6(+be^2 - 3cde + 2d^3),$$

then the equations of the nodal curve may be written

$$\mathrm{a} = 0,\ \mathrm{b} = 0,\ \mathrm{c} = 0,\ \mathrm{d} = 0,\ \mathrm{e} = 0,\ \mathrm{f} = 0,\ \mathrm{g} = 0.$$

It may be mentioned that we have identically

$$\text{ae} - 4\text{bd} + 3\text{c}^2 = 0,$$
$$\text{af} - 3\text{be} + 2\text{cd} = 0,$$
$$\text{ag} - 9\text{ce} + 8\text{d}^2 = 0,$$
$$\text{bg} - 3\text{cf} + 2\text{de} = 0,$$
$$\text{bf} - 4\text{ce} + 3\text{d}^2 = 0,$$

and moreover

$$\text{ag} - 6\text{bf} + 15\text{ce} - 10\text{d}^2$$
$$= -6\,(\text{bf} - 4\text{ce} + 3\text{d}^2), = +6\,(I^3 - 27J^2),$$

so that the equation of the developable may be written in the form

$$\text{ag} - 6\text{bf} + 15\text{ce} - 10\text{d}^2 = 0,$$

or in the more simple form

$$\text{bf} - 4\text{ce} + 3\text{d}^2 = 0,$$

each of which puts in evidence the nodal curve on the surface.

The nodal and cuspidal curves meet in the points

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e},$$

being, as it is easy to show, a system of four points. The four points in question form a tetrahedron, the equations of the faces of which may be taken to be $x=0$, $y=0$, $z=0$, $w=0$; and the equation of the surface may be expressed in this system of quadriplanar coordinates.

We introduce these coordinates *ab initio*, by taking the quartic function of t to be

$$(a,\ b,\ c,\ d,\ e\between t,\ 1)^4 = x\,(t+\alpha)^4 + y\,(t+\beta)^4 + z\,(t+\gamma)^4 + w\,(t+\delta)^4,$$

that is, by writing

$$\begin{aligned} a &= x + y + z + w, \\ b &= \alpha x + \beta y + \gamma z + \delta w, \\ c &= \alpha^2 x + \beta^2 y + \gamma^2 z + \delta^2 w, \\ d &= \alpha^3 x + \beta^3 y + \gamma^3 z + \delta^3 w, \\ e &= \alpha^4 x + \beta^4 y + \gamma^4 z + \delta^4 w. \end{aligned}$$

Observe that $(t_1,\ t_2,\ t_3,\ t_4)$ being any constant quantities, we thence have

$$\begin{aligned} e - d\Sigma t_1 + c\Sigma t_1 t_2 - b\Sigma t_1 t_2 t_3 + a t_1 t_2 t_3 t_4 & \\ = \ x\,(\alpha - t_1)(\alpha - t_2)(\alpha - t_3)(\alpha - t_4) & \\ + y\,(\beta - t_1)(\beta - t_2)(\beta - t_3)(\beta - t_4) & \\ + z\,(\gamma - t_1)(\gamma - t_2)(\gamma - t_3)(\gamma - t_4) & \\ + w\,(\delta - t_1)(\delta - t_2)(\delta - t_3)(\delta - t_4), & \end{aligned}$$

and thence in particular

$$e - d\Sigma\alpha + c\Sigma\alpha\beta - b\Sigma\alpha\beta\gamma + a\,\alpha\beta\gamma\delta = 0,$$

viz. this is the linear relation which subsists identically between (a, b, c, d, e), the five linear functions of the coordinates (x, y, z, w).

Starting from the above values of (a, b, c, d, e), we find without difficulty

$$I = (\alpha-\beta)^4\, xy + (\alpha-\gamma)^4\, xz + (\alpha-\delta)^4\, xw + (\beta-\gamma)^4\, yz + (\beta-\delta)^4\, yw + (\gamma-\delta)^4\, zw,$$

$$J = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2\, xyz + (\alpha-\beta)^2(\beta-\delta)^2(\delta-\alpha)^2\, xyw$$
$$+ (\alpha-\gamma)^2(\gamma-\delta)^2(\delta-\alpha)^2\, xzw + (\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\, yzw,$$

but we thus see the convenience of introducing constant multipliers into the expressions of the four coordinates respectively, viz. writing

$$x = (\beta\gamma\delta)^2\, x',$$
$$y = (\gamma\delta\alpha)^2\, y',$$
$$z = (\delta\alpha\beta)^2\, z',$$
$$w = (\alpha\beta\gamma)^2\, w',$$

where for shortness

$$(\beta\gamma\delta) = (\beta-\gamma)(\gamma-\delta)(\delta-\beta), \text{ \&c.},$$

or what is the same thing, taking the quartic to be

$$(a, b, c, d, e \between t, 1)^4 = x'(\beta\gamma\delta)^2(t+\alpha)^4 + y'(\gamma\delta\alpha)^2(t+\beta)^4 + z'(\delta\alpha\beta)^2(t+\gamma)^4 + w'(\alpha\beta\gamma)^2(t+\delta)^4,$$

we find

$$J = (\lambda'\mu'\nu')^2(x'y'z' + y'z'w' + z'x'w' + x'y'w'),$$
$$I = (\lambda'\mu'\nu')\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(z'w' + x'y')\},$$

where for shortness

$$\lambda' = (\alpha-\delta)^2(\beta-\gamma)^2,$$
$$\mu' = (\beta-\delta)^2(\gamma-\alpha)^2,$$
$$\nu' = (\gamma-\delta)^2(\alpha-\beta),$$

or writing

$$\surd(\lambda') = (\alpha-\delta)(\beta-\gamma),$$
$$\surd(\mu') = (\beta-\delta)(\gamma-\alpha),$$
$$\surd(\nu') = (\gamma-\delta)(\alpha-\beta),$$

we have

$$\surd(\lambda') + \surd(\mu') + \surd(\nu') = 0,$$

and the equation of the developable is thus

$$\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(x'w' + x'y')\}^3 - 27\lambda'\mu'\nu'(x'y'z' + y'z'w' + z'x'w' + x'y'w')^2 = 0.$$

Observe that $J = 0$ is a cubic surface passing through each edge of the tetrahedron, and having at each summit a conical point; $I = 0$ is a quadric surface passing through each summit of the tetrahedron, and at each of these points the tangent plane of the quadric surface touches the tangent cone of the cubic surface: to show this it is only necessary to observe that at the point ($x'=0$, $y'=0$, $z'=0$) the tangent cone is $y'z'+z'x'+x'y'=0$, and the tangent plane is $\lambda'x'+\mu'y'+\nu'z'=0$, and that these touch in virtue of the above-mentioned relation $\sqrt{(\lambda')}+\sqrt{(\mu')}+\sqrt{(\nu')}=0$. It follows that on the curve of intersection, or cuspidal edge of the developable, each of the summits is a cuspidal or stationary point, that is, the cuspidal curve has four stationary points; this agrees with the character of the curve as given, Salmon "On the Classification of Curves of Double Curvature," *Camb. and Dubl. Math. Jour.* vol. v. (1850), p. 39, viz. the character is there given

$$\text{a}=6,\ m=6,\ n=4,\ r=6,\ g=3,\ h=6,\ \alpha=0,\ \beta=4,\ x=4,\ y=6,$$

($\beta=4$, that is, there are as stated 4 stationary points).

To find the equations of the nodal curve, instead of transforming the equations as given in terms of (a, b, c, d, e), it is better to deduce these from the equation of the surface; viz. if there is a nodal curve, we must have

$$\begin{aligned}&\delta_{x'}I : \delta_{y'}I : \delta_{z'}I : \delta_{w'}I : 18J\\ =&\delta_{x'}J : \delta_{y'}J : \delta_{z'}J : \delta_{w'}J : \quad I.\end{aligned}$$

Writing these under the form $\delta_{x'}I+\theta'\delta_{x'}J=0$, &c., where θ' is regarded as an arbitrary parameter [1], we have

$$\begin{aligned}\lambda'w'+\mu'z'+\nu'y'+\theta'(y'z'+y'w'+z'w')&=0,\\ \lambda'z'+\mu'w'+\nu'x'+\theta'(z'x'+z'w'+x'w')&=0,\\ \lambda'y'+\mu'x'+\nu'w'+\theta'(x'y'+x'w'+y'w')&=0,\\ \lambda'x'+\mu'y'+\nu'z'+\theta'(y'z'+z'x'+x'y')&=0,\end{aligned}$$

which equations (eliminating θ') must be equivalent to two equations only.

I remark that the first three equations may be regarded as a set of linear equations in 1, w', θ', $\theta'w'$; and determining from them the ratios of these quantities, we have, suppose,

$$1 : -w' : \theta' : -\theta'w' = A : B : C : D,$$

where

$$A=\begin{vmatrix}\lambda', & y'z', & y'+z'\\ \mu', & z'x', & z'+x'\\ \nu', & x'y', & x'+y'\end{vmatrix},\qquad B=\begin{vmatrix}y'z', & y'+z', & \mu'z'+\nu'y'\\ z'x', & z'+x', & \nu'x'+\lambda'z'\\ x'y', & x'+y', & \lambda'y'+\mu'x'\end{vmatrix},$$

$$C=\begin{vmatrix}y'+z', & \mu'z'+\nu'y', & \lambda'\\ z'+x', & \nu'x'+\lambda'z', & \mu'\\ x'+y', & \lambda'y'+\mu'x', & \nu'\end{vmatrix},\qquad D=\begin{vmatrix}\mu'z'+\nu'y', & \lambda', & y'z'\\ \nu'x'+\lambda'z', & \mu', & z'x'\\ \lambda'y'+\mu'x', & \nu', & x'y'\end{vmatrix}.$$

[1] The value of θ' is in fact $=-\dfrac{18J}{I}$, that is, instead of the four equations involving an arbitrary parameter θ', we have really four determinate equations.

We have thence $AD - BC = 0$ and (substituting in the fourth equation) $A(\lambda'x' + \mu'y' + \nu'z') + C(y'z' + z'x' + x'y') = 0$; each of these equations must contain the equation of the cone having ($x' = 0$, $y' = 0$, $z' = 0$) for its vertex, and passing through the nodal curve. The two equations are of the orders 6 and 4 respectively; and as the curve is a quartic curve passing through the vertex in question, the equation of the cone is of the order 3. I have not effected the reduction of the sextic equation, but for the quartic equation, substituting for A, C their values, this is

$$\begin{aligned}&-(\lambda'x' + \mu'y' + \nu'z')\,[\lambda'x'^2(y'-z') + \mu'y'^2(z'-x') + \nu'z'^2(z'-x')]\\ &+(y'z' + z'x' + x'y')\,[\lambda'^2x'(y'-z') + \mu'^2y'(z'-x') + \nu'z'(x'-y')]\\ &+\{\mu'\nu'(y'-z') + \nu'\lambda'(z'-x') + \lambda'\mu'(x'-y')\}(x'+y'+z') = 0,\end{aligned}$$

which is easily reduced to

$$\begin{aligned}&\lambda'^2x'(-x'^2 + y'z' + z'x' + x'y')(y'-z')\\ &+\mu'^2y'(-y'^2 + y'z' + z'x' + x'y')(z'-x')\\ &+\nu'^2z'(-z'^2 + y'z' + z'x' + x'y')(x'-y')\\ &+\mu'\nu'[(x'+y'+z')(y'z' + z'x' + x'y') + x'y'z'](y'-z')\\ &+\nu'\lambda'[(x'+y'+z')(y'z' + z'x' + x'y') + x'y'z'](z'-x')\\ &+\lambda'\mu'[(x'+y'+z')(y'z' + z'x' + x'y') + x'y'z'](x'-y') = 0;\end{aligned}$$

and I have found that this is transformable into

$$\begin{aligned}2\{x'\surd(\lambda') + y'\surd(\mu') + z'\surd(\nu')\} \times [y'z'\surd(\lambda')(\mu'y' - \nu'z')\,z'x'\surd(\mu')(\nu'z' - \lambda'x')\,x'y'\surd(\nu')(\lambda'x' - \mu'y')\\ - x'y'z'\{\surd(\mu') - \surd(\nu')\}\{\surd(\nu') - \surd(\lambda')\}\{\surd(\lambda') - \surd(\mu')\}] = 0,\end{aligned}$$

viz. the two functions are equivalent in virtue of the relation $\surd(\lambda') + \surd(\mu') + \surd(\nu') = 0$, or, what is the same thing, they only differ by a function $(x', y', z')^4$ into the evanescent factor $\lambda'^2 + \mu'^2 + \nu'^2 - 2\mu'\nu' - 2\nu'\lambda' - 2\lambda'\mu'$. The function in { } equated to zero is therefore the equation of the cubic cone.

I do not stop to give the steps of the investigation in the above form, as the investigation may be very much simplified as follows: by linear combinations of the four equations in x', y', z', w', θ', we deduce

$$\begin{aligned}(\ \ \lambda' - \mu' - \nu')(x' + w' - y' - z') + 2\theta'(y'z' - x'w') &= 0,\\ (-\lambda' + \mu' - \nu')(y' + w' - z' - x') + 2\theta'(z'x' - y'w') &= 0,\\ (-\lambda' - \mu' + \nu')(z' + w' - x' - y') + 2\theta'(x'y' - z'w') &= 0,\\ (\ \ \lambda' + \mu' + \nu')(x' + y' + z' + w') + 2\theta'(y'z' + z'x' + x'y' + x'w' + y'w' + z'w') &= 0.\end{aligned}$$

Hence writing

$$\begin{aligned}\lambda &= \ \ \lambda' - \mu' - \nu', & x &= w' + x' - y' - z',\\ \mu &= -\lambda' + \mu' - \nu', & y &= w' - x' + y' - z',\\ \nu &= -\lambda' - \mu' + \nu', & z &= w' - x' - y' + z',\\ & & w &= w' + x' + y' + z',\end{aligned}$$

we find

$$\mu\nu = \lambda'^2 - \mu'^2 - \nu'^2 + 2\mu'\nu',$$
$$\nu\lambda = -\lambda'^2 + \mu'^2 - \nu'^2 + 2\nu'\lambda',$$
$$\lambda\mu = -\lambda'^2 - \mu'^2 + \nu' \ + 2\lambda'\mu',$$

and thence

$$\mu\nu + \nu\lambda + \lambda\mu = -(\lambda'^2 + \mu'^2 + \nu'^2 - 2\mu'\nu' - 2\nu'\lambda' - 2\lambda'\mu'), = 0,$$

that is

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 0,$$

the relation which connects the new constants λ, μ, ν. Moreover

$$yz - xw = 4(y'z' - x'w'),$$
$$zx - yw = 4(z'x' - y'w'),$$
$$xy - zw = 4(x'y' - z'w'),$$
$$3w^2 - x^2 - y^2 - z^2 = 8(y'z' + z'x' + x'y' + x'w' + y'w' + z'w'),$$

and writing for greater convenience $\theta = -\frac{2}{\theta'}$, the equations are transformed into

$$\theta\lambda x = xw - yz,$$
$$\theta\mu y = yw - zx,$$
$$\theta\nu z = zw - xy,$$
$$2\theta(\lambda + \mu + \nu) w = 3w^2 - x^2 - y^2 - z^2,$$

where

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 0,$$

viz. these equations, eliminating θ, give the equations of the nodal curve.

From the first three equations eliminating θ, we deduce

$$yzw(\mu - \nu) = x(\mu y^2 - \nu z^2),$$
$$zxw(\nu - \lambda) = y(\nu z^2 - \lambda x^2),$$
$$xyw(\lambda - \mu) = z(\lambda x^2 - \mu y^2),$$

or, as these equations may be written,

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu - \nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu - \lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda - \mu},$$

which equations, from the mode in which they are obtained, are it is clear equivalent to two equations only. Using the fourth equation, and eliminating θ by substituting therein for $\theta\lambda$, $\theta\mu$, $\theta\nu$ their values from the first three equations, we find

$$2w^2\left(3w - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}\right) = 3w^2 - x^2 - y^2 - z^2,$$

that is

$$3w^2+x^2+y^2+z^2=2w\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right),$$

or, what is the same thing,

$$xyz\,(3w^2+x^2+y^2+z^2)-2w\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

we have to show that this is in fact included in the former system, for then the four equations with θ eliminated will it is clear give two equations only.

Observe that the former system may be written

$$w=\frac{x\,(\mu y^2-\nu z^2)}{(\mu-\nu)\,yz},$$

$$(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2=0,$$

and that we have thus to show that substituting for w the value $w=\dfrac{x\,(\mu y^2-\nu z^2)}{\mu-\nu}$ in the equation

$$xyz\,(3w^2+x^2+y^2+z^2)-2w\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

the result is

$$(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2=0.$$

The substitution in question gives

$$\frac{3x^2\,(\mu y^2-\nu z^2)^2}{(\mu-\nu)^2\,yz}+yz\,(x^2+y^2+z^2)-\frac{2\,(\mu y^2-\nu z^2)}{(\mu-\nu)\,yz}\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

that is

$$3x^2\,(\mu y^2-\nu z^2)^2+(\mu-\nu)^2\,y^2z^2\,(x^2+y^2+z^2)-2\,(\mu-\nu)\,(\mu y^2-\nu z^2)\,(y^2z^2+z^2x^2+x^2y^2)=0,$$

which is in fact

$$-\mu^2y^2\,(y^2-z^2)\,(z^2-x^2)+2\mu\nu x^2\,(y^2-z^2)^2-\nu^2z^2\,(y^2-z^2)\,(x^2-y^2)=0,$$

that is, throwing out the factor y^2-z^2, it is

$$-\mu^2y^2\,(z^2-x^2)+2\mu\nu x^2\,(y^2-z^2)-\nu^2z^2\,(x^2-y^2)=0.$$

But in virtue of the equation $\dfrac{1}{\lambda}+\dfrac{1}{\mu}+\dfrac{1}{\nu}=0$, we have

$$\frac{\mu\nu}{\lambda}\,\{(\mu-\nu)\,y^2z^2+(\nu-\lambda)\,z^2x^2+(\lambda-\mu)\,x^2y^2\}$$

$$=\frac{\mu\nu}{\lambda}\,[\lambda x^2\,(y^2-z^2)+\mu y^2\,(z^2-x^2)+\nu z^2\,(x^2-y^2)],$$

$$=\mu\nu\,x^2\,(y^2-z^2)-(\mu+\nu)\,[\mu y^2\,(z^2-x^2)+\nu z^2\,(x^2-y^2)],$$

$$=-\mu^2y^2\,(z^2-x^2)+2\mu\nu\,x^2\,(y^2-z^2)-\nu^2z^2\,(x^2-y^2),$$

and the required property thus holds good.

We thus see that the equations of the nodal curve are

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu - \nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu - \lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda - \mu},$$

the nodal curve is thus the partial intersection of the two cubic scrolls (skew surfaces)

$$(\mu - \nu)\, wyz = x(\mu y^2 - \nu z^2), \quad (\nu - \lambda)\, wzx = y(\nu z^2 - \lambda x^2),$$

viz. taking A, B, C, D to be the summits of the tetrahedron the faces whereof are $x=0$, $y=0$, $z=0$, $w=0$, the first of these has AD for a nodal directrix, BC for a single directrix, BD, CD for generators; the second has BD for a nodal directrix, AC for a single directrix, AD, CD for generators; the surfaces intersect in the line AD twice, the line BD twice, and the line CD; the order of the residual curve, or nodal curve of the developable, is thus $9-(2+2+1)$, $=4$ as it should be.

I remark that the equation

$$(\mu - \nu)\, y^2z^2 + (\nu - \lambda)\, z^2x^2 + (\lambda - \mu)\, x^2y^2 = 0,$$

is the equation of the cone having its vertex at the point D, $(x=0,\ y=0,\ z=0)$, and passing through the nodal curve; the lines DA, DB, DC are each of them a nodal line of the cone, or "line through two points" of the curve; for an excubo-quartic curve the number of lines through two points which pass through a given point not on the curve is in fact $=3$.

It remains to introduce the coordinates (x, y, z, w) into the equation of the developable. We have

$$4x' = w + x - y - z,$$
$$4y' = w - x + y - z,$$
$$4z' = w - x - y + z,$$
$$4w' = w + x + y + z,$$

and thence

$$16y'z' = (w-x)^2 - (y-z)^2,$$
$$16x'w' = (w+x)^2 - (y+z)^2,$$

giving

$$8(x'w' + y'z') = w^2 + x^2 - y^2 - z^2$$

and similarly

$$8(y'w' + z'x') = w^2 - x^2 + y^2 - z^2,$$

and

$$8(z'w' + x'y') = w^2 - x^2 - y^2 + z^2.$$

Moreover

$$\begin{aligned} 16(y'z' + z'x' + x'y') = &\ (w-x)^2 - (y-z)^2 \\ &+ (w-y)^2 - (z-x)^2 \\ &+ (w-z)^2 - (x-y)^2, \\ = &\ 3w^2 - 2w(x+y+z) \\ &- x^2 - y^2 - z^2 + 2yz + 2zx + 2xy. \end{aligned}$$

Consequently

$$64w'(y'z'+z'x'+x'y') = (w+x+y+z)$$
$$\times\{3w^2-2w(x+y+z)-x^2-y^2-z^2+2yz+2zx+2xy\},$$

$$64x'y'z' = (w+x-y-z)(w-x+y-z)(w-x-y-z),$$
$$= w^3$$
$$-w^2(x+y+z)$$
$$-w(x^2+y^2+z^2-2yz-2zx-2xy)$$
$$+\quad x^3+y^3+z^3-yz^2-y^2z-zx^2-z^2x-xy^2-x^2y+2xyz.$$

Putting for shortness

$$p=x+y+z,\quad \nabla = x^2+y^2+z^2-2yz-2zx-2xy,$$

the two expressions are

$3w^3$	w^3
$+\ w^2.+p$	$+w^2.-p$
$+\ w.-2p^2-\nabla$	$-w.\nabla$
$-p\nabla$	$+x^3+y^3+z^3-y^2z-yz^2-z^2x-zx^2-x^2y-xy^2+2xyz$

or observing that $-p\nabla$ is

$$=-x^3-y^3-z^3+y^2z+yz^2+z^2x+zx^2+x^2y+xy^2+6xyz,$$

we have

$$64(w'y'z'+w'z'x'+w'x'y'+x'y'z')$$
$$= 4w^3-2w(p^2+\nabla)+8xyz,$$
$$= 4w^3$$
$$-4w(x^2+y^2+z^2)$$
$$+8xyz,$$

that is

$$16(w'y'z'+w'z'x'+w'x'y'+x'y'z') = w^3$$
$$-w(x^2+y^2+z^2)$$
$$+2xyz.$$

Moreover

$$8\{\lambda'(x'w'+y'z')+\mu'(y'x'+z'x')+\nu'(z'w'+x'y')\}$$
$$= \lambda'(w^2+x^2-y^2-z^2)$$
$$+\mu'(w^2-x^2+y^2-z^2)$$
$$+\nu'(w^2-x^2-y^2+z^2),$$
$$=-(\lambda+\mu+\nu)w^2+\lambda x^2+\mu y^2+\nu z^2;$$

and we have

$$\lambda = 2(\beta-\delta)(\gamma-\delta)(\gamma-\alpha)(\alpha-\beta),$$

$$\mu = 2(\gamma-\delta)(\alpha-\delta)(\alpha-\beta)(\beta-\gamma),$$

$$\nu = 2(\alpha-\delta)(\beta-\delta)(\beta-\gamma)(\gamma-\alpha),$$

whence $\lambda\mu\nu = 8\lambda'\mu'\nu'$.

Hence finally λ, μ, ν denoting as just mentioned, and therefore satisfying $\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{\nu}=0$, the equation of the developable is

$$\lambda\mu\nu\,\{w^3 - w(x^2+y^2+z^2) + 2xyz\}^2 + 108\,\{(\lambda+\mu+\nu)\,w^2 - \lambda x^2 - \mu y^2 - \nu z^2\}^3 = 0$$

(say this is $\lambda\mu\nu T^2 + 108S^3 = 0$), and this surface (which has obviously the cuspidal curve $S=0$, $T=0$) has also the nodal curve

$$wxyz = \frac{x^2(\mu y^2 - \nu z^2)}{\mu-\nu} = \frac{y^2(\nu z^2 - \lambda x^2)}{\nu-\lambda} = \frac{z^2(\lambda x^2 - \mu y^2)}{\lambda-\mu}.$$

I will show *à posteriori* that this is actually a nodal curve on the surface. Introducing an arbitrary parameter θ, the equations of the curve may be written *ut suprà*

$$\theta\lambda x = xw - yz,$$

$$\theta\mu y = yw - zx,$$

$$\theta\nu z = zw - xy,$$

$$2\theta(\lambda+\mu+\nu)\,w = 3w^2 - x^2 - y^2 - z^2,$$

and we have thence, as before,

$$2w\left(3w - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}\right) = 3w^2 - x^2 - y^2 - z^2.$$

Hence

$$2\theta = \frac{3w^2 - x^2 - y^2 - z^2}{(\lambda+\mu+\nu)\,w} = \frac{-2w(x^2+y^2+z^2) + 6xyz}{-\lambda x^2 - \mu y^2 - \nu z^2},$$

$$= \frac{3w^3 - w(x^2+y^2+z^2)}{(\lambda+\mu+\nu)\,w^2},$$

$$= \frac{3\,\{w^3 - w(x^2+y^2+z^2) + 2xyz\}}{(\lambda+\mu+\nu)\,w^2 - \lambda x^2 - \mu y^2 - \nu z^2},$$

$$= \frac{3T}{S}.$$

Hence writing

$$\theta(-2\lambda x) - (-2wx + 2yz) = 0,$$

$$\theta(-2\mu y) - (-2wy + 2zx) = 0,$$

$$\theta(-2\nu z) - (-2wz + 2xy) = 0,$$

$$\theta\,.\,2(\lambda+\mu+\nu)\,w - (3w^2 - x^2 - y^2 - z^2) = 0,$$

substituting for θ its value $=\dfrac{3T}{2S}$, and attending to the significations of S and T, we have

$$3T\delta_x S - 2S\delta_x T = 0,$$
$$3T\delta_y S - 2S\delta_y T = 0,$$
$$3T\delta_z S - 2S\delta_z T = 0,$$
$$3T\delta_w S - 2S\delta_w T = 0,$$

which are in fact the conditions to be satisfied in order that the point (x, y, z, w) may belong to a nodal curve of the surface $\lambda\mu\nu T^2 + 108 S^3 = 0$.

It is to be noticed that the coordinates of the before mentioned four points of intersection of the cuspidal and the nodal curves (being as already mentioned stationary points on the cuspidal curve) may be written $x, y, z, w = (1, 1, 1, 1), (1, -1, -1, 1), (-1, 1, -1, 1), (-1, -1, 1, 1)$.

We have thus far considered the developable, or torse, the equation of which is

$$\{\lambda'(x'w' + y'z') + \mu'(y'w' + z'x') + \nu'(z'w' + x'y')\}^3$$
$$- 27\lambda'\mu'\nu'(x'y'z' + x'y'w' + x'z'w' + y'z'w')^2 = 0,$$

where $\surd(\lambda') + \surd(\mu') + \surd(\nu') = 0$; or, what is the same thing, writing a, b, c, in place of $\surd(\lambda'), \surd(\mu'), \surd(\nu')$ respectively, the torse

$$\{a^2(x'w' + y'z') + b^2(y'w' + z'x') + c^2(z'w' + x'y')\}^3 - 27a^2b^2c^2(x'y'z' + x'y'w' + x'z'w' + y'z'w')^2 = 0,$$

where $a + b + c = 0$.

Inverting this by the equations $x', y', z', w' = \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}, \dfrac{1}{w}$, we obtain a sextic surface

$$\{a^2(xw + yz) + b^2(yw + zx) + c^2(zw + xy)\}^3 - 27a^2b^2c^2\, xyzw\,(x + y + z + w)^2 = 0,$$

where $a + b + c = 0$; which surface I propose [secondly] to consider in the present paper.

The surface has evidently the singular tangent planes $x = 0$, $y = 0$, $z = 0$, $w = 0$, each osculating the surface in a conic, that is, meeting it in the conic taken thrice, viz.,

$x = 0$, in a conic on the quadric cone $a^2yz + b^2yw + c^2zw = 0$,
$y = 0$, „ „ „ $a^2xw + b^2zx + c^2zw = 0$,
$z = 0$, „ „ „ $a^2xw + b^2yw + c^2xy = 0$,
$w = 0$, „ „ „ $a^2yz + b^2zx + c^3xy = 0$;

and it has also a cuspidal conic, the intersection of the plane $x+y+z+w=0$ with the quadric surface

$$a^2(xw+yz)+b^2(yw+zx)+c^2(zw+yz)=0;$$

it may be observed that the four conics of osculation are also sections of this surface.

The surface has also a nodal curve, the equations of which might be obtained by inversion of those of the nodal curve of the sextic torse above referred to; but I prefer to obtain them independently, in a synthetical manner, as follows:

Take α, β, γ arbitrary, and write

$$\begin{aligned}
-A &= (b-c)\,\alpha + b\beta - c\gamma, & F &= b\gamma - c\beta,\\
-B &= (c-a)\,\beta + c\gamma - a\alpha, & G &= c\alpha - a\gamma,\\
-C &= (b-c)\,\gamma + a\alpha - b\beta, & H &= a\beta - b\alpha,\\
M &= (b-c)\,\alpha + (c-a)\,\beta + (a-b)\,\gamma, & &\\
Q &= a^2(b-c)\,\alpha + b^2(c-a)\,\beta + c^2(a-b)\,\gamma: & &
\end{aligned}$$

then it is to be shown, that not only the equation of the surface is satisfied, but that also each of the derived equations is satisfied, by the values

$$x:y:z:w = aAGHQ : bBHFQ : cCFGQ : abcFGHM;$$

each of the quantities A, B, C, M, Q is linearly expressible in terms of F, G, H, which are themselves connected by the equation $aF+bG+cH=0$; the foregoing values of x, y, z, w are consequently proportional to quartic functions of a single variable parameter, say $F \div G$; and there is thus an excubo-quartic nodal curve.

To establish the foregoing result, we have

$$\begin{aligned}
aA+bH+cG &= 0,\\
aH+bB+cF &= 0,\\
aG+bF+cC &= 0,\\
aA+bB+cC &= 0,\\
aF+bG+cH &= 0,\\
F+G+H &= -M,\\
bcF+caG+abH &= Q,
\end{aligned}$$

$$\begin{aligned}
2bcF &= a^2A - b^2B - c^2C,\\
2caG &= -a^2A + b^2B - c^2C,\\
2abH &= -a^2A - b^2B + c^2C,
\end{aligned}$$

$$\begin{aligned}
aAGH + bHBF + cFG &= abcM\,(\alpha+\beta+\gamma)^2,\\
aGH + bHF + cFG &= -abc\,(\alpha+\beta+\gamma)^2,\\
a^3AGH + b^3HBF + c^3CFG &= -abc\,\{Q\,(\alpha+\beta+\gamma)^2 + 3FGH\},\\
aBCF + bCAG + cABH &= 2Mabc\,(\alpha+\beta+\gamma)^2,\\
Q\,(\alpha+\beta+\gamma)^2 + FGH &= ABC,
\end{aligned}$$

which are all of them identical equations; but as to some of them the verification is rather complex.

Hence we have

$$x+y+z=Q\,(aAGH+bBHF+cCFG)$$
$$=abcMQ\,(\alpha+\beta+\gamma)^2,$$
$$w=abcMFGH,$$

and thence

$$x+y+z+w=abcM\,\{Q\,(\alpha+\beta+\gamma)^2+FGH\}$$
$$=abcMABC.$$

Moreover

$$xyzw=(abc)^2\,ABC\,(FGH)^3\,Q^3M,$$

and

$$27a^2b^2c^2xyzw\,(x+y+z+w)^2=27\,(abc)^6\,(ABCFGHMQ)^3\,(*).$$

Again

$$(a^2x+b^2y+c^2z)\,w=abcFGHMQ\,(a^3AGH+b^3BHF+c^3CFG)$$
$$=(abc)^2\,FGHMQ\,\{Q\,(\alpha+\beta+\gamma)^2+3FGH\}$$
$$a^2yz+b^2zx+c^2xy=abcFGHQ^2\,(aBCF+bCAG+cABH)$$
$$=(abc)^2\,FGHMQ\,.\,2Q\,(\alpha+\beta+\gamma)^2,$$

and thence

$$a^2\,(xw+yz)+b^2\,(yw+zx)+c^2\,(xy+zw)$$
$$=3\,(abc)^2\,FGHMQ\,\{Q\,(\alpha+\beta+\gamma)^2+FGH\}$$
$$=3\,(abc)^2\,ABCFGHMQ \qquad (*),$$

and the two equations marked $(*)$ verify the equation of the surface.

To verify the derived equations, write for a moment $P=a^2(yz+xw)+b^2(zx+yw)+c^2(xy+zw)$, so that the equation of the surface is $P^3-27a^2b^2c^2xyzw\,(x+y+z+w)^2=0$, and the derived equation with respect to x is

$$\frac{3}{P}\,\frac{dP}{dx}=\frac{1}{x}+\frac{2}{x+y+z+w};$$

or substituting for P and $x+y+z+w$ their values, this is

$$\frac{dP}{dx}=\frac{(abc)^2ABCFGHMQ}{x}+2abcQFGH,$$

and similarly for y, z, and w. In particular, considering the derived equation in respect to w, this is

$$a^2x+b^2y+c^2z=abcABCQ+2abcQFGH$$
$$=abcQ\,(ABC+2FGH),$$

and we have as before

$$a^2x+b^2y+c^2z=Q\,(a^3AGH+b^3BHF+c^3CFG)$$
$$=abcQ\,\{\alpha+\beta+\gamma)^2+3FGH\}$$
$$=abcQ\,(ABC+2FGH),$$

which is thus verified; the verification of the derived equations for y, z, w can be effected, but not quite so easily.

The existence of the excubo-quartic nodal curve is thus established.