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THEOREM RELATING TO THE FOUR CONICS WHICH TOUCH THE SAME TWO LINES AND PASS THROUGH THE SAME FOUR POINTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. (1867), pp. 162—167.]

THE sides of the triangle formed by the given points meet one of the given lines in three points, say P, Q, R; and on this same line we have four points of contact, say A_1, A_2, A_3, A_4; any two pairs, say A_1, A_2; A_3, A_4, form with a properly selected pair, say Q, R, out of the above-mentioned three points, an involution; and we have thus the three involutions

$$(A_1,\ A_2;\ \ A_3,\ A_4;\ \ Q,\ R),$$
$$(A_1,\ A_3;\ \ A_4,\ A_2;\ \ R,\ P),$$
$$(A_1,\ A_4;\ \ A_2,\ A_3;\ \ P,\ Q).$$

To prove this, let $x=0$, $y=0$ be the equations of the given lines, and take for the equations of the sides of the triangle formed by the given points

$$b\,x+a\,y-a\,b=0,$$
$$b'\,x+a'\,y-a'\,b'=0,$$
$$b''x+a''y-a''b''=0:$$

the equation of any one of the four conics may be written

$$\frac{Lab}{bx+ay-ab}+\frac{L'a'b'}{b'x+a'y-a'b'}+\frac{L''a''b''}{b''x+a''y-a''b''}=0,$$

and if this touches the axis of x, say at the point $x=\alpha$, then we must have

$$\frac{La}{x-a}+\frac{L'a'}{x-a'}+\frac{L''a''}{x-a''}=\frac{-K(x-\alpha)^2}{(x-a)(x-a')(x-a'')};$$

or, assuming as we may do, $K=-(a'-a'')(a''-a)(a-a')$, this gives

$$L\,a=(a-\alpha)^2(a'-a''),$$
$$L'a'=(a'-\alpha)^2(a''-a),$$
$$L''a''=(a''-\alpha)^2(a-a').$$

But in the same manner, if the conic touch the axis of y, say at the point $y=\beta$, we have

$$L\,b=(b-\beta)^2(b'-b''),$$
$$L'b'=(b'-\beta)^2(b''-b),$$
$$L''b''=(b''-\beta)^2(b-b');$$

and thence

$$b(a-\alpha)^2(a'-a'')\,:\,b'(a'-\alpha)^2(a''-a)\,:\,b''(a''-\alpha)^2(a-a')$$
$$=a(b-\beta)^2(b'-b'')\,:\,a'(b'-\beta)^2(b''-b)\,:\,a''(b''-\beta)^2(b-b').$$

Putting

$$P=a\,b\,(a'-a'')(b'-b''),$$
$$P'=a'b'(a''-a)(b''-b),$$
$$P''=a''b''(a-a')(b-b'),$$

we have

$$(a-\alpha)^2\frac{P}{a^2}\,:\,(a'-\alpha)^2\frac{P'}{a'^2}\,:\,(a''-\alpha)^2\frac{P''}{a''^2}=(b-\beta)^2(b'-b'')^2\,:\,(b'-\beta)^2(b''-b)^2\,:\,(b''-\beta)^2(b-b')^2;$$

and thence

$$(a-\alpha)\frac{\sqrt{(P)}}{a}\,:\,(a'-\alpha)\frac{\sqrt{(P')}}{a'}\,:\,(a''-\alpha)\frac{\sqrt{(P'')}}{a''}$$
$$=(b-\beta)(b'-b'')\,:\,(b'-\beta)(b''-b)\,:\,(b''-\beta)(b-b'),$$

which gives

$$(a-\alpha)\frac{\sqrt{(P)}}{a}+(a'-\alpha)\frac{\sqrt{(P')}}{a'}+(a''-\alpha)\frac{\sqrt{(P'')}}{a''}=0,$$

and we have in like manner

$$(b-\beta)\frac{\sqrt{(P)}}{b}+(b'-\beta)\frac{\sqrt{(P')}}{b'}+(b''-\beta)\frac{\sqrt{(P'')}}{b''}=0,$$

but the first of these equations is alone required for the present purpose. Putting for shortness

$$P=a^2X,\qquad P'=a'^2X',\qquad P''=a''^2X'',$$

the equation is

$$(a-\alpha)\sqrt{(X)}+(a'-\alpha)\sqrt{(X')}+(a''-\alpha)\sqrt{(X'')},$$

and by attributing the signs + and − to the radicals, we have, corresponding to the four conics, the equations

$$\begin{aligned}(a-\alpha_1)\sqrt{(X)}+(a'-\alpha_1)\sqrt{(X')}+(a''-\alpha_1)\sqrt{(X'')}&=0,\\ -(a-\alpha_2)\sqrt{(X)}+(a'-\alpha_2)\sqrt{(X')}+(a''-\alpha_2)\sqrt{(X'')}&=0,\\ (a-\alpha_3)\sqrt{(X)}-(a'-\alpha_3)\sqrt{(X')}+(a''-\alpha_3)\sqrt{(X'')}&=0,\\ (a-\alpha_4)\sqrt{(X)}+(a'-\alpha_4)\sqrt{(X')}-(a''-\alpha_4)\sqrt{(X'')}&=0,\end{aligned}$$

where α_1, α_2, α_3, α_4 are the values of α for the four conics respectively.

Eliminating a'' we obtain the system of three equations

$$\begin{aligned}(2a-\alpha_1-\alpha_2)\sqrt{(X)}+(\alpha_2-\alpha_1)\sqrt{(X')}+(\alpha_2-\alpha_1)\sqrt{(X'')}&=0,\\ (\alpha_3-\alpha_1)\sqrt{(X)}+(2a'-\alpha_1-\alpha_3)\sqrt{(X')}+(\alpha_3-\alpha_1)\sqrt{(X'')}&=0,\\ (\alpha_1+\alpha_2-\alpha_3-\alpha_4)\sqrt{(X)}+(\alpha_1+\alpha_3-\alpha_2-\alpha_4)\sqrt{(X')}+(\alpha_1+\alpha_4-\alpha_2-\alpha_3)\sqrt{(X'')}&=0,\end{aligned}$$

and then eliminating the radicals we have

$$\begin{vmatrix} 2a-\alpha_1-\alpha_2, & \alpha_2-\alpha_1, & \alpha_2-\alpha_1 \\ \alpha_3-\alpha_1, & 2a'-\alpha_1-\alpha_3, & \alpha_3-\alpha_1 \\ \alpha_1+\alpha_2-\alpha_3-\alpha_4, & \alpha_1+\alpha_3-\alpha_2-\alpha_4, & \alpha_1+\alpha_4-\alpha_2-\alpha_3 \end{vmatrix}=0,$$

which is in fact

$$-4\begin{vmatrix} 1, & a+a', & aa' \\ 1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4 \\ 1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3 \end{vmatrix}=0,$$

as may be verified by actual expansion; the transformation of the determinant is a peculiar one.

The foregoing result was originally obtained as follows, viz. writing for a moment

$$\begin{aligned}a\sqrt{(X)}+a'\sqrt{(X')}+a''\sqrt{(X'')}&=\Theta,\\ \sqrt{(X)}+\sqrt{(X')}+\sqrt{(X'')}&=\Phi,\end{aligned}$$

the four equations are

$$\begin{aligned}\Theta-\alpha_1\Phi&=0,\\ \Theta-\alpha_2\Phi&=2(a-\alpha_2)\sqrt{(X)},\\ \Theta-\alpha_3\Phi&=2(a'-\alpha_3)\sqrt{(X')},\\ \Theta-\alpha_4\Phi&=2(a''-\alpha_4)\sqrt{(X'')};\end{aligned}$$

these give

$$\begin{aligned}(\alpha_1-\alpha_2)\Phi&=2(a-\alpha_2)\sqrt{(X)},\\ (\alpha_1-\alpha_3)\Phi&=2(a'-\alpha_3)\sqrt{(X')},\\ (\alpha_1-\alpha_4)\Phi&=2(a''-\alpha_4)\sqrt{(X'')}.\end{aligned}$$

From the last equation we have

$$(\alpha_1-\alpha_4)\,\Phi = 2\,\{\Theta - a\sqrt{(X)} - a'\sqrt{(X')}\} - 2\alpha_4\,\{\Phi - \sqrt{(X)} - \sqrt{(X')}\}$$
$$= 2\,(\alpha_1-\alpha_4)\,\Phi - 2\,(a-\alpha_4)\sqrt{(X)} - 2\,(a'-\alpha_4)\sqrt{(X')};$$

that is

$$(\alpha_1-\alpha_4)\,\Phi - 2\,(a-\alpha_4)\sqrt{(X)} - 2\,(a'-\alpha_4)\sqrt{(X')} = 0;$$

or substituting for $\sqrt{(X)}$, $\sqrt{(X')}$ their values in terms of Φ, we find

$$\alpha_1-\alpha_4 - \frac{(a-\alpha_4)(\alpha_1-\alpha_2)}{a-\alpha_2} - \frac{(a'-\alpha_4)(\alpha_1-\alpha_3)}{a'-\alpha_3} = 0,$$

which may be written

$$\alpha_1-\alpha_4-(\alpha_1-\alpha_2)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)-(\alpha_1-\alpha_3)\left(1+\frac{\alpha_3-\alpha_4}{a'-\alpha_3}\right)=0,$$

that is

$$\alpha_2+\alpha_3-\alpha_1-\alpha_4+\frac{(\alpha_2-\alpha_1)(\alpha_2-\alpha_4)}{a-\alpha_2}+\frac{(\alpha_3-\alpha_1)(\alpha_3-\alpha_4)}{a'-\alpha_3}=0;$$

or again

$$(\alpha_2-\alpha_1)\left(1+\frac{\alpha_2-\alpha_4}{a-\alpha_2}\right)+(\alpha_3-\alpha_4)\left(1+\frac{\alpha_3-\alpha_1}{a'-\alpha_3}\right)=0,$$

that is

$$(\alpha_2-\alpha_1)\frac{a-\alpha_4}{a-\alpha_2}+(\alpha_3-\alpha_4)\frac{a'-\alpha_1}{a'-\alpha_3}=0;$$

or finally

$$(\alpha_2-\alpha_1)(a-\alpha_4)(a'-\alpha_3)+(\alpha_3-\alpha_4)(a-\alpha_2)(a'-\alpha_1)=0,$$

which is a known form of the relation

$$\begin{vmatrix} 1, & a+a', & aa' \\ 1, & \alpha_1+\alpha_4, & \alpha_1\alpha_4 \\ 1, & \alpha_2+\alpha_3, & \alpha_2\alpha_3 \end{vmatrix}=0,$$

which gives the involution of the quantities a, a'; α_1, α_4; α_2, α.

We have in like manner

$$\begin{vmatrix} 1, & a'+a'', & a'a'' \\ 1, & \alpha_1+\alpha_2, & \alpha_1\alpha_2 \\ 1, & \alpha_3+\alpha_4, & \alpha_3\alpha_4 \end{vmatrix}=0,$$

and

$$\begin{vmatrix} 1, & a''+a, & a''a \\ 1, & \alpha_1+\alpha_3, & \alpha_1\alpha_3 \\ 1, & \alpha_2+\alpha_4, & \alpha_2\alpha_4 \end{vmatrix}=0,$$

which give the involutions of the systems a', a''; α_1, α_2; α_3, α_4 and a'', a; α_1, α_3; α_2, α_4 respectively.

It may be remarked that the equation of the conic passing through the three points and touching the axis of x in the point $x=\alpha$ is

$$\frac{(a-\alpha)^2(a'-a'')\,b}{bx+ay-ab}+\frac{(a'-\alpha)^2(a''-a)\,b'}{b'x+a'y-a'b'}+\frac{(a''-\alpha)^2(a-a')\,b''}{b''x+a''y-a''b''}=0,$$

and when this meets the axis of y we have

$$\frac{\frac{b}{a}(a-\alpha)^2(a'-a'')}{y-b}+\frac{\frac{b'}{a'}(a'-\alpha)^2(a''-a)}{y-b'}+\frac{\frac{b''}{a''}(a''-\alpha)^2(a-a')}{y-b''}=0.$$

Hence, if this touches the axis of y in the point $y=\beta$, the left-hand side must be

$$=\frac{\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right](y-\beta)^2}{(y-b)(y-b')(y-b'')},$$

and equating the coefficients of $\frac{1}{y^2}$, we have

$$\frac{b^2}{a}(a-\alpha)^2(a'-a'')+\frac{b'^2}{a'}(a'-\alpha)^2(a''-a)+\frac{b''^2}{a''}(a''-\alpha)^2(a-a')$$

$$=\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right](b+b'+b''-2\beta),$$

or what is the same thing,

$$\frac{b\,(b'+b'')}{a}(a-\alpha)^2(a'-a'')+\frac{b'\,(b''+b)}{a'}(a'-\alpha)^2(a''-a)+\frac{b''\,(b+b')}{a''}(a''-\alpha)^2(a-a')$$

$$=2\beta\left[\frac{b}{a}(a-\alpha)^2(a'-a'')+\frac{b'}{a'}(a'-\alpha)^2(a''-a)+\frac{b''}{a''}(a''-\alpha)^2(a-a')\right],$$

which gives β in terms of α, that is β_1, β_2, β_3, β_4 in terms of α_1, α_2, α_3, α_4 respectively.

Cambridge, 30 *November*, 1863.