389.

ON A LOCUS DERIVED FROM TWO CONICS.

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REQUIRED the locus of a point which is such that the pencil formed by the tangents through it to two given conics has a given anharmonic ratio.

Suppose, for a moment, that the equation of the tangents to the first conic is $(x-ay)(x-by)=0$, and that of the tangents to the second conic is $(x-cy)(x-dy)=0$, and write

$$A=(a-b)(c-d),$$
$$B=(a-c)(d-b),$$
$$C=(a-d)(b-c),$$

so that

$$A+B+C=0,$$

write also

$$k_1=\frac{B}{A},\quad k_2=\frac{C}{A},$$

then the anharmonic ratio of the pencil will have a given value k if

$$(k-k_1)(k-k_2)=0\,;$$

that is, if

$$k^2+k+\frac{BC}{A^2}=0,$$

or, what is the same thing, if

$$A^2(2k+1)^2+4BC-A^2=0\,;$$

that is, if

$$A^2(2k+1)^2-(B-C)^2=0,$$

where

$$A^2 = (a-b)^2 (c-d)^2,$$
$$B - C = (a+b)(c+d) - 2(ab+cd),$$

are each of them symmetrical in regard to a, b, and in regard to c, d, respectively.

Let the equations of the two conics be

$$U = (a, b, c, f, g, h \between x, y, z)^2 = 0,$$
$$U' = (a', b', c', f', g', h' \between x, y, z)^2 = 0,$$

and let (α, β, γ) be the coordinates of the variable point. Putting as usual

$$(A, B, C, F, G, H) = (bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch),$$
$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

the equation of the tangents to the first conic is

$$(A, B, C, F, G, H \between X, Y, Z)^2 = 0,$$

where

$$X = \gamma y - \beta z,\quad Y = \alpha z - \gamma x,\quad Z = \beta x - \alpha y,$$

and therefore

$$\alpha X + \beta Y + \gamma Z = 0.$$

Hence substituting for Z the value $-\frac{1}{\gamma}(\alpha X + \beta Y)$, we find, for the equation of the tangents, an equation of the form $\mathrm{a}X^2 + 2\mathrm{h}XY + \mathrm{b}Y^2 = 0$, which has, in effect, been taken to be $(X - aY)(X - bY) = 0$; that is, we have

$$1 : a+b : ab = \mathrm{a} : -2\mathrm{h} : \mathrm{b};$$

and, in like manner, if the accented letters refer to the second conic

$$1 : c+d : cd = \mathrm{a}' : -2\mathrm{h}' : \mathrm{b}'.$$

Substituting for a, h, b their values, and for a′, h′, b′ the corresponding values, we find

$1 : a+b : ab$	$1 : c+d : cd$
$= \quad A\gamma^2 - 2G\gamma\alpha + C\alpha^2$	$= \quad A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2$
$: -2(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)$	$: -2(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta)$
$: B\gamma^2 - 2F\beta\gamma + C\beta^2.$	$: B'\gamma^2 - 2F'\beta\gamma + C'\beta^2.$

We then have

$$\begin{aligned}(a-b)^2 &= (a+b)^2 - 4ab,\\ &= 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)^2\\ &\quad - 4(A\gamma^2 - 2G\gamma\alpha + C\alpha^2)(B\gamma^2 - 2F\beta\gamma + C\beta^2),\\ &= -4\gamma^2(BC - F^2, \ldots \between \alpha, \beta, \gamma)^2,\\ &= -4\gamma^2 K(a, \ldots \between \alpha, \beta, \gamma)^2,\end{aligned}$$

and similarly

$$(c-d)^2 = -4\gamma^2 K'(a', \ldots \between \alpha, \beta, \gamma)^2.$$

We have, moreover,

$$\begin{aligned}(a+b)(c+d) - 2(ab+cd)& \\ = \ & 4(H\gamma^2 - F\alpha\gamma - G\beta\gamma + C\alpha\beta)(H'\gamma^2 - F'\alpha\gamma - G'\beta\gamma + C'\alpha\beta) \\ & - 2(B\gamma^2 - 2F\beta\gamma + C\beta^2 \qquad)(A'\gamma^2 - 2G'\gamma\alpha + C'\alpha^2) \\ & - 2(B'\gamma^2 - 2F'\beta\gamma + C'\beta^2 \qquad)(A\gamma^2 - 2G\gamma\alpha + C\alpha^2), \\ = & - 2\gamma^2(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2,\end{aligned}$$

and substituting the foregoing values, we find

$$4(2k+1)^2 KK'(a, \ldots \between \alpha, \beta, \gamma)^2 (a', \ldots \between \alpha, \beta, \gamma)^2 - \{(BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2\}^2 = 0,$$

or putting for shortness

$$\Theta = (BC' + B'C - 2FF', \ldots, GH' + G'H - AF' - A'F, \ldots \between \alpha, \beta, \gamma)^2,$$

the equation of the locus is

$$4(2k+1)^2 KK' \,.\, UU' - \Theta^2 = 0,$$

where (α, β, γ) are current coordinates. The locus is thus a quartic curve having quadruple contact with each of the conics $U=0$, $U'=0$; viz. it touches them at their points of intersection with the conic $\Theta = 0$, *which is the locus of the point such that the four tangents form a harmonic pencil.*

The equation may be written somewhat more elegantly under the form

$$4(2k+1)^2 \,.\, KU \,.\, K'U' - \Theta^2 = 0;$$

viz. in this equation we have

$$\begin{aligned} KU &= (BC - F^2, \ldots \qquad \between \alpha, \beta, \gamma)^2, \\ K'U' &= (B'C' - F'^2, \ldots \qquad \between \alpha, \beta, \gamma)^2, \\ \Theta &= (BC' + B'C - 2FF', \ldots \between \alpha, \beta, \gamma)^2. \end{aligned}$$

In the last form the equation is expressed in terms of the coefficients $(A, \ldots)$, $(A', \ldots)$ of the *line* equations of the conics, viz. these may be taken to be

$$(A, \ldots \between \xi, \eta, \zeta)^2 = 0, \quad (A', \ldots \between \xi, \eta, \zeta)^2 = 0.$$

In particular, if each of the conics break up into a pair of points, viz. (l, m, n) and (p, q, r) for the first conic, (l', m', n') and (p', q', r') for the second conic, then the line equations are

$$2(l\xi + m\eta + n\zeta)(p\xi + q\eta + r\zeta) = 0,$$
$$2(l'\xi + m'\eta + n'\zeta)(p'\xi + q'\eta + r'\zeta) = 0,$$

so that

$$A = 2lp, \; .. \; F = mr + nq, \; ..$$
$$A' = 2l'p', \; .. \; F' = m'r' + n'q', \; ..$$

$$(BC - F^2, \; ...) = -(mr - nq \, , \; np - lr \, , \; lq - mp \,)^2,$$
$$(B'C' - F'^2, \; ...) = -(m'r' - n'q', \; n'p' - l'r', \; l'q' - m'p')^2,$$
$$BC' + B'C - 2FF' = 2\{(mn' - m'n)(qr' - q'r) - (mr' - nq')(m'r - n'q), \; ...\},$$

and substituting these values the equation is

$$(2k+1)^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix}^2 \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}^2 - \left\{ \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ p, & q, & r \\ p', & q', & r' \end{vmatrix} - \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix} \right\}^2 = 0,$$

which, if A, B, C denote

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p, & q, & r \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p', & q', & r' \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ p', & q', & r' \\ p, & q, & r \end{vmatrix}, \quad \begin{vmatrix} \alpha, & \beta, & \gamma \\ l, & m, & n \\ p', & q', & r' \end{vmatrix} \begin{vmatrix} \alpha, & \beta, & \gamma \\ l', & m', & n' \\ p, & q, & r \end{vmatrix}$$

respectively, $(A + B + C = 0)$ is, in fact, the equation

$$(2k+1)^2 A^2 - (B - C)^2 = 0,$$

or, what is the same thing,

$$\left(k - \frac{B}{A}\right)\left(k - \frac{C}{A}\right) = 0,$$

that is

$$k = \frac{B}{A} \text{ or } k = \frac{C}{A},$$

either of which expresses the anharmonic property of the points of a conic in the form given by the theorem *ad quatuor lineas*.

Reverting to the case of two conics, then if these be referred to a set of conjugate axes, the equations will be

$$a\,x^2 + b\,y^2 + c\,z^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 = 0,$$

we have $K = abc$, $K' = a'b'c'$,

$$\Theta = (bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2,$$

and the equation of the quartic curve is

$$4\,(2k+1)^2\,abca'b'c'\,(ax^2 + by^2 + cz^2)\,(a'x^2 + b'y^2 + c'z^2)$$
$$- \{(bc' + b'c)\,aa'x^2 + (ca' + c'a)\,bb'y^2 + (ab' + a'b)\,cc'z^2\}^2 = 0.$$

I suppose in particular that the two conics are

$$x^2 + my^2 - 1 = 0,$$

$$mx^2 + \quad y^2 - 1 = 0,$$

the equation of the quartic is

$$4(2k+1)^2 m^2 (x^2 + my^2 - 1)(mx^2 + y^2 - 1) - \{(m^2 + m)(x^2 + y^2) - m^2 - 1\}^2 = 0;$$

or putting $\lambda = \dfrac{(m+1)^2}{4(2k+1)^2}$, this is

$$\lambda \left(x^2 + y^2 - \frac{m^2+1}{m^2+m}\right)^2 - (x^2 + my^2 - 1)(mx^2 + y^2 - 1) = 0.$$

To fix the ideas, suppose that m is positive and >1, so that each of the conics is an ellipse, the major semi-axis being $=1$, and the minor semi-axis being $= \dfrac{1}{\sqrt{(m)}}$. For any real value of k the coefficient λ is positive, and it may accordingly be assumed that λ is positive.

We have $\dfrac{m^2+1}{m(m+1)} > \dfrac{1}{m} < 1$, or the radius of the circle is intermediate between the semi-axes of the ellipses, hence the points of contact on each ellipse are real points.

Writing for shortness

$$\alpha = \frac{m^2+1}{m^2+m},$$

the equation is

$$(x^2 + my^2 - 1)(mx^2 + y^2 - 1) - \lambda(x^2 + y^2 - \alpha)^2 = 0.$$

For the points on the axis of x, we have

$$(x^2 - 1)(mx^2 - 1) - \lambda(x^2 - \alpha)^2 = 0,$$

that is

$$(m - \lambda)x^4 + \{-(1+m) + 2\lambda\alpha\}x^2 + (1 - \lambda\alpha^2) = 0,$$

and thence

$$(m - \lambda)x^2 = \tfrac{1}{2}(1+m) - \lambda\alpha \pm \tfrac{1}{2}\sqrt{\{(m-1)^2 + 4\lambda(1-\alpha)(1-m\alpha)\}},$$

or, substituting for α its value, this is

$$(m - \lambda)x^2 = \tfrac{1}{2}(m+1) - \frac{\lambda\left(m + \dfrac{1}{m}\right)}{m+1} \pm \frac{\tfrac{1}{2}(m-1)}{m+1}\sqrt{\{(m+1)^2 - 4\lambda\}}.$$

Remarking that the values $\dfrac{(m+1)^2}{\left(m + \dfrac{1}{m}\right)^2}$, m, $\tfrac{1}{4}(m+1)^2$ are in the order of increasing magnitude, and considering successive values of λ; first the value $\lambda = \frac{1}{\alpha^2}, = \frac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, we have

$$(m-\lambda)\,x^2 = \tfrac{1}{2}(m+1) - \frac{m+1}{m+\frac{1}{m}} \pm \frac{\tfrac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)}$$

$$= \frac{(m+1)\tfrac{1}{2}\left(m+\frac{1}{m}-2\right) \pm \tfrac{1}{2}(m-1)\left(m-\frac{1}{m}\right)}{\left(m+\frac{1}{m}\right)};$$

or observing that

$$(m+1)\left(m+\frac{1}{m}-2\right) = (m+1)\frac{1}{m}(m-1)^2 = \frac{1}{m}(m-1)(m^2-1) = (m-1)\left(m-\frac{1}{m}\right),$$

this is

$$(m-\lambda)\,x^2 = 0, \text{ or } \frac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}},$$

or, what is the same thing,

$$\frac{(m-1)(m^3+2m^2-1)}{m\left(m+\frac{1}{m}\right)^2}\,x^2 = 0, \text{ or } \frac{(m-1)\left(m-\frac{1}{m}\right)}{m+\frac{1}{m}}, \quad x^2 = 0, \text{ or } \frac{\left(m^2-\frac{1}{m^2}\right)m}{m^3+2m^2-1}.$$

The next critical value is $\lambda = m$. The curve here is

$$(x^2+my^2-1)(mx^2+y^2-1) - m(x^2+y^2-\alpha)^2 = 0,$$

that is

$$m(x^4+y^4) + (1+m^2)\,x^2y^2 - (m+1)(x^2+y^2) + 1$$
$$-\,m(x^4+y^4) - 2m\;x^2y^2 + 2m\alpha\;(x^2+y^2) - m\alpha^2 = 0,$$

that is

$$(m-1)^2\,x^2y^2 + (2m\alpha - m - 1)(x^2+y^2) + 1 - m\alpha^2 = 0,$$

or, substituting for α its value,

$$2m\alpha - m - 1 = \frac{2m^2+2}{m+1} - (m+1) = \frac{(m-1)^2}{m+1},$$

$$1 - m\alpha^2 = 1 - \frac{(m^2+1)^2}{m(m+1)^2} = -\frac{(m-1)^2(m^2+m+1)}{m(m+1)^2};$$

the equation is

$$x^2y^2 + \frac{1}{m+1}(x^2+y^2) - \frac{m^2+m+1}{m(m+1)^2} = 0,$$

or, as this may also be written,

$$\left(x^2+\frac{1}{m+1}\right)\left(y^2+\frac{1}{m+1}\right) - \frac{1}{m} = 0,$$

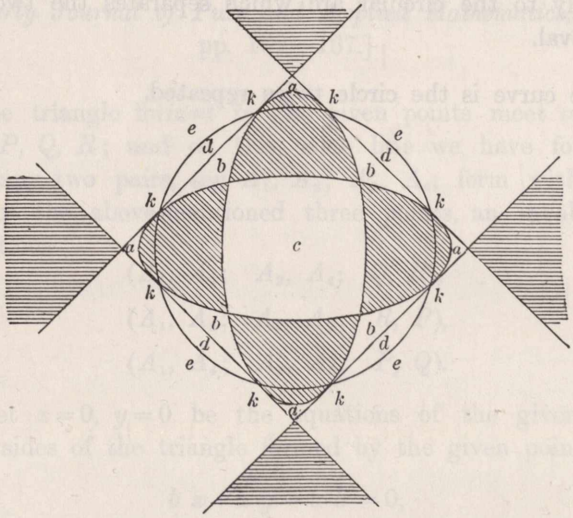
which has a pair of imaginary asymptotes parallel to the axis of x, and a like pair parallel to the axis of y, or what is the same thing, the curve has two isolated points at infinity, one on each axis.

The next critical value is $\lambda = \frac{1}{4}(m+1)^2$; the curve here reduces itself to the four lines

$$\left\{(x+y)^2 - \frac{m+1}{m}\right\}\left\{(x-y)^2 - \frac{m+1}{m}\right\} = 0;$$

and it is to be observed that when λ exceeds this value, or say $\lambda > \frac{1}{4}(m+1)^2$, the curve has no real point on either axis; but when $\lambda = \infty$, the curve reduces itself to $(x^2+y^2-\alpha)^2 = 0$, i.e. to the circle $x^2+y^2-\alpha = 0$ twice repeated, having in this special case real points on the two axes.

It is now easy to trace the curve for the different values of λ. The curve lies in every case within the unshaded regions of the figure (except in the limiting cases after-mentioned); and it also touches the two ellipses and the four lines at the eight points k, at which points it also cuts the circle; but it does not cut or touch the



four lines, the two ellipses, or the circle, except at the points k. Considering λ as varying by successive steps from 0 to ∞;

$\lambda = 0$, the curve is the two ellipses.

$\lambda < \dfrac{(m+1)^2}{\left(m+\frac{1}{m}\right)^2}$, the curve consists of two ovals, an exterior sinuous oval lying in the four regions a and the four regions b; and an interior oval lying in the region c.

$\lambda = \dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2}$, there is still a sinuous oval as above, but the interior oval has dwindled to a conjugate point at the centre.

$\lambda > \dfrac{(m+1)^2}{\left(m+\dfrac{1}{m}\right)^2} < m$; $\lambda = m$; $\lambda > m < \dfrac{(m+1)^2}{4}$; there is no interior oval, but only a sinuous oval as above; which, as λ increases, approaches continually nearer to the four sides of the square. For the critical value $\lambda = m$, there is no change in the general form, but the curve has for this value of λ, two conjugate points, one on each axis at infinity.

$\lambda = \frac{1}{4}(m+1)^2$, the curve becomes the four lines.

$\lambda > \frac{1}{4}(m+1)^2$, the curve lies wholly in the four regions *a* and the four regions *e*, consisting thereof of four detached sinuous ovals. As λ deviates less from the value $\frac{1}{4}(m+1)^2$, each oval approaches more nearly to the infinite trilateral formed by the side and infinite line-portions which bound the regions *d*, *e* to which the oval belongs. And as λ departs from the limit $\frac{1}{4}(m+1)^2$, and approaches to ∞, each sinuous oval approaches more nearly to the circular arc which separates the two regions *d, e,* which contains the sinuous oval.

Finally, $\lambda = 0$, the curve is the circle twice repeated.