XXIII.

CORRESPONDENCE

[Note Book 39.]

[*To Professor Powell.**]

Observatory, Dublin,
Oct. 24*th*, 1835.

MY DEAR SIR

I received your letter a few days ago, & had even previously been intending to write to you on the subject of dispersion. My opinion of the value of your researches on that subject has not undergone any change, & I continue to consider you successful, and believe you to be the first who has been so, in the arduous & important enterprise of practically establishing the existence of a near agreement between the facts of dispersion & a formula suggested by theory.

But it occurred to me some time ago, and I have been intending to mention it to you, that this formula does not (in my judgment) follow as a *rigorous*, though it does as an *approximate*, consequence from Cauchy's mathematical investigations. In short, I think that you are not strictly warranted in omitting a certain sign of summation, although the final result is only slightly affected by that omission.

To simplify the question, let us conceive a plane wave perpendicular to the axis of x, with vibrations parallel to the axis of y. Then the displacements ξ and ζ will vanish, & we shall have the formula

$$\frac{d^2\eta}{dt^2} = S\left\{m\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\Delta\eta\right\}, \tag{1}$$

by Cauchy's theory, or by your equations (12) in the Phil. Mag. for Jan. last; η being the value at the time t of the varying displacement of the molecule m which has x, y, z for its rectangular coordinates of equilibrium; & $\eta + \Delta\eta$ being the displacement, at the same moment t, of another molecule m, which has for its rectangular coordinates of equilibrium $x + \Delta x$, $y + \Delta y$, $z + \Delta z$; while r is the distance $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ between these two molecules in their positions of equilibrium, & β is the angle between this distance & the axis of y; & finally $\mathrm{f}(r)$ and $f(r)$ are functions of r, of which the former (if positive) expresses the law of attraction, or (if negative) the law of repulsion, and the latter is derived from it by the rule

$$f(r) = r\mathrm{f}'(r) - \mathrm{f}(r):$$

and S is a sign of summation, relative to the actions (attractive or repulsive) of all the molecules m.

If now we suppose
$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{2}$$

η_0, η_1, t_0 & λ, μ being constants, we shall have

$$\frac{d^2\eta}{dt^2} = -\left(\frac{2\pi}{\lambda}\right)^2 \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{3}$$

* [Baden Powell, 1796–1860. Savilian Professor of Geometry, Oxford, 1827–60. Published researches on Optics and Radiation.]

and

$$\Delta\eta = -2\eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right)\left(\sin\frac{\pi\mu\Delta x}{\lambda}\right)^2 - \eta_1 \sin\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right)\sin\frac{2\pi\mu\Delta x}{\lambda}; \qquad (4)$$

so that, the 2nd part of $\Delta\eta$ disappearing in the summation, the differential equation of vibration (1) is satisfied, if we establish the following relation between the constants λ, μ,

$$\left(\frac{2\pi}{\lambda}\right)^2 = S\left\{2m\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\left(\sin\frac{\pi\mu\Delta x}{\lambda}\right)^2\right\}. \qquad (5)$$

Such is, I think, the law of the velocity $\frac{1}{\mu}$ of propagation of a wave, considered as depending on the periodical time λ of the vibrations of the molecules, & deduced from the principles of Cauchy. If we put for abridgment

$$\frac{\pi\mu\Delta x}{\lambda} = \theta, \qquad (6)$$

and

$$\frac{m}{2}\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\Delta x^2 = H^2, \qquad (7)$$

(although this quantity H^2 is not necessarily nor always positive), the law becomes

$$\left(\frac{1}{\mu}\right)^2 = S \,.\, H^2\left(\frac{\sin\theta}{\theta}\right)^2, \qquad (8)$$

in which it appears necessary, for rigour, to retain the sign of summation. To omit that sign would come to considering the action of only two near molecules; and even to apply the sign to H^2 only, without treating θ as variable, would come to considering only the action of two near layers of molecules, for which Δx^2 is the same: whereas it is likely that the number of such near layers, within the extent of sensible action, is on the contrary very great. Yet, after all, the law of their total action is almost the same in kind as the law of the action of those two. For if we suppose, as it seems to be permitted to do, that θ (which is the *semidifference of phases* of two near molecules) is small, though not insensible, within the extent of sensible action; we may then develope, according to the ascending powers of this small arc, the square of the ratio of its sine to itself,

$$\left(\frac{\sin\theta}{\theta}\right)^2 = 1 - \frac{\theta^2}{3} + \frac{2\theta^4}{45} - \&\text{c}. \qquad (9)$$

$$= 1 - \frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2\Delta x^2 + \frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4\Delta x^4 - \&\text{c}.,$$

and thus the law (8) will become

$$\left(\frac{1}{\mu}\right)^2 = S(H^2) - \frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2 S(H^2\Delta x^2) + \frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4 S(H^2\Delta x^4) - \&\text{c}. \qquad (10)$$

And *if this series converge rapidly enough*, it will give, *nearly*,

$$\left(\frac{1}{\mu}\right)^2 = S(H^2) - \frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{S(H^2)}, \qquad (11)$$

$$\frac{1}{\mu} = \sqrt{S(H^2)}\left\{1 - \frac{1}{6}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{\{S(H^2)\}^2}\right\}, \qquad (12)$$

and finally

$$\frac{1}{\mu} = H,\frac{\sin\theta,}{\theta,}, \qquad (13)$$

if we put for abridgment

$$H_{,}=\sqrt{S(H^2)},\tag{14}$$

and

$$\theta_{,}=\frac{\pi\sqrt{S(H^2\Delta x^2)}}{\lambda S(H^2)}.\tag{15}$$

But if we develope $\left(\frac{1}{\mu}\right)^2$ according to the powers of $\left(\frac{\pi}{\lambda}\right)^2$ by the approximate formula (13), we get

$$\left(\frac{1}{\mu}\right)^2=S(H^2)-\frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{S(H^2)}+\frac{2}{45}\left(\frac{\pi}{\lambda}\right)^4\frac{\{S(H^2\Delta x^2)\}^2}{\{S(H^2)\}^3}-\&c.;\tag{16}$$

whereas if we develope it by the more accurate formula (10), we find

$$\left(\frac{1}{\mu}\right)^2=S(H^2)-\frac{1}{3}\left(\frac{\pi}{\lambda}\right)^2\frac{S(H^2\Delta x^2)}{S(H^2)}+\left(\frac{\pi}{\lambda}\right)^4\left\{\frac{2}{45}\frac{S(H^2\Delta x^4)}{\{S(H^2)\}^2}-\frac{1}{9}\frac{\{S(H^2\Delta x^2)\}^2}{\{S(H^2)\}^3}\right\}-\&c.;\tag{17}$$

and it seems possible enough that the difference of the third terms may be sensible.—Had we taken as our approximate formula

$$\frac{1}{\mu}=H_{\prime\prime}\frac{\sin\theta_{\prime\prime}}{\theta_{\prime\prime}},\tag{18}$$

in which

$$H_{\prime\prime}=H_{,}=\sqrt{S(H^2)},\tag{19}$$

and

$$\theta_{\prime\prime}=\frac{\pi\mu}{\lambda}\sqrt{\frac{S(H^2\Delta x^2)}{S(H^2)}},\tag{20}$$

we should have had the approximate development

$$\left(\frac{1}{\mu}\right)^2=S(H^2)-\frac{1}{3}\left(\frac{\pi\mu}{\lambda}\right)^2S(H^2\Delta x^2)+\frac{2}{45}\left(\frac{\pi\mu}{\lambda}\right)^4\frac{\{S(H^2\Delta x^2)\}^2}{S(H^2)}-\&c.,\tag{21}$$

which would resemble more the original development (10), but still might differ sensibly from it, because the two expressions,

$$S(H^2\Delta x^4)\quad\text{and}\quad\frac{\{S(H^2\Delta x^2)\}^2}{S(H^2)},$$

are not in general coincident.—I conclude therefore that your simplification of Cauchy's results, obtained by omitting the sign of summation, or by attending only to the action of one pair of points, or at most one pair of layers, may be attended with some loss of accuracy, & indeed is likely to be so *if the third term* $\left(\text{or the term proportional to }\frac{1}{\lambda^4}\right)$ *in the development of* $\frac{1}{\mu}$ *be sensible*. And perhaps some of the differences which you have met, between the observed and calculated indices, may have their source in the insufficient expression of this third term. You will judge whether attention to this suggestion might not increase the value of your already valuable researches.—

For my own part, when my attention had been awakened by your success in reconciling so far as you have done the facts with the theory of dispersion, & when I came to perceive that this theory required in rigour the retaining of the sign of summation, (which I did only about a month ago, having been occupied till then by other things, especially by Halley's Comet,) I instituted several numerical calculations, some account of which you may like to have. The immediate result of theory appears, for the reasons that I have given, to be a law which may be expressed by a series of the form

$$\left(\frac{1}{\mu}\right)^2 = A_0 - A_1\left(\frac{\mu}{\lambda}\right)^2 + A_2\left(\frac{\mu}{\lambda}\right)^4 - \&c., \tag{22}$$

the general term being

$$\frac{\left(-\frac{4\pi^2\mu^2}{\lambda^2}\right)^i}{1\,.\,2\,.\,3\,.\,4\ldots(2i+2)} S\left\{m\frac{\mathrm{f}(r)+\cos\beta^2 f(r)}{r}\Delta x^{2i+2}\right\}, \tag{23}$$

in which we may, if we choose, change Δx to $r\cos\alpha$. And it seems likely that in all useful applications of the law this series will converge; and also that the terms will be alternately positive and negative, or at least that the first term A_0 will be positive and the second term negative. Moreover it may be expected that the resultant development of μ itself, according to the ascending integer powers of $\left(\frac{1}{\lambda}\right)^2$, will also be convergent; namely the series

$$\mu = a_0 + a_1\lambda^{-2} + a_2\lambda^{-4} + \&c., \tag{24}$$

of which the coefficients may be determined by equating to zero the coefficients of the several powers of λ^{-2} in the following identical development, deduced from (22) by substitution of the value (24) for μ, namely

$$0 = -1 + A_0(a_0 + a_1\lambda^{-2} + a_2\lambda^{-4} + \&c.)^2 - A_1\lambda^{-2}(a_0 + a_1\lambda^{-2} + \&c.)^4 + A_2\lambda^{-4}(a_0 + \&c.)^6 - \&c. \tag{25}$$

For the first three of the new coefficients, we have thus the three equations,

$$\left.\begin{aligned} 0 &= -1 + A_0 a_0^2, \\ 0 &= 2A_0 a_0 a_1 - A_1 a_0^4, \\ 0 &= 2A_0 a_0 a_2 + A_0 a_1^2 - 4A_1 a_0^3 a_1 + A_2 a_0^6; \end{aligned}\right\} \tag{26}$$

which give

$$\left.\begin{aligned} a_0 &= A_0^{-\frac{1}{2}}, \\ a_1 &= \tfrac{1}{2}A_1 A_0^{-\frac{5}{2}}, \\ a_2 &= -\tfrac{1}{2}A_2 A_0^{-\frac{7}{2}} + \tfrac{7}{8}A_1^2 A_0^{-\frac{9}{2}}. \end{aligned}\right\} \tag{27}$$

If we may confine ourselves to these three, the law of dispersion becomes simply

$$\mu = a_0 + a_1\lambda^{-2} + a_2\lambda^{-4}, \tag{28}$$

in which a_0 & a_1 at least may be expected to be positive, while all the three coefficients a_0, a_1, a_2 are constant for any single medium, but vary in passing from one such medium to another. Consider now the four rays B, D, F, H. We shall have, for these four rays, and for any single medium, the system of the four equations

$$\left.\begin{aligned} \mu_B &= a_0 + a_1\lambda_B^{-2} + a_2\lambda_B^{-4}, \\ \mu_D &= a_0 + a_1\lambda_D^{-2} + a_2\lambda_D^{-4}, \\ \mu_F &= a_0 + a_1\lambda_F^{-2} + a_2\lambda_F^{-4}, \\ \mu_H &= a_0 + a_1\lambda_H^{-2} + a_2\lambda_H^{-4}; \end{aligned}\right\} \tag{29}$$

between which it is possible to eliminate the three medium-constants a_0, a_1, a_2, and so to deduce a general relation, valid for all media, between the 4 indices μ_B, μ_D, μ_F, μ_H and the four periodical times λ_B, λ_D, λ_F, λ_H.

A slight consideration shows that this relation can only involve the two ratios of the three differences of the four indices, & the two ratios of the three differences of the four reciprocals of the squares of the periodical times; so that if we put for abridgment

$$\frac{\mu_D-\mu_B}{\mu_H-\mu_B}=s_D, \quad \frac{\mu_F-\mu_B}{\mu_H-\mu_B}=s_F, \tag{30}$$

and

$$\frac{\lambda_D^{-2}-\lambda_B^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}}=t_D, \quad \frac{\lambda_F^{-2}-\lambda_B^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}}=t_F, \tag{31}$$

the result of the elimination will be a relation between the four quantities s_D, s_F, t_D, t_F only, and will not involve the four other quantities $\mu_B, \mu_H, \lambda_B, \lambda_H$, if we previously substitute for $\mu_D, \mu_F, \lambda_D^{-2}, \lambda_F^{-2}$ their expressions deduced from (30) & (31), namely

$$\mu_D=\mu_B+s_D(\mu_H-\mu_B), \qquad \mu_F=\mu_B+s_F(\mu_H-\mu_B), \tag{32}$$

and

$$\lambda_D^{-2}=\lambda_B^{-2}+t_D(\lambda_H^{-2}-\lambda_B^{-2}), \quad \lambda_F^{-2}=\lambda_B^{-2}+t_F(\lambda_H^{-2}-\lambda_B^{-2}). \tag{33}$$

This result of elimination will $\because$ be the same as if we supposed, in the equations (29),

$$\left.\begin{array}{llll}\lambda_B^{-2}=0, & \lambda_H^{-2}=1, & \lambda_D^{-2}=t_D, & \lambda_F^{-2}=t_F,\\ \mu_B=0, & \mu_H=1, & \mu_D=s_D, & \mu_F=s_F,\\ a_0=0, & a_1=b, & a_2=c; & \end{array}\right\} \tag{34}$$

that is, the same as if we eliminated any two new quantities b & c between the three new equations

$$s_D=bt_D+ct_D^2, \quad s_F=bt_F+ct_F^2, \quad 1=b+c. \tag{35}$$

This last elimination is easy, & gives, as the relation sought, the following:

$$\frac{s_D}{t_D}\frac{1-t_F}{t_D-t_F}+\frac{s_F}{t_F}\frac{1-t_D}{t_F-t_D}=1. \tag{36}$$

As a verification, we may expand this relation, by substituting the values (30) & (31), so as to put it under the form

$$\begin{aligned}0= \quad &(\mu_D-\mu_B)(\lambda_H^{-2}-\lambda_B^{-2})(\lambda_F^{-2}-\lambda_B^{-2})(\lambda_H^{-2}-\lambda_F^{-2})\\ &-(\mu_F-\mu_B)(\lambda_H^{-2}-\lambda_B^{-2})(\lambda_D^{-2}-\lambda_B^{-2})(\lambda_H^{-2}-\lambda_D^{-2})\\ &+(\mu_H-\mu_B)(\lambda_F^{-2}-\lambda_B^{-2})(\lambda_D^{-2}-\lambda_B^{-2})(\lambda_F^{-2}-\lambda_D^{-2});\end{aligned} \tag{37}$$

for it will then be found to be satisfied, independently of the three medium-constants a_0, a_1, a_2, by the expressions (29) for the four indices $\mu_B, \mu_D, \mu_F, \mu_H$.

Having found from theory that this relation (36) or (37) was likely to hold good for all media, I proceeded to try it on the ten media observed by Fraunhofer. Adopting first his values of λ, which are for these four rays

$$\lambda_B=0{,}2541; \quad \lambda_D=0{,}2175; \quad \lambda_F=0{,}1794; \quad \lambda_H=0{,}1464; \tag{38}$$

(the unit of time being here the time which light takes to traverse, in what is called a vacuum, the ten thousandth part of a Paris inch*;) I perceived that within the limits of the errors of these determinations, & indeed to a very great accuracy, the square of the period λ_F is the harmonic mean between the squares of the extreme periods λ_B & λ_H; so that

$$\lambda_F^{-2}=\tfrac{1}{2}(\lambda_H^{-2}+\lambda_B^{-2}), \tag{39}$$

* [The metre superseded the French foot as the legal unit of length in France on Nov. 2nd, 1801. The Paris inch equals 0·0254 of a metre.]

or in the notation (31),

$$t_F = \tfrac{1}{2}. \tag{40}$$

Availing myself of this circumstance, I put the relation (36) or (37) under the simpler form

$$4s_F\, t_D(1-t_D) - s_D = t_D(1-2t_D), \tag{41}$$

or

$$\mu_D - \mu_F = a_D(\mu_F - \mu_B) + b_D(\mu_H - 2\mu_F + \mu_B), \tag{42}$$

a_D and b_D being coefficients independent of the medium, which serve to calculate the index μ_D when the indices μ_B, μ_F, μ_H have been observed, & which have for their expressions, as functions of t_D, or of λ_B, λ_D, λ_H, the following:

$$a_D = -(1-2t_D) = -\frac{\lambda_H^{-2} - 2\lambda_D^{-2} + \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}}; \tag{43}$$

$$b_D = -t_D(1-2t_D) = -\frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} \cdot \frac{\lambda_H^{-2} - 2\lambda_D^{-2} + \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}}; \tag{44}$$

supposing, as already mentioned, that the observed relation (39) or (40) is accurate. Employing also the observed values (38) of λ_B, λ_D, λ_H, I found

$$\left.\begin{aligned} \log(-a_D) &= \bar{1},80441; \\ \log(-b_D) &= \bar{1},06281. \end{aligned}\right\} \tag{45}$$

For Flint Glass of the kind which he called Nr 13, Fraunhofer had found by observation:

$$\mu_B = 1,6277; \quad \mu_F = 1,6483; \quad \mu_H = 1,6711; \tag{46}$$

and hence, by (42) and (45), I calculated

$$\mu_D = 1,63492. \tag{47}$$

The value observed by Fraunhofer was

$$\mu_D = 1,6350. \tag{48}$$

The difference between calculation and observation is ∵ in this case very small. A similar calculation for each of the ten media, the data being taken to four decimal places from your paper in the Transactions, & the logarithms being taken to five places, (even fewer might here have sufficed,) gave, very easily, results which may thus be collected:

Medium	μ_D *Calcul*[d]	μ_D *Obs*[d]	
Flint Glass Nr 13	1,63492	1,6350	(49)
Flint Glass Nr 23	1,63350	1,6337	
Flint Glass Nr 30	1,63051	1,6306	
Flint Glass Nr 3	1,60825	1,6085	
Crown Glass *M*	1,55901	1,5591	
Crown Glass Nr 13	1,52788	1,5280	
Crown Glass Nr 9	1,52945	1,5296	
Oil of turpentine	1,47444	1,4744	
Solution of potash	1,40270	1,4028	
Water	1,33346	1,3336.	

The agreement between the ten observed & the ten calculated values of μ_D is close enough; yet it is worth remarking that in almost every case the calculated value is a little less than the observed. This circumstance led me to think that by altering a little the multipliers a_D, b_D

in the formula (42), the agreement might be improved so far as the *indices* are concerned. But in order to justify an alteration of these multipliers, it is necessary to suppose that the values of λ may be a little altered, or in other words that the *periodical times* of vibration for the four rays have not been determined so very accurately that the errors of their determinations may not have some sensible effect in the calculation of the indices; because the multipliers a_D, b_D may be expressed in general as the following functions of those periodical times,

$$\left.\begin{aligned} a_D &= b_D + (t_D - t_F)\frac{1-t_D}{t_F}, \\ b_D &= (t_D - t_F)\frac{t_D}{1-t_F}; \end{aligned}\right\} \tag{50}$$

or thus more fully,

$$a_D = \frac{\lambda_D^{-2} - \lambda_F^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}}\left\{\frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_F^{-2}} + \frac{\lambda_H^{-2} - \lambda_D^{-2}}{\lambda_F^{-2} - \lambda_B^{-2}}\right\}, \tag{51}$$

and

$$b_D = \frac{\lambda_D^{-2} - \lambda_F^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} \cdot \frac{\lambda_D^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_F^{-2}}; \tag{52}$$

as appears from the comparison of the formula (42) with the relation (36) or (37). However it seemed to me worth while to try the effect of supposing the values of λ to be quite unknown, & of determining the multipliers a_D, b_D from observations on the indices alone. The formula (42), (which is not dependent on the particular supposition (39) or (40), provided that we conceive a_D, b_D to represent the general & as yet unknown values (51) & (52),) gives for any two different media the two equations:

$$\left.\begin{aligned} \mu_D - \mu_F &= (a_D - b_D)(\mu_F - \mu_B) + b_D(\mu_H - \mu_F), \\ \mu'_D - \mu'_F &= (a_D - b_D)(\mu'_F - \mu'_B) + b_D(\mu'_H - \mu'_F); \end{aligned}\right\} \tag{53}$$

the accented indices belonging to the second medium: & hence we have, by elimination,

$$\left.\begin{aligned} a_D - b_D &= \frac{(\mu'_H - \mu'_F)(\mu_D - \mu_F) - (\mu_H - \mu_F)(\mu'_D - \mu'_F)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}, \\ b_D &= \frac{(\mu'_D - \mu'_F)(\mu_F - \mu_B) - (\mu_D - \mu_F)(\mu'_F - \mu'_B)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}. \end{aligned}\right\} \tag{54}$$

To apply these expressions I chose the two media Water & Flint Glass, and among the various specimens of the latter, examined by Fraunhofer, I chose his Nr 23, (specific gravity = 3,724,) because it appears from his Table of indices, extracted by Herschel in article 437 of the Treatise on Light, that he has observed and recorded the phenomena of each of these two media, as they presented themselves in two distinct experiments....

[Numerical details are here omitted and the resulting table simplified.]

...and substituting these values in the expressions (54), I found

$$b_D = -0{,}1547788; \quad a_D - b_D = -0{,}4732852; \quad a_D = -0{,}6280640; \tag{61}$$

or

$$\log(-b_D) = \bar{1}{,}1897115; \quad \log(b_D - a_D) = \bar{1}{,}6751230; \quad \log(-a_D) = \bar{1}{,}7980039. \tag{62}$$

Using these two new multipliers $a_D - b_D$ & b_D, and employing equations of the form (53) & the observed values of μ_B, μ_F, μ_H as stated in that Table of Fraunhofer to which I just now referred, I found as follows:

Med^m	$\mu_F-\mu_D$ *Calc*^{d}	$\mu_F-\mu_D$ *Obs*^{d}	*Calc*^{d} − *Obs*^{d}	
F.G. 13	0,0132368	0,0132240	+0,0000128	
F.G. 23	0,0131015	0,0131015	0,0000000	
F.G. 30	0,0129154	0,0128810	+0,0000344	
F.G. 3	0,0116659	0,0115480	+0,0001179	
C.G. *M*	0,0076340	0,0076660	−0,0000320	(63)
C.G. 13	0,0063462	0,0063550	−0,0000088	
C.G. 9	0,0064643	0,0064650	−0,0000007	
Oil of T.	0,0071984	0,0073020	−0,0001036	
Sol^n of P.	0,0052832	0,0052770	+0,0000062	
Water	0,0042260	0,0042260	0,0000000.	

Hence
$$\left.\begin{array}{ll} \text{Sum of positive differences} & = +0{,}0001713, \\ \text{Sum of negative differences} & = -0{,}0001451, \\ \text{Sum of all differences} & = +0{,}0000262, \\ \text{Mean of all differences} & = +0{,}0000026. \end{array}\right\} \tag{64}$$

This mean difference may be regarded as insensible; which cannot so well be said of the mean of the differences (49); & therefore the new multipliers $a_D - b_D$ & b_D are probably not far from the truth, & may be thought to be somewhat more exact than the multipliers (45).

To see now what relations they suppose between the periodical times of vibration, we are to resolve the equations (51) & (52), or more simply the two equations (50), so as to deduce the values of t_D and t_F, or of the equivalent expressions
$$\frac{\lambda_D^{-2}-\lambda_B^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}} \text{ and } \frac{\lambda_F^{-2}-\lambda_B^{-2}}{\lambda_H^{-2}-\lambda_B^{-2}}.$$

The equations (50) give, when cleared of their divisors,
$$\left.\begin{array}{l} (a_D-b_D)\,t_F=(t_D-t_F)(1-t_D), \\ b_D(1-t_F)=(t_D-t_F)\,t_D; \end{array}\right\} \tag{65}$$
and therefore, by addition,
$$(a_D-b_D)\,t_F+b_D(1-t_F)+t_F=t_D. \tag{66}$$

Substituting this expression for t_D in the second equation (65), we find
$$b_D(1-t_F)=\{(a_D-b_D)\,t_F+b_D(1-t_F)\}\{(a_D-b_D)\,t_F+b_D(1-t_F)+t_F\}; \tag{67}$$
that is, we have to resolve the following quadratic for t_F,
$$t_F^2(a_D-2b_D)(a_D-2b_D+1)+2t_Fb_D(a_D-2b_D+1)+b_D(b_D-1)=0. \tag{68}$$

Hence, by the numeric values (61) or (62), we find
$$0=-0{,}2170601\,t_F^2-0{,}2109614\,t_F+0{,}1787352; \tag{69}$$
that is,
$$0=t_F^2+0{,}9719029\,t_F-0{,}8234368; \tag{70}$$
and $\because$, attending only to the positive root,
$$t_F=+0{,}5434102; \tag{71}$$
from which we derive, by (66),
$$t_D=+0{,}2155518. \tag{72}$$

The values of these two ratios deduced from the observed values of λ, (38), by the formula (31), are sensibly smaller, namely

$$t_F = +0{,}4999527, \tag{73}$$

and

$$t_D = +0{,}1813010. \tag{74}$$

Yet perhaps when the great difference of the methods is considered, the agreement of results, such as it is, may be thought to give some support to the theory. However, the fairest way to judge of the degree to which the results of these two methods approach to or diverge from each other is to compare the values of the periods λ_D and λ_F, which were deduced by Fraunhofer from a spectrum of interference, formed by a fine grating, (&, if I remember rightly, without any use of the knowledge of the refractive indices,) with the values of the same two mean periods λ_D and λ_F, calculated from Fraunhofer's similarly deduced values of the two extreme periods λ_B and λ_H, by employing the theoretical law of dispersion (28) and the 8 refractive indices $\mu_B, \mu_D, \mu_F, \mu_H, \mu'_B, \mu'_D, \mu'_F, \mu'_H$, for water and for a specimen of flint glass, which were determined by the same optician for the same four definite rays, through a process which did not suppose any knowledge of the periods of vibration. Calculating ∵ the two mean periods λ_D and λ_F, by the method above explained, that is by the formulae (33), (71), (72), from the two extreme periods λ_B, λ_H considered as known by observation of the diffraction-spectrum, namely $\lambda_B = 0{,}2541$, $\lambda_H = 0{,}1464$ in (38), and from the 4 water and 4 flint glass indices, in equations (59) & (60), considered as known by observation of two prismatic spectra, I find

$$\lambda_D = 0{,}21221; \quad \lambda_F = 0{,}17561: \tag{75}$$

values which do not differ very greatly from those which Fraunhofer found from the diffraction-spectrum, namely,

$$\lambda_D = 0{,}2175; \quad \lambda_F = 0{,}1794: \tag{38}$$

since the differences, namely $\quad -0{,}00529$ and $-0{,}00379$, (76)

correspond each, in space, to less than the millionth part of an inch &, as compared with the corresponding period or undulation itself, to less than the fortieth part of such period. I even think that by pursuing, with greater accuracy & to greater extent than I have done, this train of investigation, it will be possible to deduce, from the phenomena of dispersion alone, those relations between the several periods or undulations which answer to my fractions t_D, t_F, &c., namely the ratios of the differences of the inverse squares of those periods, more exactly than from the phenomena of interference. And perhaps I may be tempted hereafter to resume this particular inquiry: since I see that many improvements might be made in the plan of the foregoing calculations.

Meanwhile, as I have been, I am likely to continue to be, diverted for some time from this inquiry, I shall lay before you a few other results of past calculations, which look a little promising. I have hitherto mentioned only combinations of the four rays B, D, F, H. But by combining in exactly the same way the four rays B, F, G, H, I arrived at analogous conclusions. Thus, adopting Fraunhofer's observed value of λ_G,

$$\lambda_G = 0{,}1587, \tag{77}$$

(in which the unit is still the ten thousandth part of a Paris inch, or the time light takes to traverse it in a space void of ponderable matter,) & retaining his values of λ_B, λ_H, I found

$$t_G = \frac{\lambda_G^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} = +0{,}7769594; \tag{78}$$

and hence, approximately, in the formula

$$\mu_G - \mu_F = a_G(\mu_F - \mu_B) + b_G(\mu_H - 2\mu_F + \mu_B), \tag{79}$$

which is analogous to (42), I found, by supposing

$$t_F = \tfrac{1}{2}, \tag{40}$$

these numerical expressions for the multipliers a_G, b_G,

$$\left.\begin{aligned} a_G &= -1 + 2t_G = +0{,}55392, \\ b_G &= t_G(-1 + 2t_G) = +0{,}43037, \end{aligned}\right\} \tag{80}$$

or

$$\left.\begin{aligned} \log a_G &= \bar{1}{,}74345, \\ \log b_G &= \bar{1}{,}63384. \end{aligned}\right\} \tag{81}$$

Adopting, next, the approximate values of μ_B, μ_F, μ_H, as stated, from Fraunhofer's observations, in your paper, & collected in table (49) of this letter, for the ten media mentioned in that paper or in that table, I found for those media, in their order, results which may be thus collected:

Medium	μ_G *Calcul*^{d}	μ_G *Obs*^{d}	*Calcul*^{d} − *Obs*^{d}	
F.G. 13	1,66066	1,6603	+0,00036	
F.G. 23	1,65910	1,6588	+0,00030	
F.G. 30	1,65569	1,6554	+0,00029	
F.G. 3	1,63101	1,6308	+0,00021	
C.G. *M*	1,57368	1,5735	+0,00018	(82)
C.G. 13	1,54001	1,5399	+0,00011	
C.G. 9	1,54182	1,5416	+0,00022	
Oil of T.	1,48833	1,4882	+0,00013	
Sol^n of P.	1,41272	1,4126	+0,00012	
Water	1,34140	1,3413	+0,00010.	

Here

$$\left.\begin{aligned} \text{Sum of differences} &= +0{,}00202; \\ \text{Mean of differences} &= +0{,}000202. \end{aligned}\right\} \tag{83}$$

This mean is not insensible; nor does it seem likely that it would be made so, by redressing the little errors of these comparatively rough calculations: for instance I happened to employ the value $\bar{1}{,}74344$ for $\log a_G$ instead of $\bar{1}{,}74345$; but if the latter more accurate logarithm of 0,55392 had been employed it would have tended to increase (though very slightly) the excess of the calculated over the observed μ_G. I proceeded ∵ to calculate the multipliers $a_G - b_G$ & b_G, in the formula

$$\mu_G - \mu_F = (a_G - b_G)(\mu_F - \mu_B) + b_G(\mu_H - \mu_F), \tag{84}$$

by employing the expressions

$$\left.\begin{aligned} a_G - b_G &= \frac{(\mu'_H - \mu'_F)(\mu_G - \mu_F) - (\mu_H - \mu_F)(\mu'_G - \mu'_F)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}, \\ b_G &= \frac{(\mu'_G - \mu'_F)(\mu_F - \mu_B) - (\mu_G - \mu_F)(\mu'_F - \mu'_B)}{(\mu'_H - \mu'_F)(\mu_F - \mu_B) - (\mu_H - \mu_F)(\mu'_F - \mu'_B)}, \end{aligned}\right\} \tag{85}$$

which are analogous to (54), & in which the four indices μ_B, μ_F, μ_G, μ_H are those determined for water, and μ'_B, μ'_F, μ'_G, μ'_H are those determined for flint glass (N^r 23), by the means of the two experiments of Fraunhofer on each of those two media. Those mean values of μ_B, μ_F, μ_H

for water, & of μ'_B, μ'_F, μ'_H for that specimen of flint glass, have been already stated in equations (59) & (60) of this letter; with respect to μ_G & μ'_G, they were taken out in the same way, from Fraunhofer's table extracted by Herschel, which gave

$$\left.\begin{aligned}\mu_G &= 1,3412770,\\ \mu'_G &= 1,6588485.\end{aligned}\right\} \tag{86}$$

In this manner I found, by (85),

$$a_G - b_G = +0,09514275; \quad b_G = +0,4433440; \quad a_G = 0,53848675; \tag{87}$$

$$\log(a_G - b_G) = \bar{2},9783757; \quad \log b_G = \bar{1},6467408. \tag{88}$$

With these new multipliers & the observed values of $\mu_F - \mu_B$, $\mu_H - \mu_F$, as stated in table (63) of this letter for each of the ten media, I found for those media the following values of $\mu_G - \mu_F$, calculated by the formula (84), & the following differences between calculation & observation:

Medium	$\mu_G - \mu_F$ *Calc*[d]	$\mu_G - \mu_F$ *Obs*[d]	*Calc*[d] − *Obs*[d]	
F.G. 13	0,0120606	0,0120250	+0,0000356	(89)
F.G. 23	0,0120799	0,0120805	−0,0000006	
F.G. 30	0,0119152	0,0119400	−0,0000248	
F.G. 3	0,0107262	0,0107300	−0,0000038	
C.G. *M*	0,0067819	0,0067940	−0,0000121	
C.G. 13	0,0055411	0,0055710	−0,0000299	
C.G. 9	0,0056337	0,0056050	+0,0000287	
Oil of T.	0,0064507	0,0064620	−0,0000113	
Sol[n] of P.	0,0044777	0,0044970	−0,0000193	
Water	0,0034739	0,0034740	−0,0000001.	

Here

$$\left.\begin{aligned}&\text{Sum of positive differences} = +0,0000643;\\ &\text{Sum of negative differences} = -0,0001019;\\ &\text{Sum of all the differences} = -0,0000376;\\ &\text{Mean of all the differences} = -0,0000038;\end{aligned}\right\} \tag{90}$$

so that the separate differences are small, much smaller than those in table (82), and the mean of all is quite insensible & is less than the fiftieth part of the mean in (83). There is reason ∵ to consider the new values of the multipliers (87) as more exact than the old ones (80), from which however they do not very greatly differ. And substituting the new values of $a_G - b_G$ and b_G in the equations analogous to (50), namely in the following:

$$\left.\begin{aligned}a_G - b_G &= (t_G - t_F)\frac{1 - t_G}{t_F},\\ b_G &= (t_G - t_F)\frac{t_G}{1 - t_F},\end{aligned}\right\} \tag{91}$$

or in these others, derived from them,

$$t_G = t_F(a_G - 2b_G + 1) + b_G, \tag{92}$$

and $$t_F^2(a_G - 2b_G)(a_G - 2b_G + 1) + 2t_F b_G(a_G - 2b_G + 1) + b_G(b_G - 1) = 0, \tag{93}$$

I found first this quadratic equation for t_F,

$$0 = -0,2269571t_F^2 + 0,5779421t_F - 0,2467901, \tag{94}$$

that is, $$0 = t_F^2 - 2,5464830t_F + 1,0873863, \tag{95}$$

or, confining ourselves to the root which lies between 0 and 1,

$$t_F = 0,5426557; \tag{96}$$

and then, $$t_G = 0,7970463. \tag{97}$$

These values, deduced solely from the indices of four rays observed in two prismatic spectra, differ sensibly from the values

$$t_F = +0,4999527 \tag{73}$$

and $$t_G = +0,7769594, \tag{78}$$

which were deduced solely from the spectrum of diffraction; but it may be considered as somewhat confirming the theory that they do not differ still more. And if we employ the prismatic values (96), (97) of t_F, t_G to calculate the two mean periods or undulations λ_F, λ_G from the two extreme periods λ_B, λ_H considered as known by observations on diffraction, we find, by the method already explained,

$$\lambda_F = 0,17568; \quad \lambda_G = 0,15746, \tag{98}$$

values which differ from those deduced from the observations on diffraction alone, namely from the following

(38) $$\lambda_F = 0,1794 \quad \text{and} \quad \lambda_G = 0,1587, \tag{77}$$

only by the little differences

$$-0,00372 \quad \text{and} \quad -0,00124, \tag{99}$$

of which each is much less that the millionth part of an inch, if expressed in space, & of which the former is about the fiftieth part, & the latter less than the hundredth part of the whole corresponding period or undulation λ. But what appears to me the most remarkable and encouraging result of these calculations, is the near agreement of the two prismatic results, (71) & (96), which give respectively

$$\left.\begin{aligned} \frac{\lambda_F^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} &= +0,5434102, \\ \frac{\lambda_F^{-2} - \lambda_B^{-2}}{\lambda_H^{-2} - \lambda_B^{-2}} &= +0,5426557; \end{aligned}\right\} \tag{100}$$

the first having been obtained by combining the rays B, D, F, H & the second by combining the rays B, F, G, H, as observed in two different spectra produced by prismatic dispersion. So close is this agreement, that, if Fraunhofer's values be taken for λ_B and λ_H, the two resulting values of λ_F, namely

(75) $$\lambda_F = 0,17561 \quad \& \quad \lambda_F = 0,17568, \tag{98}$$

when reduced to what they represent in space, do not differ by so much as the hundred-millionth part of an inch in the length of the concluded undulation. If methods more refined had been employed,—for example, the method of least squares in combining more media than two,—& if greater care had been given to the numerical part of the calculation, it seems reasonable to think that the agreement might have been found closer still. But I have reason to think that, for highly dispersive media, even the *fourth* term ($= a_3\lambda^{-6}$) of the series (24) for the index μ is sensible; and that it must be allowed for, if we would obtain a sufficiently exact expression for the dependence of that index on the period or undulation λ.

If you should not adopt these views; or if, adopting them, you should still choose to continue the use of the formula

$$\frac{1}{\mu}=\frac{H\sin\theta}{\theta}, \tag{101}$$

(in which θ varies inversely as λ, and H is a constant quantity,) as giving a good approximation to the truth; you may possibly find it convenient to employ occasionally this simple rule, which will give quickly a near approach to the value of θ_B:

$$\left.\begin{aligned}\log\theta_B&=3{,}77348+\log\sin\phi,\\ \log\sec\phi&=\tfrac{1}{2}(\log\mu_H-\log\mu_B).\end{aligned}\right\} \tag{102}$$

The value of θ_B thus found is expressed in minutes. For example, in the case of Flint Glass Nr 13,

$\mu_B=1{,}6277$	$\log=0{,}21157$	Const...	$3{,}77348$
$\mu_H=1{,}6711$	$\log=0{,}22300$	[log] $\sin\phi$...	$\bar{1}{,}20691$
	$2\,\boxed{0{,}01143}$		$2{,}98039$
$\phi=9°\,16'$	$0{,}00571/5$	$\theta_B=956'=15°\,56'$.	

This value $\theta_B=15°\,56'$ is very nearly correct, as a solution of the equations

$$\lambda_B\theta_B=\lambda_H\theta_H,\quad \mu_B\lambda_B\sin\theta_B=\mu_H\lambda_H\sin\theta_H, \tag{103}$$

λ_B and λ_H having the values 0,2541 & 0,1464, & μ_B, μ_H having those stated above. A process of correction, with which I shall not trouble you at present, gave me as a next approximation $\theta_B=15°\,57'\,53''{,}4$; on calculating θ_H from which, by the second equation (103), and then calculating θ_B from it by the first of those two equations (103), I was brought back almost exactly to what I set out with, as the second approximation, for I found $\theta_B=15°\,57'\,53''{,}5$. But this exactness appears to be useless here. My formula for the first approximation is

$$\theta_B=\sqrt{\frac{6\left(1-\frac{\mu_B}{\mu_H}\right)}{\left(\frac{\lambda_B}{\lambda_H}\right)^2-1}}. \tag{104}$$

I forget whether that which I gave you extempore in the section room* was exactly the same as this.

I became greatly interested lately in the Dynamics of Light, & thought that I could in some things improve the theory of Cauchy; especially by bringing the results to agree more closely with those of Fresnel, respecting the laws of double refraction & polarisation; though I was obliged to let some differences remain, such as that striking one concerning the relation of the direction of vibration to the plane of polarisation, & a few others, no one of which seemed capable of being soon decided by experiment. It is likely that I shall resume the whole subject, after the end of my next course of lectures on Astronomy, which always distract me from study. But before that time I hope to see Professor Lloyd, who has not yet (I believe) returned from a continental excursion; and I shall give him your message about the arragonite.—This letter has swelled to an Essay, but I expect to send it in an official frank. Accept my thanks for the new indices in your last letter, & believe me to remain, my dear Sir,

Very truly yours,

William R. Hamilton.

Prof. Powell.

* [British Association Meeting at Dublin, 1835.]

[*To Professor Powell.*]

Observatory, Dublin,
*Dec*r. 17*th*, 1835.

MY DEAR SIR

Your letter reached me yesterday or the day before, & I have great pleasure in trying to remove any obscurity into which I may have fallen, in my former long epistle. I ought perhaps to have justified my assumption of the formula

$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right); \tag{2}$$

or at all events I am bound to prevent any mistake of the meaning which I attach to that assumption. For, you will observe that I did not pretend to prove that *no other* formula would satisfy the differential equation

$$\frac{d^2\eta}{dt^2} = S\left\{m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta\eta\right\}; \tag{1}$$

I only showed that this *particular* formula (2) was *one solution* of that differential equation, provided that λ & μ are supposed to be connected by the relation

$$\left(\frac{2\pi}{\lambda}\right)^2 = S\left\{2m\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\left(\sin\frac{\pi\mu\Delta x}{\lambda}\right)^2\right\}; \tag{5}$$

& thence concluded that periodical *vibrations of the particular kind expressed by the formula* (2) MAY *be propagated* (on Cauchy's principles) *through a system of attracting & repelling points*, PROVIDED *that the slowness μ of propagation is connected with & varies with the periodical time λ of vibration by the law or relation* (5); and even that *this law* or relation is a *necessary condition* for the propagation of *such* vibrations: without pretending to decide whether vibrations of *some other sort* may or may not be propagated according to *other laws* of velocity. I confined myself to the study of vibrations of that *particular* sort which I thought that *induction from phenomena* had led mathematicians to consider; & did not presume to say that I had followed out *deductively* all *possible* mathematical consequences from the differential equation of vibration. Yet, as I have hitherto studied much less the physics than the mathematics of light, I am anxious to have your opinion as to the correctness of my assumption (2), considered as suggested by *induction*; & to know whether it really appears to you to contain anything new or unusual, when I explain fully my meanings for the symbols.

To represent the *phenomena of interference*, it is supposed, in the wave-theory of light, that certain points, or molecules of ether, *vibrate periodically in space and time*; in such a manner that if we consider the propagation of an indefinite series of plane waves parallel to the plane of *yz*, all the vibrating points, which at any one moment t have any one common x, have also equal and parallel displacements ξ, η, ζ, and velocities $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$, of vibration; the same displacements & velocities recurring periodically, *in time* after all the multiples 1λ, 2λ, 3λ, &c. of some one fixed time-period λ, & *in space* after all the multiples $1\lambda'$, $2\lambda'$, $3\lambda'$, &c. of some one fixed space-period λ': so that we may put

$$\xi = \phi(x,t), \quad \eta = \chi(x,t), \quad \zeta = \psi(x,t),$$

the functions ϕ, χ, ψ being such as to satisfy these conditions of periodicity

$$\phi(x,t)=\phi(x,t+\lambda)=\phi(x+\lambda',t),$$
$$\chi(x,t)=\chi(x,t+\lambda)=\chi(x+\lambda',t),$$
$$\psi(x,t)=\psi(x,t+\lambda)=\psi(x+\lambda',t).$$

Besides, by the *conception of the velocity* (v) *of propagation* of a wave, it is equal to the *ratio* $\frac{\lambda'}{\lambda}$ *of the period* λ' *in space to the period* λ *in time*; & accordingly it is thus that the periods in time, at least for the planetary spaces, have been inductively concluded from the periods in space observed in what we call a void (through the phenomenon of interference), and then divided by the common velocity of propagation considered as also known from astronomical observation: and more generally, the *conception of a uniform velocity of propagation* conducts to these formulae of displacement,

$$\xi=\phi\left(\frac{x}{v}-t\right), \quad \eta=\chi\left(\frac{x}{v}-t\right), \quad \zeta=\psi\left(\frac{x}{v}-t\right),$$

in which $v=\frac{\lambda'}{\lambda}$, & $\Delta\xi=0$, $\Delta\eta=0$, $\Delta\zeta=0$, if $\Delta\left(\frac{x}{v}-t\right)=$ any multiple of λ. Again, the *phenomena of refraction*, as explained by the theory of waves, conduct to the conclusion that the *velocity* v *of propagation of light*, of any one colour, in any transparent terrestrial substance, *is less* than in the planetary spaces & less *in the ratio of* 1 *to* μ, μ being the index of refraction: we must $\because$ suppose that either the period in space is diminished, or the period in time increased, or at least that the period in space becomes less as compared with the period in time, on entering a transparent terrestrial substance, than it had been in the planetary spaces. And the *phenomena of dispersion* show that *this diminution* of the ratio $\left(\frac{\text{space-period}}{\text{time-period}}\right)$ *is greater for violet than for red light*, because they show that the index of refraction is greater. Besides it was concluded by Newton, though with some diffidence, & it has (*I believe*) been confirmed by subsequent observation of the *phenomena of thin plates*; (but I beg you to set me right if I am mistaken on this point of the induction;) that whatever the colour of the light & whatever the transparent medium may be, the length of a fit,* or (as we would say) the length of an undulation, that is the *period in space*, which I have here called λ', is *inversely proportional to the index of refraction*, so that the product $\mu\lambda'$ is constant for any one colour, & does not vary with the medium, though it varies with the colour, and is less for violet than for red light. And hence I gather, what indeed is very natural & seems to be almost necessary à priori in the theory of waves, and what I thought was generally supposed by undulationists, though I do not remember seeing it expressly stated in any book, and beg of you to set me right if you know of any facts which militate against the conclusion: namely that the *periodical time of vibration*, which I have called λ, in this letter and in the last, *is constant for any one colour and does not vary with the medium*. For if we take as the unit of velocity the velocity of propagation of light in the planetary spaces, we shall have in general $v=\frac{1}{\mu}$; & since we have also $v=\frac{\lambda'}{\lambda}$, we may conclude that $\lambda=\mu\lambda'=$ a quantity depending on colour only. Again to represent the *phenomena of plane*

* [This is Newton's phrase. See *Newton's Opticks*, Book Two, Part III, Prop. XII, new edition, Bell (1931), p. 281.]

or rectilinear polarisation, and the *interference of polarised rays*, we suppose the *displacements* of the vibrating molecules to be either exactly or nearly *parallel to or contained upon the plane surface of the wave*, & to have a *line of direction independent of the time*; I therefore, who am not now considering circular polarisation, nor indeed extraordinary media, but who may suppose if I choose that the light is rectilinearly polarised, am at liberty to suppose that while the wave is parallel to the plane of yz the vibration is parallel to the axis of y, & ∵ to take for my formulae of displacement the following:

$$\xi = 0, \quad \eta = \chi(\mu x - t), \quad \zeta = 0,$$

in which the function χ must satisfy the condition $\Delta\eta = \Delta\chi(\mu x - t) = 0$, if $\Delta(\mu x - t) =$ any multiple of λ. The *simplest* way of satisfying this condition, & the one which seemed to me to *agree best with the received way of representing phenomena in the theory of waves*, was to take

$$\eta = A + B\cos\frac{2\pi(\mu x - t)}{\lambda} + C\sin\frac{2\pi(\mu x - t)}{\lambda},$$

or, which comes to the same thing,

$$\eta = \eta_0 + \eta_1 \cos\left(\frac{2\pi}{\lambda}(\mu x - t + t_0)\right), \tag{2}$$

t_0 being entirely arbitrary & η_0, η_1 being arbitrary but small. The constant η_0 is introduced for greater generality; I have some reasons for thinking it equal to $-\eta_1$, so that the *absolute displacement* would vary as the *versed sine of the phase*: but in the present investigation, it is sufficient if we can rightly represent the laws of *relative displacement*, which are not affected by the introduction of an arbitrary common term; so that if you do not like my η_0 you may for the present expunge it. Having thus shown, perhaps at too great length, my reasons *from induction* for *assuming* the formula (2) as a *probable law of vibration* and one which I thought differed little (if at all) from the assumptions of other undulationists, I may refer to my former letter for the process, taken from Cauchy, or (at the least) suggested by him, by which *I compared that assumed law of vibration with his differential equations of motion* & showed that it was at least a *particular solution* of those equations and ∵ *dynamically possible, if μ be supposed to depend on λ by the relation* (5).

After all, if you choose rather to consider μ as depending on the period in space, which is different for different media, as well as for different colours, & which I shall still call λ', you have only to change λ to $\mu\lambda'$ in that relation (5), & it then becomes

$$\left(\frac{2\pi}{\mu\lambda'}\right)^2 = S\left\{2m\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\left(\sin\frac{\pi\Delta x}{\lambda'}\right)^2\right\};$$

that is, as in formula (8),

$$\left(\frac{1}{\mu}\right)^2 = S\,.\,H^2\left(\frac{\sin\theta}{\theta}\right)^2,$$

in which H^2 is still the (not necessarily positive) quantity

$$\frac{m}{2}\,\frac{\mathrm{f}(r) + \cos\beta^2 f(r)}{r}\,\Delta x^2,$$

and θ is still the semi-difference of phases of 2 near molecules, represented now by the expression $\theta = \dfrac{\pi\Delta x}{\lambda'}$, which perfectly resembles Cauchy's. My *essential objection is to the omitting of the* *sign of summation S* and *not* to the considering of periods in space, though I think *periods in time more convenient*, because they do not seem to change in passing from one medium to another. On this last point of physics, I repeat my request that you will set me right if you think me wrong; & as to the mathematics connected with the sign of summation, I shall gladly write again, if you think there is the least occasion for it. My Lectures are just over; but, as I guessed, I feel more anxious at this moment to pursue my general view of dynamics & to expand the principles, connected therewith, of my Calculus of Principal Functions than to resume for some time my suspended optical researches. Believe me to remain very truly yours

W. R. HAMILTON.

[The remainder of the correspondence with Professor Powell is of no scientific interest.]

*To Sir John Herschel.**

Observatory, Dublin,
Feb. 8^{th}, 1839.

MY DEAR SIR

When we parted last Summer on the banks of Ullswater, you were pleased to encourage me to pursue my researches respecting the dynamics of light; & I think, from what you said on the occasion, that you are not likely to consider it as an intrusion, if I sometimes endeavour to take advantage of your greater knowledge of what has been done in mathematics abroad, so as to learn whether my results have been anticipated; & whether, even where this may not have been done, the researches and results of others are sufficiently analogous to my own to deserve and demand my attention.

It is only since the beginning of the present year that I have resumed my speculations upon light; and the problem which I have been recently considering is the *propagation of vibration*, properly so called, as distinguished from the mere *preservation* of a mode of vibration already established. Most of my own former calculations, & nearly all of those which I had seen in the writings of others, at least all those which had appeared to me to be sufficiently exact, were directed to the examination of the conditions under which a mode of vibration, thus once established, might thus be preserved for ever, through an indefinite extent of space and time; they related to the discovery of particular integrals of the general differential equations, giving periodical expressions for the disturbances as functions of x, y, z, t, which were *not discontinuous functions*, at least within the range of any single medium. The whole of each such medium was supposed to be agitated at once; & the question was, what mode of agitation could be *permanent*, the acting forces being taken into account. It was not shown, at least not to my own satisfaction, how the vibration could *spread*, according to the generally admitted laws, from one part of a medium to another part of the same, leaving all beyond and all behind it at rest. Much had been done, perhaps, in the dynamics of *light*; little, I thought, in the dynamics of *darkness*.

To give a very simple example of the difficulty that appeared to me to exist, & at the same

* [Sir John Frederick William Herschel, 1792–1871. For a general account of Hamilton's correspondence with Herschel, see Graves, *Life of Hamilton*.]

time of the questions that I have resolved, let us consider the case of a single row of equal particles, arranged at equal & finite distances on one indefinite straight line,

$$\begin{matrix} . & . & . & . & . & . & . \\ -3 & -2 & -1 & 0 & +1 & +2 & +3 \end{matrix}$$

so that their abscissae x may be considered as integer numbers, positive, negative or null, & that their ordinates y at first all vanish. Under these conditions, equilibrium subsists and motion will never arise. But if we now remove some or all of these particles,

$$m_{-\infty}, \quad \ldots \quad m_{-3}, \quad m_{-2}, \quad m_{-1}, \quad m_0, \quad m_1, \quad m_2, \quad m_3, \quad \ldots \quad m_\infty,$$

indefinitely little on either side from their natural positions on the axis of x, in directions perpendicular to that axis; so as to give to each particle m_x an indefinitely small ordinate y_x, (positive or negative or null,) without changing its abscissa x; the equilibrium will in general be broken, and motion will commence, whether we do or do not give any initial velocities y'_x to the particles. And if for greater simplicity we suppose that the force $m\mathrm{f}(r)$ which any one of these particles exerts on any other is attractive at the distance $r=1$, & insensible at the distance $r=2$ and at all greater distances; we shall have this general differential equation, as regulating the motion of any particle m_x,

$$\frac{d^2}{dt^2}y_{x,t} = m\mathrm{f}(1)\,(y_{x+1,t}-y_{x,t}) + m\mathrm{f}(1)\,(y_{x-1,t}-y_{x,t}),$$

or more concisely

$$D_t^2 y_{x,t} = \frac{a^2\Delta_x^2}{1+\Delta_x} y_{x,t}, \tag{1}$$

$a^2 = m\mathrm{f}(1)$ being the attraction at the unit of distance, that is, the attraction exerted by any particle on either of the two immediately adjacent. A particular integral of this equation is

$$y_{x,t} = \text{const.} + \text{const.} \times \sin(2x\nu - 2at\sin\nu + \text{const.}),$$

and a still more particular one is therefore

$$y_{x,t} = \eta \operatorname{vers}(2x\nu - 2at\sin\nu), \tag{2}$$

η and ν being any arbitrary constants. This last expression represents therefore a mode of vibration of which the *permanence is possible*; and which would continue for ever, *if it were once established for all values of x.* If even we suppose that, at the origin of t, the values of $y_{x,t}$ & of its differential coefficient $y'_{x,t} = D_t y_{x,t}$ agree with this expression, or in other words that the initial displacements & initial velocities are all such as to agree with this law; so that we have, for all values of x, the conditions

$$y_{x,0} = \eta \operatorname{vers} 2x\nu, \quad \text{and} \quad y'_{x,0} = -2a\eta \sin\nu \sin 2x\nu; \tag{3}$$

then the expression (2) for $y_{x,t}$, or the mode of vibration which it represents, will hold good for all values of t and of x, that is, at every moment & for every particle. And this is a fair *specimen*, I think, of the *sort* of results which I had obtained myself, & of the most satisfactory among those which I had seen as obtained by others, respecting the dynamics of light as distinguished from the dynamics of darkness, or the preservation rather than the propagation of vibrations. It gives, for example, in this particular vibratory motion, the expression

$$a\frac{\sin\nu}{\nu}$$

for the *rapidity with which a given phase travels*; and this is the simplest (though not the most accurate) form of Cauchy's Law of Dispersion. It is in fact the form which Mr Powell at first supposed to follow from Cauchy's investigations on that subject.

But let us now suppose that the initial conditions (3) are satisfied only for a finite range of x, or in other words that only a finite number of the particles have their initial displacements & velocities such as to agree with the law (2). To fix our conceptions, let the length of a wave be an exact multiple n of the molecular interval, so that $\nu=\frac{\pi}{n}$; and let a large but finite number i of such wave-lengths, immediately behind the origin of x, be the extent of initial disturbance; so that the particles $m_0, m_{-1}, m_{-2}, \ldots m_{-in}$, have their initial displacements & initial velocities determined by the conditions (3); the other particles $m_1, m_2, m_3, \ldots m_\infty$, and $m_{-in-1}, m_{-in-2}, \ldots m_{-\infty}$, (and also, by (3), the particles $m_0, m_{-n}, m_{-2n}, \ldots m_{-in+n}, m_{-in}$,) having their initial displacements & initial velocities null. In other words, let the conditions (3) be satisfied for all values of x which are not greater than 0, and not less than $-in$; but let $y_{x,0}$ & $y'_{x,0}$ vanish for all other values of x, and even, by the conditions (3) themselves, for

$$x=0, \quad x=-n, \quad x=-2n, \ldots x=-in+n, \quad x=-in.$$

These things being supposed, it is clear that the law (2) cannot hold good for *all* values of x & t; but the astronomical phenomena of the actual transfer of luminiferous vibration from one part of space to another, & the terrestrial phenomena of Sound, &c., together with the calculations already made respecting these phenomena, appear to make it *probable* that we shall find, with some reasonable degree of approximation, that the law (2) still holds for *those* particular values of x and t which are connected by the relations

$$x<at\frac{\sin\nu}{\nu}, \quad x+in>at\frac{\sin\nu}{\nu}. \tag{2'}$$

In fact, if this were rigorously true, it would imply that the *vibration travelled onwards*, with a *constant velocity* $=a\frac{\sin\nu}{\nu}$, and a *constant amplitude* $=2\eta$, occupying always i *wave-lengths*, or agitating always a row of in particles, but leaving all which are beyond and all which are behind these undisturbed. I thought it therefore important to inquire, *how nearly* these conclusions follow from these assumptions; or, in other words, how nearly the differential equation (1), combined with the initial conditions (3) for all values of x from 0 to $-in$, and with the conditions

$$y_{x,0}=0, \quad y'_{x,0}=0, \tag{3'}$$

for all other values of x, conduct to the law (2), combined with the restrictions (2)′. And I have found that from these premises results do follow, which have a certain resemblance to the results thus expected or guessed at, but which also differ from them sensibly.

After long circuits of analysis, I obtained this form for the general & rigorous integral of the equation in mixed differences (1):

$$y_{x,t}=\frac{2}{\pi}\Sigma_{(l)-\infty}^{\infty}\left(y_{x+l,0}+y'_{x+l,0}\int_0^t dt\right)\int_0^{\frac{\pi}{2}} d\theta\cos(2at\cos\theta)\cos(2l\theta+l\pi); \tag{1'}$$

which I afterwards developed into this other form

$$y_{x,t}=y_{x,0}+\frac{a^2t^2}{1.2}\Delta_x^2 y_{x-1,0}+\frac{a^4t^4}{1.2.3.4}\Delta_x^4 y_{x-2,0}+\&c.$$
$$+ty'_{x,0}+\frac{a^2t^3}{1.2.3}\Delta_x^2 y'_{x-1,0}+\frac{a^4t^5}{1.2.3.4.5}\Delta_x^4 y'_{x-2,0}+\&c.; \quad (1)''$$

$y_{x,0}$ & $y'_{x,0}$ being two quite arbitrary functions, representing the arbitrary initial displacements and velocities. Either of these two forms, $(1)'$ or $(1)''$, may easily be shown à posteriori to represent generally and rigorously the integral of the equation (1); & the latter of them may receive this elegant symbolic transformation:

$$y_{x,t}=\left\{1-\frac{a^2\Delta_x^2}{1+\Delta_x}\left(\int_0^t dt\right)^2\right\}^{-1}(y_{x,0}+ty'_{x,0}); \quad (1)'''$$

which might have been directly obtained, by a simple process, from the proposed equation (1).

Introducing now the initial conditions, or determining the arbitrary functions so as to have

$$y_{x,0}=0, \quad y'_{x,0}=0, \quad \text{if } x>0 \text{ or } <-in, \quad (3)'$$

but

$$\left.\begin{array}{c} y_{x,0}=\eta\,\text{vers}\,\dfrac{2x\pi}{n}, \quad y'_{x,0}=-2a\eta\sin\dfrac{\pi}{n}\sin\dfrac{2x\pi}{n}, \\ \text{if } x \ngtr 0, \text{ and } \nless -in; \end{array}\right\} \quad (3)''$$

I find this other expression, also rigorous, but adapted to this particular question,

$$y_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin in\theta}{\sin\theta}\frac{\cos(2x\theta+in\theta-2at\sin\theta)\,d\theta}{\cos\theta-\cos\dfrac{\pi}{n}}; \quad (2)''$$

which may be shown à posteriori to be correct, especially if we observe that

$$\frac{\sin\nu}{\pi}\int_0^\pi\frac{\cos k\theta\,d\theta}{\cos\theta-\cos\nu}=+\sin k\nu,\ 0,\ \text{or}\ -\sin k\nu,$$

according as the integer k is >0, $=0$, or <0.

It remains therefore to discuss the rigorous integral $(2)''$, & to examine *how nearly* it reproduces the law of vibration (2) with the restrictions $(2)'$.

The expression $(2)''$ may be separated into two parts, of which one changes sign with a or t, while the other remains unaltered when the sign of a or t is changed; the latter of these two parts expresses the effect $Y_{x,t}$ of the initial displacements, and the former expresses the effect $Z_{x,t}$ of the initial velocities; we have, therefore,

Effect of initial displacements

$$=Y_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin in\theta}{\sin\theta}\frac{\cos(2x\theta+in\theta)\cos(2at\sin\theta)\,d\theta}{\cos\theta-\cos\dfrac{\pi}{n}}; \quad (2)'''$$

and

Effect of initial velocities

$$=Z_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin in\theta}{\sin\theta}\frac{\sin(2x\theta+in\theta)\sin(2at\sin\theta)\,d\theta}{\cos\theta-\cos\dfrac{\pi}{n}}. \quad (2)^{\text{iv}}$$

But if it be true that the vibration spreads in one determined direction, that direction must depend on the initial velocities, & must change when a changes sign; so that while $Y_{x,t}+Z_{x,t}$ is the displacement, whether $=0$ or >0, of the particle m_x at the time t, x & t being supposed for simplicity to be each >0, & the vibration being supposed to spread from m_0 towards m_∞, it seems that we ought to have a displacement either exactly or sensibly null, $Y_{x,t}-Z_{x,t}=0$, when, by changing the sign of a, we suppose the vibration to spread the contrary way towards $m_{-\infty}$. We are therefore to inquire whether, for positive values of x and t, the functions $Y_{x,t}$ & $Z_{x,t}$ are exactly, or even nearly equal; that is, whether the effects of the initial displacements & velocities are exactly or nearly the same.

Now, these two functions cannot be *exactly equal* for all positive values of x & t; because one of them, $Y_{x,t}$, is an even function, & the other, $Z_{x,t}$, is an odd function of t. Yet I have found that these two functions approach to a numerical equality, as t or x or either tends to $+\infty$; although one changes sign with t and the other does not.

This result is of the same character as the theorem:

$$\int_0^\infty \frac{\sin ax}{x}\,dx = +\frac{\pi}{2}, \text{ or } 0, \text{ or } -\frac{\pi}{2},$$

according as a is >0, or $=0$, or <0; and indeed I make a frequent use of this last cited theorem in the general discussion of the foregoing integrals.

It is instructive to consider first the case of $i=\infty$, that is, the case when all the particles behind m_0 are supposed to be agitated according to the law (2), at the origin of t. This case is simpler in itself than the case of i finite, & the latter can be reduced to the former. The effect of a finite number of initial waves may be considered as the difference of the effects of two systems of such waves, each infinite in number but terminating differently.

Making $i=\infty$, I find that the integrals become

$$Y_{x,t}=\frac{\eta}{2}\left\{1-\cos\frac{2x\pi}{n}\cos\left(2at\sin\frac{\pi}{n}\right)\right\}-\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin 2x\theta\cos(2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\dfrac{\pi}{n}\right)},$$

$$Z_{x,t}=-\frac{\eta}{2}\sin\frac{2x\pi}{n}\sin\left(2at\sin\frac{\pi}{n}\right)+\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\cos 2x\theta\sin(2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\dfrac{\pi}{n}\right)},$$

$$y_{x,t}=\frac{\eta}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2at\sin\frac{\pi}{n}\right)-\frac{\eta}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi\frac{\sin(2x\theta-2at\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\dfrac{\pi}{n}\right)}.$$

As n, though large, is finite, it is permitted to suppose that at is a good deal greater than n^3; I shall suppose it to be a large multiple r^2 of $\dfrac{s^2\sin\dfrac{\pi}{n}}{\left(2\operatorname{vers}\dfrac{\pi}{n}\right)^2}$,

$$at=\frac{r^2s^2\sin\dfrac{\pi}{n}}{\left(2\operatorname{vers}\dfrac{\pi}{n}\right)^2},$$

in which s is a positive number so large that we have, sensibly,

$$\int_{-\infty}^{\infty} \cos\left(\phi \pm \frac{\psi^2}{s^2}\right) \frac{\sin\psi}{\psi} d\psi = \pi \cos\phi.$$

Then, for all values of x between the limits

$$\pm\left(at\cos\frac{\pi}{n} - \frac{s}{2}\sqrt{at\sin\frac{\pi}{n}}\right) = \pm\left\{at - \left(1+\frac{1}{r}\right)at\,\mathrm{vers}\frac{\pi}{n}\right\},$$

I find that we have, nearly, (& more and more exactly as t or r increases without limit,)

$$\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\sin 2x\theta \cos(2at\sin\theta)\, d\theta}{\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)} = \tfrac{1}{2}\sin\frac{2x\pi}{n}\sin(2at\sin\theta),$$

$$\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2 \int_0^\pi \frac{\cos 2x\theta \sin(2at\sin\theta)\, d\theta}{\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)} = \tfrac{1}{2}\cos\frac{\pi}{n} - \tfrac{1}{2}\cos\frac{2x\pi}{n}\cos(2at\sin\theta);$$

and therefore

$$Y_{x,t} = \frac{\eta}{2}\,\mathrm{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right),$$

$$Z_{x,t} = \frac{\eta}{2}\,\mathrm{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \eta\left(\sin\frac{\pi}{2n}\right)^2;$$

so that these two functions are in fact nearly equal, if n be large, that is, if the wave-length contain many molecular intervals, & if at the same time we take t sufficiently large and confine x within the limits just assigned. On the same suppositions, the displacement becomes*

$$y_{x,t} = \eta\,\mathrm{vers}\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right) - \eta\left(\sin\frac{\pi}{2n}\right)^2;$$

it differs therefore from that expressed by the law (2), only by a small and constant quantity. But the limits found for x differ sensibly from those anticipated in (2)′, & conduct to a sensibly different law of the *velocity of propagation*. In fact, on taking $x = at\cos\frac{\pi}{n}$, I find

$$y_{x,t} = \eta\left(\cos\frac{\pi}{2n}\right)^2 - \frac{\eta}{2}\cos\left(\frac{2x\pi}{n} - 2at\sin\frac{\pi}{n}\right),$$

the coefficient of the cosine of the phase being reduced to half its former amount; and this coefficient becomes insensible on taking

$$x = at\cos\frac{\pi}{n} + \frac{s}{2}\sqrt{at\sin\frac{\pi}{n}},$$

which assumption gives

$$y_{x,t} = \eta\left(\cos\frac{\pi}{2n}\right)^2.$$

This value of the displacement remains sensibly constant, till we pass to $x = at - \frac{1}{2}s'(at)^{\frac{1}{3}}$, s' being another positive number so large as to give sensibly

$$\int_{-\infty}^{\infty} \sin\left(\psi \pm \frac{\psi^3}{3s'^3}\right)\frac{d\psi}{\psi} = \pi.$$

* [See Appendix, Note 12, pp. 640, 641.]

But while x increases from $at - \frac{1}{2}s'(at)^{\frac{1}{3}}$ to $at + \frac{1}{2}s'(at)^{\frac{1}{3}}$, the displacement $y_{x,t}$ decreases from $\eta\left(\cos\frac{\pi}{2n}\right)^2$ to 0, being $= \frac{1}{3}\eta\left(\cos\frac{\pi}{2n}\right)^2$ when x is $= at$; & when x is greater than $at + \frac{1}{2}s'(at)^{\frac{1}{3}}$, the displacement $y_{x,t}$ is insensible. Now, by the suppositions which we have made, the limit conjectured in (2)′, namely

$$x = at\frac{\sin\nu}{\nu} = \frac{atn}{\pi}\sin\frac{\pi}{n},$$

is greater than $at\cos\frac{\pi}{n} + \frac{s}{2}\sqrt{at\sin\frac{\pi}{n}}$ & less than $at - \frac{1}{2}s'(at)^{\frac{1}{3}}$; we may even consider its excess over the one, & its defect below the other, as increasing nearly proportionally to t. The velocity of propagation therefore cannot be considered as $= a\frac{\sin\nu}{\nu}$; but must rather be represented by

$$a\cos\nu = a\cos\frac{\pi}{n}.$$

In short, if we were to consider $a\frac{\sin\nu}{\nu}$ as the velocity of propagation, we should be obliged to consider as the front of the wave, or rather of the wave-system, a particle m_x which is not properly *vibrating* at all but is only *displaced* by an amount $= \eta\left(\cos\frac{\pi}{2n}\right)^2$, which is common to a great and increasing number of particles beyond and behind it. But when we consider $a\cos\nu$ as the velocity, we get for the front of the wave-system a particle which is not merely displaced but vibrating, with a phase determined by the law (2), though the amplitude of its excursions is only half the amplitude which that law would assign. It is situated in the middle of an interval which does indeed continually increase, but bears continually a less and less ratio to the time t; the particles beyond which interval have no amplitude of excursion, while the particles behind it have the full amplitude required by the law (2). And whatever particle we take, throughout this interval, the ratio of its abscissa x to the time t tends ultimately to the limit

$$a\cos\nu \quad \text{or} \quad a\cos\frac{\pi}{n},$$

and *not* to the limit

$$\frac{a\sin\nu}{\nu} \quad \text{or} \quad \frac{an}{\pi}\sin\frac{\pi}{n}.$$

I think then that a distinction is completely established, at least in this particular question, between the *rapidity of progress of a given phase*, in a space *already occupied* by an *indefinite* system of waves, & the *velocity of propagation of vibration*, by which a bounded system *comes to occupy new parts of space*. The two velocities differ little, it is true, if the number of molecular intervals contained in one wave-length be great; but the difference is finite and follows a simple law, namely that of the difference between the cosine of a small arc and the ratio of the sine of the same arc to that arc itself; the latter being the old, & the former being the new result—at least, the result is entirely new to me; I am anxious to learn whether you know of its having been obtained before. Of course I am aware that it would be precarious and premature to speculate on any physical consequences of it, until it shall have been made a nearer approach to nature, by our considering, as no doubt we ought to do, the action on any one particle as being the resultant of the actions not of *two* but of *many* others.

It is remarkable that the same reasonings conduct us to consider still a third different velocity, namely that with which the *disturbance* (as distinguished from the *vibration*) is propagated. This *velocity of propagation of disturbance* is simply $=a$; it is therefore independent of the length of the wave, & is equal to the square root of the attraction exerted by any one particle on that adjacent, the distance between these two adjacent particles being taken as the unit of length.

Disturbance is also propagated *backward* with the same velocity a; in such a manner that (when t is large and positive as before) all negative values of x which are algebraically

$$> -at + \tfrac{1}{2}s'(at)^{\frac{1}{3}}$$

give
$$y_{x,t} = \eta \operatorname{vers}\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n}\right) - \eta\left(\sin\frac{\pi}{2n}\right)^2;$$

the value $x = -at$ gives

$$y_{x,t} = \eta \operatorname{vers}\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n}\right) - \tfrac{1}{3}\eta\left(\sin\frac{\pi}{2n}\right)^2;$$

and values algebraically $< -at - \tfrac{1}{2}s'(at)^{\frac{1}{3}}$ give sensibly

$$y_{x,t} = \eta \operatorname{vers}\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n}\right),$$

that is, they reproduce the law (2) without any sensible error, the effect of the discontinuity of the initial state of the medium not having yet had time to show itself.

It seems unnecessary to trouble you at present with any further details; except to mention generally that when I return to supposing the number i of the wave-lengths primitively agitated to be not infinite but only *very large*, I find results entirely analogous. Disturbance is propagated forward from the front and backward from the rere of the initial multiple wave with a velocity which may still be considered as $=a$; there is also a forward (but not a backward) propagation of vibration, & its velocity is still $=a\cos\frac{\pi}{n}$; the amplitude of the excursion is for a long time undiminished throughout the much larger part of the travelling multiple wave, of which the extent may long be considered as retaining its initial value $=in$; but however great the value of i may be, that is, however many simple waves we suppose to be in vibration at the origin of t, if only this number be finite, the progress of time will at last cause the *terminal diffusion* to encroach more and more upon the advancing i-fold wave, till its vibratory motion will have spent itself to all sense on the particles before and behind it, according to a law which I have been able to assign, and which gives, ultimately for the central particle, $\left(\text{whose abscissa is } -\frac{in}{2} + at\cos\frac{\pi}{n}\right)$, an amplitude of excursion which varies inversely as the square-root of the time.—The formulae allow us also to trace the motion backwards in *time*, so as to say what must *have been* the disturbances at any earlier epoch, in order to produce the assumed state at the moment $t=0$.

Feb. 9th.

It would be unreasonable to ask you to examine at present into the *correctness* of the foregoing results; which may indeed be affected by some mistake of transcription; but perhaps with little trouble you could give me an opinion with regard to their *novelty*, which might prevent me from claiming as my own what had been already discovered by others; or, in the contrary case, might encourage me to pursue the subject farther—I shall perhaps say something on the matter to the R.I.A. on ☽ next; in which event, it will be expected that I should allow some notice of it to be printed in the "Proceedings" or Monthly Notices.

Now, any communication from you which should reach me in about a *week from this date* might be of use in the construction of such a notice; & that without mentioning your name unless you expressly desired or allowed it.—Received Diploma & Letter for Miss H.*

W. R. H.

* [Miss Caroline Herschel. See Graves, *Life of Sir W. R. Hamilton*, II, pp. 290, 291, for Herschel's reply to this letter.]