XX.

PROPAGATION OF MOTION IN ELASTIC MEDIUM—DISCRETE MOLECULES*

[1839.]

[Note Book 53.]

[CONTENTS

[*Statement of problem.*]

(Jan. 9th, 1839.)

Let there be proposed the equation in mixed differences:

$$\frac{d^2}{dt^2}\eta_{x,t}=a^2(\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t});\qquad(1)$$

in which a is constant, real & positive. One integral of this equation is

$$\eta_{x,t}=\text{vers}\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_,}-t\right)\right\},\qquad(2)$$

λ, $a_,$ being constants, and a being connected with λ, $a_,$ by the relation

$$a^2=\left(\frac{\pi}{\lambda}\right)^2\left(\sin\frac{\pi}{\lambda a_,}\right)^{-2};\qquad(3)$$

* [For a synopsis of this paper see pp. 576, 577 and also Letter to Herschel, p. 599.]

because the equation (2) gives

$$\frac{d^2}{dt^2}\eta_{x,t}=\left(\frac{2\pi}{\lambda}\right)^2\cos\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_,}-t\right)\right\},$$

and

$$\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t}=2\operatorname{vers}\frac{2\pi}{\lambda a_,}\cos\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_,}-t\right)\right\}.$$

So that we may suppose

$$\frac{\pi}{\lambda a}=\sin\frac{\pi}{\lambda a_,}, \tag{3}'$$

and a and $a_,$ are nearly equal, if $\frac{\pi}{\lambda a}$ be small.

We may put $$\tau=ta\sqrt{2};$$

(so that if τ now represent the time, the total attraction is equal to the distance from the position of equilibrium); & then the original equation (1) becomes

$$\left(\frac{d^2}{d\tau^2}+1\right)\eta_{x,\tau}=\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau}); \tag{1}'$$

while the particular integral (2) becomes

$$\eta_{x,\tau}=\operatorname{vers}\left\{\frac{2\pi}{\lambda}\left(\frac{x}{a_,}-\frac{\tau}{a\sqrt{2}}\right)\right\}; \tag{2}'$$

the relation (3) or (3)′ still holding good. We may also put, for abridgment,

$$\mu=\frac{2\pi}{\lambda a_,},\quad m=\frac{\pi\sqrt{2}}{\lambda a}=\sqrt{\operatorname{vers}\mu}; \tag{3}''$$

& then the particular integral (2)′ becomes

$$\eta_{x,\tau}=\operatorname{vers}(\mu x-m\tau). \tag{2}''$$

That is, (1)′ has (2)″ for a particular integral, if the relation (3)″ hold good.

The particular integral (2)″ gives by differentiation

$$\eta'_{x,\tau}=\frac{d}{d\tau}\eta_{x,\tau}=-m\sin(\mu x-m\tau); \tag{2}'''$$

$$\eta''_{x,\tau}=\left(\frac{d}{d\tau}\right)^2\eta_{x,\tau}=m^2\cos(\mu x-m\tau); \tag{2}^{\text{iv}}$$

also

$$\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau})=1-\cos(\mu x-m\tau)\cos\mu=\eta_{x,\tau}+\operatorname{vers}\mu\cos(\mu x-m\tau); \tag{2}^{\text{v}}$$

& accordingly these give, by (3)″,

$$\eta''_{x,\tau}+\eta_{x,\tau}=\tfrac{1}{2}(\eta_{x+1,\tau}+\eta_{x-1,\tau}). \tag{1}''$$

Now let it be supposed that when $\tau=0$ the values of $\eta_{x,\tau}$ & $\frac{d}{d\tau}\eta_{x,\tau}$ vanish for $x=0$ & for all positive values of x, & are consistent with the equations (2)″ (2)‴ for all negative values of x; so that we have

$$\eta_{x,0}=0,\quad \eta'_{x,0}=0,\quad \text{if } x \not< 0, \tag{4}_1$$

but

$$\eta_{x,0}=\operatorname{vers}\mu x,\quad \eta'_{x,0}=-m\sin\mu x,\quad \text{if } x \not> 0; \tag{4}_2$$

and let it be required to determine generally the function $\eta_{x,\tau}$, x being at present restricted to integer values.

This question corresponds to the propagation of vibration in an elastic medium, & must illustrate the spreading of such vibration from one part of the medium to another, while the said vibration remains, or may remain, (approximately) confined between 2 parallel and indefinite planes which move (& why move? Sept. 30th/57) with one common velocity. It ought to turn out that if the velocity of propagation $\frac{m}{\mu}$ is positive (τ here being taken to represent the time), so that $\eta'_{x,0}$ is positive for small negative values of x, then $\eta_{x,\tau}$ will take sensibly the value (2)″ as soon as $\frac{m\tau}{\mu}$ is sensibly greater than x; but that if $\frac{m}{\mu}$ be supposed negative, then $\eta_{x,\tau}$ will sensibly vanish when $-\frac{m\tau}{\mu}$ is sensibly $>-x$, the value of $\eta'_{x,0}$ being in this case negative for small negative values of x. And the *physical explanation* is doubtless of this kind: μ being in both cases supposed for simplicity to be such that $\frac{\pi}{2\mu}$ is a large positive integer number, $\nu=\frac{\pi}{2\mu}=\frac{\lambda a}{4}$, we have in both cases positive displacements $\eta_{x,0}$ for all the values $x=-1, x=-2, \ldots x=-(4\nu-1)$, the greatest being that which corresponds to $x=-2\nu=-\frac{\pi}{\mu}$; but in the first case, in which m is >0, the displacements corresponding to $x=-1, x=-2, \ldots x=-(2\nu-1)$ are all *increasing*, while in the 2nd case, in which $m<0$, the same displacements are all *decreasing*, at the same original moment $\tau=0$; it is clear then that in the first case the system is departing more and more from equilibrium, while in the 2nd case it is approaching more and more to equilibrium, at the moment $\tau=0$, & within the extent included between the limits $x=0$ and $x=-2\nu=-\frac{\pi}{\mu}$; and that therefore the originally quiescent part on the positive side of the limit $x=0$ must be much more disturbed in the first case than in the second. But it is interesting to investigate the law & quantity of this inequality of disturbance, by resolving, at least approximately, the algebraic problem proposed in the preceding page.

Perhaps the *theorem of Fourier** might help to resolve this problem; but I prefer at present to proceed in a way of my own, as follows.

Let $$y_{-x,\tau}=\eta_{x,\tau}-\text{vers}(\mu x-m\tau); \tag{5}$$

* [Hamilton appends the following note. "Fourier's theorem is that

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\cos(\alpha x-\alpha x')f(x')\,d\alpha\right)dx',$$

or, as it may be better stated,

$$f(x)=\frac{1}{2\pi}\underset{h=0}{\Lambda}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x-\alpha x')\,d\alpha\right)f(x')\,dx'.$$

In fact, as I have remarked in an old book (Oct. 1835), we have

$$\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x-\alpha x')\,d\alpha=\sqrt{\frac{\pi}{h}}\,e^{-\frac{(x'-x)^2}{4h}};$$

$\because$ multiplying this by $f(x')\,dx'$ and changing x' to $x+2t\sqrt{h}$, we have

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-h\alpha^2}\cos(\alpha x-\alpha x')\,d\alpha\right)f(x')\,dx'=2\sqrt{\pi}\int_{-\infty}^{\infty}e^{-t^2}f(x+2t\sqrt{h})\,dt,$$

of which the limit relatively to $h=0$ is $2\pi f(x)$."]

& let us introduce, instead of the old system of functions $\eta_{-n,\tau}$, $\eta_{n-1,\tau}$ (n being >0), a new system, $y_{n,\tau}$, $\eta_{n-1,\tau}$, of which each vanishes, as also does its 1st differential coefficient relatively to τ, at the moment $\tau=0$. We shall then have

$$y''_{-x,\tau} + y_{-x,\tau} = \tfrac{1}{2}y_{-x-1,\tau} + \tfrac{1}{2}y_{-x+1,\tau}; \qquad (1)'''$$

$$\therefore \begin{cases} y''_{n+1,\tau} + y_{n+1,\tau} = \tfrac{1}{2}(y_{n+2,\tau} + y_{n,\tau}); & (1)'''_1 \\ y''_{1,\tau} + y_{1,\tau} = \tfrac{1}{2}(y_{2,\tau} + \eta_{0,\tau} - \operatorname{vers} m\tau); & (1)'''_2 \\ \eta''_{0,\tau} + \eta_{0,\tau} = \tfrac{1}{2}(\eta_{1,\tau} + y_{1,\tau} + \operatorname{vers}\overline{\mu + m\tau}); & (1)'''_3 \\ \eta''_{n,\tau} + \eta_{n,\tau} = \tfrac{1}{2}(\eta_{n+1,\tau} + \eta_{n-1,\tau}); & (1)'''_4 \end{cases}$$

n being, in each of these 4 last formulae, an integer >0. Also

$$0 = y_{n,0} = y'_{n,0} = \eta_{n-1,0} = \eta'_{n-1,0}; \quad n \text{ being still } > 0. \qquad (4)'$$

We have also
$$\operatorname{vers}(\mu + m\tau) = 1 - \cos\mu\cos m\tau + \sin\mu\sin m\tau$$
$$= 1 - (1-m^2)\cos m\tau + m\sqrt{2-m^2}\sin m\tau; \qquad (3)'''$$

so that m may be considered as the only arbitrary constant of the problem & may be supposed to be small and positive: attending only at first to the case of forward propagation.

[*Properties of operator* ∇_τ.]

Let the notation $$\nabla_\tau f_\tau = \phi_\tau \qquad (6)$$
imply that $$\phi''_\tau + \phi_\tau = f_\tau; \qquad (6)'$$
& that $$\phi_0 = 0, \quad \phi'_0 = 0. \qquad (6)'', (6)'''$$
Then if we put $$\tfrac{1}{2}\{1-(1-m^2)\cos m\tau\} = \alpha_\tau, \qquad (7)$$
we shall have $$\tfrac{1}{2}\operatorname{vers} m\tau = \nabla_\tau\alpha_\tau; \qquad (7)'$$
& putting also $$\tfrac{1}{2}m\sqrt{2-m^2}\sin m\tau = \beta_\tau, \qquad (7)''$$
we have the equations

$$\begin{cases} y_{n+1,\tau} = \tfrac{1}{2}\nabla_\tau(y_{n+2,\tau} + y_{n,\tau}); & (8)_1 \\ y_{1,\tau} = \tfrac{1}{2}\nabla_\tau(y_{2,\tau} + \eta_{0,\tau}) - \nabla^2_\tau\alpha_\tau; & (8)_2 \\ \eta_{0,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{1,\tau} + y_{1,\tau}) + \nabla_\tau(\alpha_\tau + \beta_\tau); & (8)_3 \\ \eta_{n,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n+1,\tau} + \eta_{n-1,\tau}); & (8)_4 \end{cases}$$

n being still an integer >0. Make
$$\eta_{0,\tau} - \nabla_\tau\alpha_\tau = H_\tau; \qquad (9)$$
then
$$\begin{cases} y_{n+1,\tau} = \tfrac{1}{2}\nabla_\tau(y_{n+2,\tau} + y_{n,\tau}); \quad \eta_{n+1,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n+2,\tau} + \eta_{n,\tau}); & (9)_1, (9)_5 \\ y_{1,\tau} = \tfrac{1}{2}\nabla_\tau(y_{2,\tau} + H_\tau - \nabla_\tau\alpha_\tau); \quad \eta_{1,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{2,\tau} + H_\tau + \nabla_\tau\alpha_\tau); & (9)_2, (9)_4 \\ H_\tau = \tfrac{1}{2}\nabla_\tau(\eta_{1,\tau} + y_{1,\tau} + 2\beta_\tau); & (9)_3 \end{cases}$$

& these would give*
$$\left.\begin{matrix} H_\tau = 0, \quad y_{n,\tau} = -\eta_{n,\tau}, \\ \eta_{0,\tau} = \nabla_\tau\alpha_\tau = \tfrac{1}{2}\operatorname{vers} m\tau, \end{matrix}\right\} \text{if } \beta_\tau \text{ were } = 0. \qquad (10)$$

* [If the term β_τ is absent, H_τ, $y_{1,\tau}+\eta_{1,\tau}$, $y_{2,\tau}+\eta_{2,\tau}$, ... represent the transverse displacements of points P_1, P_2, P_3, ... of a stretched string having equal masses at P_1, P_2, P_3, ... and $P_0P_1 = P_1P_2 =$ etc. If P_0 is always fixed and if the others are initially at rest in the equilibrium position, then there are no solutions except

$$H_\tau = 0, \; y_{1,\tau} + \eta_{1,\tau} = 0, \text{ etc.}]$$

It results from the investigations of Oct. 1835, that the same supposition ($\beta_\tau = 0$) would give generally*

$$-y_{n,\tau} = \eta_{n,\tau} = \frac{n}{2^n} \Sigma_{(k)0}^{\infty} \frac{[n+2k-1]^{k-1}}{[k]^k} \frac{\nabla_\tau^{n+2k+1}\alpha_\tau}{2^{2k}}, \qquad (10)'$$

the square brackets denoting factorials; so that, more fully,

$$-y_{n,\tau} = \eta_{n,\tau} = 2^{-n}\left\{\nabla_\tau^{n+1}\alpha_\tau + n\,2^{-2}\nabla_\tau^{n+3}\alpha_\tau + \frac{n(n+3)}{1.2}2^{-4}\nabla_\tau^{n+5}\alpha_\tau \right.$$

$$\left. + \frac{n(n+4)(n+5)}{1.2.3}2^{-6}\nabla_\tau^{n+7}\alpha_\tau + \frac{n(n+5)(n+6)(n+7)}{1.2.3.4}2^{-8}\nabla_\tau^{n+9}\alpha_\tau + \&c.\right\}, \qquad (10)''$$

& in particular (developing as far as $\nabla_\tau^{10}\alpha_\tau$ inclusive)

$$\left\{\begin{aligned}
-y_{1,\tau} &= \eta_{1,\tau} = \tfrac{1}{2}\nabla_\tau^2\alpha_\tau + \tfrac{1}{8}\nabla_\tau^4\alpha_\tau + \tfrac{1}{16}\nabla_\tau^6\alpha_\tau + \tfrac{5}{128}\nabla_\tau^8\alpha_\tau + \tfrac{7}{256}\nabla_\tau^{10}\alpha_\tau + \&c.; && (10)_1''\\
-y_{2,\tau} &= \eta_{2,\tau} = \tfrac{1}{4}\nabla_\tau^3\alpha_\tau + \tfrac{1}{8}\nabla_\tau^5\alpha_\tau + \tfrac{5}{64}\nabla_\tau^7\alpha_\tau + \tfrac{7}{128}\nabla_\tau^9\alpha_\tau + \&c.; && (10)_2''\\
-y_{3,\tau} &= \eta_{3,\tau} = \tfrac{1}{8}\nabla_\tau^4\alpha_\tau + \tfrac{3}{32}\nabla_\tau^6\alpha_\tau + \tfrac{9}{128}\nabla_\tau^8\alpha_\tau + \tfrac{7}{128}\nabla_\tau^{10}\alpha_\tau + \&c.; && (10)_3''\\
-y_{4,\tau} &= \eta_{4,\tau} = \tfrac{1}{16}\nabla_\tau^5\alpha_\tau + \tfrac{1}{16}\nabla_\tau^7\alpha_\tau + \tfrac{7}{128}\nabla_\tau^9\alpha_\tau + \&c.; && (10)_4''\\
-y_{5,\tau} &= \eta_{5,\tau} = \tfrac{1}{32}\nabla_\tau^6\alpha_\tau + \tfrac{5}{128}\nabla_\tau^8\alpha_\tau + \tfrac{5}{128}\nabla_\tau^{10}\alpha_\tau + \&c.; && (10)_5''\\
-y_{6,\tau} &= \eta_{6,\tau} = \tfrac{1}{64}\nabla_\tau^7\alpha_\tau + \tfrac{3}{128}\nabla_\tau^9\alpha_\tau + \&c.; && (10)_6''\\
-y_{7,\tau} &= \eta_{7,\tau} = \tfrac{1}{128}\nabla_\tau^8\alpha_\tau + \tfrac{7}{512}\nabla_\tau^{10}\alpha_\tau + \&c.; && (10)_7''\\
-y_{8,\tau} &= \eta_{8,\tau} = \tfrac{1}{256}\nabla_\tau^9\alpha_\tau + \&c.; && (10)_8''\\
-y_{9,\tau} &= \eta_{9,\tau} = \tfrac{1}{512}\nabla_\tau^{10}\alpha_\tau + \&c. && (10)_9''
\end{aligned}\right.$$

In fact these developments verify, to the same degree of accuracy, the conditions

$$\eta_{1,\tau} - \tfrac{1}{2}\nabla_\tau\eta_{2,\tau} = \tfrac{1}{2}\nabla_\tau^2\alpha_\tau; \quad \eta_{2,\tau} - \tfrac{1}{2}\nabla_\tau(\eta_{1,\tau} + \eta_{3,\tau}) = 0; \quad \eta_{3,\tau} - \tfrac{1}{2}\nabla_\tau(\eta_{2,\tau} + \eta_{4,\tau}) = 0; \quad \&c.$$

And, taking the whole series $(10)'$, we have

$$\eta_{1,\tau} = \Sigma_{(k)0}^{\infty}\frac{[2k]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+2}\alpha_\tau}{2^{2k+1}} = \tfrac{1}{2}\nabla_\tau^2\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{[2k+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+4}\alpha_\tau}{2^{2k+2}}; \qquad (10)_1'$$

$$\eta_{2,\tau} = 2\,\Sigma_{(k)0}^{\infty}\frac{[2k+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+3}\alpha_\tau}{2^{2k+2}} = \tfrac{1}{4}\nabla_\tau^3\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{[2k+3]^k}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+5}\alpha_\tau}{2^{2k+3}}; \qquad (10)_2'$$

$$\eta_{3,\tau} = 3\,\Sigma_{k(0)}^{\infty}\frac{[2k+2]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+4}\alpha_\tau}{2^{2k+3}} = \qquad ; \qquad (10)_3'$$

$$\eta_{n+1,\tau} = (n+1)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+n+2}\alpha_\tau}{2^{2k+n+1}}$$

$$= \frac{1}{2^{n+1}}\nabla_\tau^{n+2}\alpha_\tau + (n+1)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+2]^k}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+n+4}\alpha_\tau}{2^{2k+n+3}}; \qquad (10)_{n+1}'$$

$$\eta_{n+2,\tau} = (n+2)\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+1]^{k-1}}{[k]^k}\frac{\nabla_\tau^{2k+n+3}\alpha_\tau}{2^{2k+n+2}}; \qquad (10)_{n+2}'$$

$$\eta_{n,\tau} = \frac{1}{2^n}\nabla_\tau^{n+1}\alpha_\tau + n\,\Sigma_{(k)0}^{\infty}\frac{[2k+n+1]^k}{[k+1]^{k+1}}\frac{\nabla_\tau^{2k+n+3}\alpha_\tau}{2^{2k+n+2}}; \qquad (10)_n'$$

and since

$$n[2k+n+1]^k + (n+2)[2k+n+1]^{k-1}(k+1) = \{n(k+n+2) + (n+2)(k+1)\}[2k+n+1]^{k-1}$$

$$= (n+1)(n+2k+2)[2k+n+1]^{k-1} = (n+1)[2k+n+2]^k,$$

we have, as we ought,

$$\eta_{n+1,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n,\tau} + \eta_{n+1,\tau}); \quad \eta_{1,\tau} = \tfrac{1}{2}\nabla_\tau(\nabla_\tau\alpha_\tau + \eta_{2,\tau}).$$

* [See top of page 532.]

[Here follow some investigations concerning finite expressions for series like (10)′. In some pencilled notes added later Hamilton points out that (10)′ is equal to $(1-\sqrt{1-\nabla_\tau^2})^n \nabla_\tau^{-n+1}\alpha_\tau$. It is easy to see that this expression satisfies $(8)_4$ and, when $n=0$, is equal to $\nabla_\tau \alpha_\tau$. Moreover from the definition of the operator ∇_τ, since (10)′ contains only positive powers of ∇_τ, $\eta_{n,\tau}$ and $\eta'_{n,\tau}$ are zero when $\tau=0$.]

(Jan. 10th, 1839.)

In the next place let us consider the system of equations

$$\left.\begin{aligned} \alpha_\tau = 0, \quad y_{n,\tau} = \eta_{n,\tau} = \tfrac{1}{2}\nabla_\tau(\eta_{n+1,\tau}+\eta_{n-1,\tau}), \qquad (n>0), \\ \eta_{0,\tau} = \nabla_\tau \eta_{1,\tau} + \nabla_\tau \beta_\tau. \end{aligned}\right\} \tag{11}$$

...[Here $\eta_{n,\tau}$ is obtained by ordinary methods. Its value however could be inferred from (10)′]....

Hence

$$\eta_{n,\tau} = 2^{-n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}[n+2k]^k[0]^{-k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau.$$

Hence the equations (8) give, when both α_τ and β_τ are retained, the expressions

$$\eta_{0,\tau} = \nabla_\tau\alpha_\tau + \Sigma_{(k)0}^{\infty}\frac{1.3.5\ldots(2k-1)}{2.4.6\ldots 2k}\nabla_\tau^{2k+1}\beta_\tau; \tag{12$_1$}$$

$$\eta_{n,\tau} = \frac{n}{2^n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}\frac{[n+2k-1]^{k-1}}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\left(\beta_\tau+\alpha_\tau+\frac{2k}{n}\beta_\tau\right); \tag{12$_2$}$$

$$y_{n,\tau} = \frac{n}{2^n}\nabla_\tau^{n+1}\Sigma_{(k)0}^{\infty}\frac{[n+2k-1]^{k-1}}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\left(\beta_\tau-\alpha_\tau+\frac{2k}{n}\beta_\tau\right). \tag{12$_3$}$$

The expression for $\eta_{0,\tau}$ may be concisely represented thus:

$$\eta_{0,\tau} = \nabla_\tau\alpha_\tau + \nabla_\tau(1-\nabla_\tau^2)^{-\frac{1}{2}}\beta_\tau = \nabla_\tau\alpha_\tau + (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau. \tag{13}$$

But

$$\nabla_\tau = \left\{1+\left(\frac{d}{d\tau}\right)^2\right\}^{-1};$$

$$\therefore \quad \nabla_\tau^{-2}-1 = \left\{1+\left(\frac{d}{d\tau}\right)^2\right\}^2 - 1 = \left(\frac{d}{d\tau}\right)^2\left\{2+\left(\frac{d}{d\tau}\right)^2\right\};$$

$$\therefore \quad (\nabla_\tau^{-2}-1)^{-\frac{1}{2}} = \left(\frac{d}{d\tau}\right)^{-1}\left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}} = \int_0^\tau\left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}}d\tau. \tag{14}$$

We have

$$\eta_{0,\tau} = \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau\left\{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}}\sin m\tau\, d\tau$$

$$= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau\Sigma_{(k)0}^{\infty}\frac{[-\frac{1}{2}]^k 2^{-k}}{[k]^k}\left(\frac{d}{d\tau}\right)^{2k}\sin m\tau\, d\tau$$

$$= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}\sqrt{1-\tfrac{1}{2}m^2}\int_0^\tau\Sigma_{(k)0}^{\infty}\frac{[-\frac{1}{2}]^k}{[k]^k}\left(-\frac{m^2}{2}\right)^k\sin m\tau\, d\tau$$

$$= \tfrac{1}{2}\operatorname{vers} m\tau + \frac{m}{2}(1-\tfrac{1}{2}m^2)^{\frac{1}{2}}(1-\tfrac{1}{2}m^2)^{-\frac{1}{2}}\int_0^\tau\sin m\tau\, d\tau = \operatorname{vers} m\tau. \tag{15}$$

We seem then to be conducted to the equation

$$\eta_{0,\tau} = \text{vers}\, m\tau \qquad (15)_1$$

as the exact sum of the series $(12)_1$. On the other hand we should have had exactly

$$\eta_{0,\tau} = 0, \qquad (15)_2$$

if we had changed the sign of the radical $\sqrt{2-m^2}$ in β_τ.

Perhaps also we should be able to prove by some similar reasoning that the part depending on β_τ should be considered as negative for negative values of τ, & ∵ that for such values $\eta_{0,\tau}$ vanishes, the radical $\sqrt{2-m^2}$ or the ratio m/μ, which expresses the velocity of propagation, being still taken with the positive sign.

On the whole it is perhaps possible to prove by the foregoing investigation or by some one very similar that $\eta_{0,\tau}$ may become exactly $=\text{vers}\, m\tau$ for all positive values of τ after having been exactly $=0$ for all negative values thereof. And $\eta''_{0,0}$ is the mean of the two values 0 and m^2. Can we apply a similar process to the expression for $\eta_{n,\tau}$?

$$\left[\nabla_\tau \textit{ and } \frac{d}{d\tau} \textit{ not commutative.}\right]$$

The part depending on β_τ in $\eta_{1,\tau}$ is

$$\tfrac{1}{2}\nabla_\tau^2 \Sigma_{(k)\,0}^{\infty} \frac{[2k+1]^k}{[k]^k} (\tfrac{1}{2}\nabla_\tau)^{2k} \beta_\tau.$$

Here

$$[2k+1]^k = (2k+1)\,2k\,(2k-1)\dots(k+2) = \frac{[2k+1]^{2k+1}}{[k+1]^{k+1}}$$

$$= \frac{2^k (2k+1)(2k-1)\dots 3\,.\,1}{k+1} = \frac{2^{2k+1}(-1)^{k+1}[-\frac{1}{2}]^{k+1}}{k+1};$$

∵ the part depending on β_τ in $\eta_{1,\tau}$ is

$$\Sigma_{(k)0}^{\infty} \frac{[-\frac{1}{2}]^{k+1}}{[k+1]^{k+1}} (-\nabla_\tau^2)^{k+1} \beta_\tau = \{-1 + (1-\nabla_\tau^2)^{-\frac{1}{2}}\}\beta_\tau;$$

which accordingly agrees with the equation (11),

$$\eta_{0,\tau} = \nabla_\tau(\eta_{1,\tau} + \beta_\tau),$$

α_τ being supposed to vanish. This equation gives also

$$\eta_{1,\tau} = -\beta_\tau + \eta''_{0,\tau} + \eta_{0,\tau},$$

in which $\eta_{0,\tau}$ is to be made only $\frac{1}{2}\text{vers}\, m\tau$, because we have supposed α_τ to vanish; thus

$$\eta''_{0,\tau} + \eta_{0,\tau} = \frac{1}{2} - \frac{1-m^2}{2}\cos m\tau = \frac{1-\cos\mu\cos m\tau}{2}; \quad \& \quad -\beta_\tau = -\frac{\sin\mu\sin m\tau}{2};$$

it seems ∵ that $\eta_{1,\tau} = \dfrac{\text{vers}\,(m\tau-\mu)}{2}$, so far as it depends on β_τ alone. Yet this value of $\eta_{1,\tau}$ does not vanish with τ, though it ought to contain ∇_τ as a factor. Perhaps the development of $\eta_{1,\tau}$ diverges when $\tau < \mu/m$. But we must observe that the equation $\eta''_{0,\tau} + \eta_{0,\tau} = \eta_{1,\tau} + \beta_\tau$ requires that $\eta''_{0,0}$ should vanish, which the foregoing formulae do not make it do.

We can certainly conceive that $\eta_{0,\tau}$, $\eta_{n,\tau}$ are functions such that $\eta_{0,0}=\eta_{n,0}=\eta'_{0,0}=\eta'_{n,0}=0$, and that

$$\eta''_{0,\tau}+\eta_{0,\tau}-\eta_{1,\tau}=\beta_\tau=\tfrac{1}{2}\sin\mu\sin m\tau,$$

$$\eta''_{n,\tau}+\eta_{n,\tau}=\tfrac{1}{2}(\eta_{n+1,\tau}+\eta_{n-1,\tau});\quad (m^2=\text{vers}\,\mu,\ n>0);$$

but will these functions $\eta_{0,\tau}$, $\eta_{n,\tau}$ be convergently expressed by the series

$$\eta_{0,\tau}=\nabla_\tau\Sigma^\infty_{(k)\,0}\frac{[2k]^k}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau,$$

$$\eta_{n,\tau}=\frac{\nabla^{n+1}_\tau}{2^n}\Sigma^\infty_{(k)\,0}\frac{[n+2k]^k}{[k]^k}(\tfrac{1}{2}\nabla_\tau)^{2k}\beta_\tau\,?$$

And when we have found thus that

$$\eta''_{0,\tau}+\eta_{0,\tau}=(1-\nabla^2_\tau)^{-\frac{1}{2}}\beta_\tau,$$

can we effect this operation $(1-\nabla^2_\tau)^{-\frac{1}{2}}$?

We just now made $\nabla_\tau(1-\nabla^2_\tau)^{-\frac{1}{2}}C_\tau=(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}C_\tau$ & transformed this to

$$\int_0^\tau\left\{2+\left(\frac{d}{d\tau}\right)^2\right\}^{-\frac{1}{2}}C_\tau d\tau=\frac{1}{\sqrt{2}}\int_0^\tau\frac{C_\tau d\tau}{\sqrt{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}};$$

but it is an obvious objection to this transformation that the $\frac{d}{d\tau}$ of the last expression does not *in general* vanish with τ. However this objection does not apply in the case where $C_\tau=\beta_\tau=\frac{1}{2}\sin\mu\sin m\tau$, because $C_0^{(2k+1)}=0$; but on the other hand we ought to have

$$(1-\nabla^2_\tau)^{-\frac{1}{2}}C_\tau=\frac{1}{\sqrt{2}}\left\{1+\left(\frac{d}{d\tau}\right)^2\right\}\int_0^\tau\frac{C_\tau d\tau}{\sqrt{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}},$$

& the $\frac{d}{d\tau}$ of $\frac{\beta_\tau}{\sqrt{1+\frac{1}{2}\left(\frac{d}{d\tau}\right)^2}}$ does not vanish with τ.

The transformation therefore is erroneous, and although its elegance appears to point to some true process, intimately connected with it,* yet I believe it may be better to employ for the present a different method & one more analogous to that of my old investigations.

Let us then admit that, (β_τ being still $=\frac{1}{2}\sin\mu\sin m\tau$,)

$$\eta_{0,\tau}=\nabla_\tau\beta_\tau+\tfrac{1}{2}\nabla^3_\tau\beta_\tau+\tfrac{3}{8}\nabla^5_\tau\beta_\tau+\&c.$$

and endeavour to calculate finite trigonometrical expressions for at least some of the terms of this development. For this purpose we are to calculate if possible a general expression for $\nabla^k_\tau\sin m\tau$ or at least expressions for this function, for several moderate values of k.

....

* [The error arises from the fact that the operations $\frac{d}{d\tau}$ and ∇_τ are not commutative. Correct results are arrived at later because $\int_0^\tau d\tau$ and ∇_τ *are* commutative.]

If we were to stop at this degree of approximation, [the first three terms of the last expression for $\eta_{0,\tau}$,] we should have

$$\nabla_\tau \sin m\tau = \frac{\sin m\tau - m\sin\tau}{1-m^2},$$

$$\tfrac{1}{2}\nabla_\tau^3 \sin m\tau = \frac{\sin m\tau - m\sin\tau}{2(1-m^2)^3} - \frac{m(\sin\tau - \tau\cos\tau)}{4(1-m^2)^2} - \frac{m\{(3-\tau^2)\sin\tau - 3\tau\cos\tau\}}{16(1-m^2)},$$

$$\tfrac{3}{8}\nabla_\tau^5 \sin m\tau = \frac{3(\sin m\tau - m\sin\tau)}{8(1-m^2)^5} - \frac{3m(\sin\tau - \tau\cos\tau)}{16(1-m^2)^4} - \frac{3m\{(3-\tau^2)\sin\tau - 3\tau\cos\tau\}}{64(1-m^2)^3}$$
$$- \frac{m\{(15-6\tau^2)\sin\tau - (15-\tau^2)\tau\cos\tau\}}{128(1-m^2)^2}$$
$$- \frac{m\{(105-45\tau^2+\tau^4)\sin\tau - (105-10\tau^2)\tau\cos\tau\}}{1024(1-m^2)};*$$

which 3 expressions indeed are rigorous: & the sum of these three expressions, multiplied by $\frac{1}{2}m\sqrt{2-m^2}$, will give an *accurate* expression for $(\nabla_\tau + \frac{1}{2}\nabla_\tau^3 + \frac{3}{8}\nabla_\tau^5)\beta_\tau$, & ∵ (probably) an *approximate* expression for that part of $\eta_{0,\tau}$ which depends on β_τ, & which is to be added to $\frac{1}{2}\operatorname{vers} m\tau$.

[Neglecting powers of m above the square, Hamilton works out numerically the value of this expression for

$$\tau = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ 4,\ 5;$$

e.g. for $\tau = 5$ he finds that $\eta_{0,5} = \frac{17{,}5169}{17{,}6776}\operatorname{vers} 5m = \left(1-\frac{1}{110}\right)\operatorname{vers} 5m$ approximately.]

$$[(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^i.]$$

(Jan. 12th, 1839.)

We found that $$\eta_{0,\tau} = \nabla_\tau \alpha_\tau + (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau,$$

in which $\nabla_\tau \alpha_\tau = \frac{1}{2}\operatorname{vers} m\tau$ and $\beta_\tau = \frac{1}{2}m\sqrt{2-m^2}\sin m\tau$.

Now the operation $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}$ may be transformed in the following way, which occurred to me late last night,

$$\nabla_\tau = \left\{1+\left(\frac{d}{d\tau}\right)^2\right\}^{-1} = \left(\int_0^\tau d\tau\right)^2\left\{1+\left(\int_0^\tau d\tau\right)^2\right\}^{-1}$$
$$= \left(\int_0^\tau d\tau\right)^2 - \left(\int_0^\tau d\tau\right)^4 + \left(\int_0^\tau d\tau\right)^6 - \&c.$$

For example

$$\nabla_\tau \tau = \frac{\tau^3}{6} - \frac{\tau^5}{120} + \&c. = \tau - \sin\tau; \quad \nabla_\tau 1 = \frac{\tau^2}{2} - \frac{\tau^4}{24} + \&c. = \operatorname{vers}\tau;$$

$$\nabla_\tau \tau^2 = \frac{\tau^4}{4.3} - \frac{\tau^6}{6.5.4.3} + \frac{\tau^8}{8.7.6.5.4.3} - \&c. = 2\left(\cos\tau - 1 + \frac{\tau^2}{2}\right);$$

$$\nabla_\tau \tau^3 = \frac{\tau^5}{5.4} - \frac{\tau^7}{7.6.5.4} + \frac{\tau^9}{9.8.7.6.5.4} - \&c. = 6\left(\sin\tau - \tau + \frac{\tau^3}{6}\right);$$

* [The law of formation is obvious, e.g.

$$(105-45\tau^2+\tau^4)\sin\tau - (105-10\tau^2)\tau\cos\tau = \tau^8\left(\frac{d}{d\tau}\frac{1}{\tau}\right)^4\sin\tau.]$$

& generally

$$\nabla_\tau \tau^{2k} = \frac{\tau^{2k+2}}{(2k+2)(2k+1)} - \frac{\tau^{2k+4}}{[2k+4]^4} + \&c. = [2k]^{2k}(-1)^k\left(1 - \frac{\tau^2}{2} + \frac{\tau^4}{24} - \ldots + \frac{(-\tau^2)^k}{[2k]^{2k}} - \cos\tau\right);$$

$$\nabla_\tau \tau^{2k+1} = [2k+1]^{2k+1}(-1)^k\left(\tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \ldots + \frac{(-1)^k\tau^{2k+1}}{[2k+1]^{2k+1}} - \sin\tau\right);$$

$$\nabla_\tau \sin m\tau = m(\tau - \sin\tau) + m^3\left(\tau - \frac{\tau^3}{6} - \sin\tau\right) + m^5\left(\tau - \frac{\tau^3}{6} + \frac{\tau^5}{120} - \sin\tau\right) + \&c.$$

$$= \frac{m}{1-m^2}(\tau - \sin\tau) - \frac{m^3\tau^3}{6(1-m^2)} + \frac{m^5\tau^5}{120(1-m^2)} - \&c. = \frac{\sin m\tau - m\sin\tau}{1-m^2};$$

$$\nabla_\tau \cos m\tau = 1 - \cos\tau + m^2\left(1 - \frac{\tau^2}{2} - \cos\tau\right) + m^4\left(1 - \frac{\tau^2}{2} + \frac{\tau^4}{24} - \cos\tau\right) + \&c. = \frac{\cos m\tau - \cos\tau}{1-m^2}.$$

Hence

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}} = \left(\int_0^\tau d\tau\right)^2\left\{\left(1 + \left(\int_0^\tau d\tau\right)^2\right)^2 - \left(\int_0^\tau d\tau\right)^4\right\}^{-\frac{1}{2}} = \left(\int_0^\tau d\tau\right)^2\left\{1 + 2\left(\int_0^\tau d\tau\right)^2\right\}^{-\frac{1}{2}}$$

$$= \Sigma_{(k)\,0}^{\infty}[-\tfrac{1}{2}]^k[0]^{-k}2^k\left(\int_0^\tau d\tau\right)^{2k+2} = \Sigma_{(k)\,0}^{\infty}\frac{1.3.5\ldots(2k-1)}{1.2.3\ldots k}(-1)^k\left(\int_0^\tau d\tau\right)^{2k+2}.$$

Hence

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau^i = [i]^i\,\Sigma_{(k)\,0}^{\infty}[-\tfrac{1}{2}]^k[0]^{-k}2^k[0]^{-(2k+2+i)}\tau^{2k+2+i}$$

$$= [i]^i\left(\int_0^\tau d\tau\right)^{i+2}\Sigma_{(k)\,0}^{\infty}([0]^{-k})^2\left(-\frac{\tau^2}{2}\right)^k = \ldots$$

$$= \frac{2}{\pi}[i]^i\left(\int_0^\tau d\tau\right)^{i+2}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta)$$

and we may change the order of the signs of integration.

The expression which I deduced (in pencil, late last night,) from a result in Poisson's Memoir on Waves was

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{2}{3\pi}\left(\int_0^\tau d\tau\right)^3\int_0^1 da\,(1-a^2)^{\frac{3}{2}}\{(8 - 2a^2\tau^2)\cos(a\tau\sqrt{2}) - 7a\tau\sqrt{2}\sin(a\tau\sqrt{2})\}$$

$$= \frac{2}{3\pi}\left(\int_0^\tau d\tau\right)^3\int_0^{\frac{\pi}{2}} d\theta\sin\theta^4\{(8 - 2\tau^2\cos\theta^2)\cos(\tau\sqrt{2}\cos\theta) - 7\tau\sqrt{2}\cos\theta\sin(\tau\sqrt{2}\cos\theta)\}.$$

Here also I deduced

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{1}{3\pi\sqrt{2}}\int_0^1 da\,\frac{(1-a^2)^{\frac{3}{2}}}{a^3}\{(1 + 2a^2\tau^2)\sin(a\tau\sqrt{2}) - a\tau\sqrt{2}\cos(a\tau\sqrt{2})\}$$

$$= \frac{\tau^2\sqrt{2}}{3\pi}\int_0^{\tau\sqrt{2}}\frac{dx}{x^3}\left(1 - \frac{x^2}{2\tau^2}\right)^{\frac{3}{2}}\{(1 + x^2)\sin x - x\cos x\}\,dx.$$

This gives approximately, when τ is large,

$$(\nabla_\tau^{-2} - 1)^{-\frac{1}{2}}\tau = \frac{\tau^2\sqrt{2}}{3\pi}\int_0^\infty\frac{dx}{x^3}\{(1 + x^2)\sin x - x\cos x\} = \frac{\tau^2}{2\sqrt{2}};$$

because

$$\int_0^\infty\frac{\sin x}{x}\,dx = \frac{\pi}{2}, \quad\text{and}\quad \int_0^\infty\frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}.$$

[Extract from pencil note.] (Poisson has shown in his Memoir on Waves, pages 168, 169, that

$$\frac{8}{3\pi}\int_0^1 (1-a^2)^{\frac{3}{2}} \cos(2a\sqrt{p})\, da = \frac{8P}{3\pi} = \frac{1}{1.2} - \frac{p}{2.3} + \frac{p^2}{3.4.(1.2)^2} - \frac{p^3}{4.5.(1.2.3)^2} + \text{\&c.}$$

Hence

$$\frac{8}{3\pi}\left(\frac{d}{dp}\right)^2 (Pp^2) = 1 - p + \frac{p^2}{(1.2)^2} - \frac{p^3}{(1.2.3)^2} + \text{\&c.}$$

In this change p to $\frac{1}{2}\tau^2$, & perform the operation $\int_0^\tau d\tau$ three times; and we shall have my

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau.)$$

To use now the function given at the middle of the previous page, it is necessary to have an expression for $\left(\int_0^x dx\right)^r \cos x$. This

$$= \left(\int_0^x dx\right)^r \Sigma_{(k)0}^{\infty} \frac{(-x^2)^k}{[2k]^{2k}}$$

$$= \Sigma_{(k)0}^{\infty} \frac{(-1)^k}{[2k]^{2k}} \frac{x^{2k+r}}{[2k+r]^r} = \Sigma_{(k)0}^{\infty} \frac{(-1)^k x^{2k+r}}{[2k+r]^{2k+r}}.$$

Hence

$$\left(\int_0^x dx\right)^{2r} \cos x = (-1)^r \left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{2.3.4} + \ldots - \frac{(-x^2)^{r-1}}{[2r-2]^{2r-2}}\right);$$

and

$$\left(\int_0^x dx\right)^{2r+1} \cos x = (-1)^r \left(\sin x - x + \frac{x^3}{2.3} - \frac{x^5}{2.3.4.5} + \ldots - \frac{(-x^2)^{r-1} x}{[2r-1]^{2r-1}}\right).$$

Hence

$$\left(\int_0^\tau d\tau\right) \cos n\tau = \frac{1}{n} \sin n\tau; \quad \left(\int_0^\tau d\tau\right)^2 \cos n\tau = -\frac{1}{n^2}(\cos n\tau - 1);$$

$$\left(\int_0^\tau d\tau\right)^3 \cos n\tau = -\frac{1}{n^3}(\sin n\tau - n\tau); \text{ \&c.}$$

Hence

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^i = \frac{1}{\pi}[i]^i \left(\int_0^\tau d\tau\right)^i \int_0^{\frac{\pi}{2}} d\theta \sec\theta^2 \operatorname{vers}(\tau\sqrt{2}\cos\theta).$$

This relation gives

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau = \frac{1}{\pi\sqrt{2}} \int_0^{\frac{\pi}{2}} d\theta \sec\theta^3 \{\tau\sqrt{2}\cos\theta - \sin(\tau\sqrt{2}\cos\theta)\};$$

which ought to turn out to be nearly $= \dfrac{\tau^2}{2\sqrt{2}}$, when τ is large.

Hence we ought to find

$$\lim_{\tau=\infty} \int_0^1 da\, (1-a^2)^{-\frac{1}{2}} \frac{\tau\sqrt{2}a - \sin(\tau\sqrt{2}a)}{\tau^2 a^3} = \frac{\pi}{2};$$

that is,

$$\frac{\pi}{4} = \lim_{\tau=\infty} \int_0^{\tau\sqrt{2}} dx \left(1 - \frac{x^2}{2r^2}\right)^{-\frac{1}{2}} \frac{x - \sin x}{x^3} = \int_0^\infty \frac{x - \sin x}{x^3} dx.$$

This will require $\left(\text{because we found } \int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}\right)$ that we should find

$$\int_0^\infty \frac{\operatorname{vers} x}{x^2}\,dx = \frac{\pi}{2}.$$

And reciprocally, if we can establish this last proposition, we shall be able to infer the former.

$$\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \lim_{y=\infty}\int_0^\infty e^{-\frac{x^2}{y^2}}\frac{x-\sin x}{x^3}\,dx$$

$$= \lim_{y=\infty} y^{-2}\int_0^\infty e^{-z^2}\frac{yz-\sin yz}{z^3}\,dz = \lim_{y=\infty}\frac{1}{2y}\int_0^\infty e^{-z^2}z^{-2}\operatorname{vers}(yz)\,dz$$

$$= \tfrac{1}{2}\lim_{y=\infty}\int_0^\infty e^{-z^2}z^{-1}\sin(yz)\,dz = \lim_{y=\infty}\tfrac{1}{2}\int_0^\infty e^{-\frac{x^2}{y^2}}x^{-1}\sin x\,dx$$

$$= \tfrac{1}{2}\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{4}.$$

Hence, going back, we ought to have

$$\frac{\pi}{4} = \lim_{\tau=\infty}\int_0^{\tau\sqrt{2}} dx\left(1-\frac{x^2}{2\tau^2}\right)^{-\frac{1}{2}}\frac{x-\sin x}{x^3}$$

$$= \lim_{\tau=\infty}\int_0^1 da\,(1-a^2)^{-\frac{1}{2}}\frac{\tau\sqrt{2}a - \sin(\tau\sqrt{2}a)}{2\tau^2a^3}$$

$$= \lim_{\tau=\infty}\int_0^{\frac{\pi}{2}} d\theta\sec\theta^3\,\frac{\tau\sqrt{2}\cos\theta - \sin(\tau\sqrt{2}\cos\theta)}{2\tau^2}$$

$$= \frac{\pi}{\sqrt{2}}\times\lim_{\tau=\infty}\frac{1}{\tau^2}(\nabla_\tau^{-2}\quad 1)^{-\frac{1}{2}}\tau;$$

$\because$ this last limit $= \frac{1}{2\sqrt{2}}$, as before. Thus the difference $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau - \frac{\tau^2}{2\sqrt{2}}$ becomes an indefinitely small fraction of τ^2, when τ becomes indefinitely great. It is however conceivable that this difference may itself be constant or even may increase indefinitely. I think indeed that this will not be found to be the case: but it requires to be examined.

[*Expression for* $\eta_{0,\tau}$.]

(Jan. 18th, 1839.)

It seems probable that the part $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau$, of the expression on page 535 for $\eta_{0,\tau}$, bears ultimately the ratio of equality to the other part $\nabla_\tau\alpha_\tau$ of the same expression, when τ increases without limit; & $\because$ that

$$\frac{1}{m^2}\frac{\operatorname{vers} m\tau}{\sqrt{2-m^2}} = (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{\sin m\tau}{m}\ \text{(ultimately)};$$

$$\sqrt{\tfrac{1}{2}}\,\Sigma_{(k_\prime)0}^{\infty}[-\tfrac{1}{2}]^{k_\prime}[0]^{-k_\prime}\left(-\frac{m^2}{2}\right)^{k_\prime}.\Sigma_{(k_{\prime\prime})0}^{\infty}\frac{(-m^2\tau^2)^{k_{\prime\prime}}\tau^2}{[2k_{\prime\prime}+2]^{2k_{\prime\prime}+2}} = (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\,\Sigma_{(k)0}^{\infty}\frac{(-m^2\tau^2)^k\tau}{[2k+1]^{2k+1}}\ \text{(ultimately)};$$

$$\because\ \text{(ultimately)}\ \frac{(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^{2k+1}}{[2k+1]^{2k+1}} = \sqrt{\tfrac{1}{2}}\,\Sigma_{(k_\prime)0}^{k}[-\tfrac{1}{2}]^{k_\prime}[0]^{-k_\prime}(\tfrac{1}{2})^{k_\prime}[0]^{-2(k-k_\prime+1)}\tau^{2(k-k_\prime+1)};$$

in which we know that the 1st member

$$= \frac{2}{\pi}\left(\int_0^\tau d\tau\right)^{2k+3}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta).$$

For example, making $k=0$, we have, as before,

$$\frac{2}{\pi}\left(\int_0^\tau d\tau\right)^3 \int_0^{\frac{\pi}{2}} d\theta \cos(\tau\sqrt{2}\cos\theta) = \frac{\tau^2}{2\sqrt{2}},$$

τ tending to ∞; & if the same law hold for $k=1$, we shall have ultimately

$$\frac{2}{\pi}\left(\int_0^\tau d\tau\right)^5 \int_0^{\frac{\pi}{2}} d\theta \cos(\tau\sqrt{2}\cos\theta) = \frac{\tau^4}{24\sqrt{2}} - \frac{\tau^2}{8\sqrt{2}}.$$

Changing τ to $t\sqrt{2}$, we ought $\because$ to have, ultimately, as t tends to infinity,

$$\frac{2}{\pi}\left(\int_0^t dt\right)^3 \int_0^{\frac{\pi}{2}} d\theta \cos(2t\cos\theta) = \frac{t^2}{4}; \quad \left(-\frac{1}{16}\right)^*;$$

$$\frac{2}{\pi}\left(\int_0^t dt\right)^5 \int_0^{\frac{\pi}{2}} d\theta \cos(2t\cos\theta) = \frac{t^4}{48} - \frac{t^2}{32}. \quad \left(+\frac{3}{256}\right)^*.$$

(Jan. 19th, 1839.)

$$\left\{\left(\frac{d}{d\,.\,m\tau}\right)^2 + 1\right\}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sec\theta \sin(\tau\sqrt{2}\cos\theta);\dagger$$

$$\because \quad (\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)}{1-\frac{2}{m^2}\cos\theta^2} - \frac{\sqrt{2}}{m\pi}\sin m\tau \int_0^{\frac{\pi}{2}} \frac{d\theta}{1-\frac{2\cos\theta^2}{m^2}}$$

$$= \frac{m}{\pi}\int_0^{\frac{\pi}{2}} \frac{m\sin(\tau\sqrt{2}\cos\theta) - \sqrt{2}\cos\theta\sin m\tau}{(m^2-2\cos\theta^2)\cos\theta}\,d\theta = \frac{m^2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} \frac{m\sin n\tau - n\sin m\tau}{(m^2-n^2)\,mn}\,d\theta$$

$$= \frac{m^2\sqrt{2}}{\pi}\int_0^\tau \left(\int_0^{\frac{\pi}{2}} \frac{\cos n\tau - \cos m\tau}{m^2-n^2}\,d\theta\right) d\tau, \quad (\text{if } n = \sqrt{2}\cos\theta),$$

$$= \frac{m^2}{\pi}\int_0^\tau \left(\int_0^{\sqrt{2}} \frac{\cos n\tau - \cos m\tau}{m^2-n^2}\,\frac{dn}{\sqrt{1-\frac{n^2}{2}}}\right) d\tau; \quad \because \text{making } \frac{n}{m} = p,\ m\tau = v,$$

$$= \frac{1}{\pi}\int_0^{m\tau}\left(\int_0^{\frac{\sqrt{2}}{m}} \frac{\cos pv - \cos v}{1-p^2}\,\frac{dp}{\sqrt{1-\frac{1}{2}m^2p^2}}\right) dv.$$

* [The figures in round brackets are added in pencil in the manuscript (see page 551). Hamilton later on arrived at a complete form for the values of these integrals when t is large. He seemed to have known it about 1840 but after much numerical testing did not publish it until Nov. 1857, *Phil. Mag.* XIV, pp. 375–382. He gives the result as follows:

If
$$f(t) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\omega\cos(2t\cos\omega),$$
then
$$\left(\int_0^t dt\right)^n f(t) - \left(-\int_t^\infty dt\right)^n f(t) = \Sigma_{(m)}\, 2^{-2m-1}\,[-\tfrac{1}{2}]^m\,[0]^{-m}\,[0]^{-(n-2m-1)}\,t^{n-2m-1},$$

the summation to extend to all positive integral values of m such that $n-2m-1 \geqslant 0$. The asymptotic expressions can be derived by integration of the asymptotic expression for $J_0(\tau\sqrt{2})$.]

† $\Big[\left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\sin m\tau = \left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\frac{m}{\sqrt{2}}\Sigma_{(k)0}^{\infty}\frac{(-m^2\tau^2)^k\,m\tau}{[2k+1]^{2k+1}}$

$$= \frac{\sqrt{2}}{\pi}\left\{\left(\frac{d}{m\,d\tau}\right)^2+1\right\}\Sigma_{(k)0}^{\infty}(-1)^k m^{2k+2}\left(\int_0^\tau d\tau\right)^{2k+3}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\sec\theta\sin(\tau\sqrt{2}\cos\theta)\,d\theta.\Big]$$

Hence, *rigorously*,

$$\boxed{\eta_{0,\tau} = \tfrac{1}{2}\,\text{vers}\, v + \frac{1}{\pi}\int_0^v \left(\int_0^{\frac{\sqrt{2}}{m}} \frac{\cos pv - \cos v}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,dp\right)dv, \quad \text{if } v = m\tau.}$$

As m tends to 0, $v = m\tau$ remaining given, this expression tends to the limit

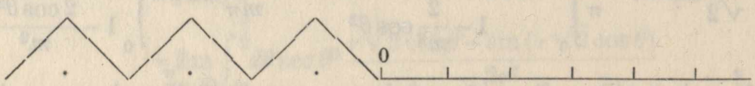
$$\eta_{0,\tau} = \tfrac{1}{2}\,\text{vers}\, v + \frac{1}{\pi}\int_0^v \left(\int_0^\infty \frac{\cos pv - \cos v}{1-p^2}\,dp\right)dv;$$

I therefore think it *probable* that the integral $\int_0^v \left(\int_0^\infty \frac{\cos pv - \cos v}{1-p^2}\,dp\right)dv$ is *rigorously* $= \frac{\pi}{2}\,\text{vers}\, v$.

If we take $m =$ its extreme value, namely $\sqrt{2}$, then the integral vanishes & the expression for $\eta_{0,\tau}$ reduces itself exactly to $\frac{1}{2}\,\text{vers}\, v = \frac{1}{2}\,\text{vers}\,(\tau\sqrt{2})$. In this case $\mu = \pi = \text{vers}^{-1} m^2$; & the equations $(4)_2$ become

$$\eta_{x,0} = \text{vers}\, \pi x, \quad \eta'_{x,0} = -\sqrt{2}\sin \pi x,$$

x being here any integer $\not> 0$; $\because\ \eta'_{x,0} = 0$; & $\eta_{x,0} = 0$ or $= 2$, according as $-x$ is even or odd. In this case we have initial displacements of the form



but no initial velocities; & the origin 0 receives only *half* the disturbance which it would do for the other extreme value of m, namely 0, corresponding to an infinitely long wave.* In fact, if we conceive a great but finite succession of initial displacements (without initial velocities) of the kind corresponding to the short wave lately mentioned, there is no reason why they should propagate themselves in one direction rather than in the opposite; they will therefore divide themselves between both.

(Jan. 26th, 1839.)

Is the integral $\int_0^\infty dp\, \frac{\sin pv - p \sin v}{p\,(1-p^2)} = \frac{\pi}{2}\,\text{vers}\, v$?

In the 1st place we may consider it as = some finite quantity, and may treat this as a function of v; but when we add to this function its own 2nd differential coefficient we get

$$\left\{1 + \left(\frac{d}{dv}\right)^2\right\}\int_0^\infty \frac{\sin pv - p\sin v}{p\,(1-p^2)}\,dp = \int_0^\infty \frac{\sin pv}{p}\,dp = -\frac{\pi}{2} \text{ or } 0 \text{ or } \frac{\pi}{2},$$

according as v is < 0 or $= 0$ or > 0; therefore we have *rigorously*

$$\int_0^\infty \frac{\sin pv - p\sin v}{p\,(1-p^2)}\,dp = \pm\frac{\pi}{2}\,\text{vers}\, v,$$

the upper or lower sign being taken according as v is $>$ or $<$ 0.

* [The wave-length is $2\pi/\mu$; $m\tau = v$, $m^2 = \text{vers}\,\mu$.]

*Hence the function** $\quad \frac{1}{2}\operatorname{vers} v + \frac{1}{\pi}\int_0^\infty \frac{\sin pv - p\sin v}{p\,(1-p^2)}\,dp$

is rigorously $=0$ *when* v *is* <0 *and* $=\operatorname{vers} v$ *when* $v>0$. *And this particular case of discontinuity appears to me to be of great importance in the dynamics of light and darkness.*

The rigorous equation $(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\tau^i = [i]^i\left(\int_0^\tau d\tau\right)^i \gamma_\tau$, (page 537), in which

$$\gamma_\tau = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \sec\theta^2 \operatorname{vers}(\tau\sqrt{2}\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\tau\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)$$

$$= \int_0^\tau \gamma'_\tau d\tau, \text{ in which } \gamma'_\tau = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta),$$

combined with the general development

$$\beta_\tau = \beta_0 + \tau\beta'_0 + [0]^{-2}\tau^2\beta''_0 + \&c.$$

gives, generally,

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \beta_0\gamma_\tau + \beta'_0\int_0^\tau d\tau\gamma_\tau + \beta''_0\left(\int_0^\tau d\tau\right)^2\gamma_\tau + \&c.$$

$$= \underset{x=0}{\Sigma^\infty_{(h)0}}\left(\frac{d}{dx}\right)^k\beta_x\left(\int_0^\tau d\tau\right)^k\gamma_\tau = \underset{x=0}{\Sigma^\infty_{(k)0}}\left(\frac{d}{dx}\right)^k\beta_x\left(\int_0^\tau d\tau\right)^{k+1}\gamma'_\tau$$

$$= \frac{\int_0^\tau d\tau}{1-\underset{x=0}{\frac{d}{dx}}\int_0^\tau d\tau}\cdot\beta_x\gamma'_\tau;\quad \because \left(\frac{d}{d\tau}-\frac{d}{dx}\right)_{x=0}(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \underset{x=0}{\beta_x}\gamma'_\tau;$$

but, suppressing for the present the condition $x=0$, the partial differential equation of the 1st order

$$\left(\frac{d}{d\tau}-\frac{d}{dx}\right)\Phi_{x,\tau} = F_{x,\tau}$$

gives

$$\Phi_{x,\tau} = \int_0^\tau F_{\tau,-\tau,\tau}d\tau + \operatorname{funct}(x+\tau),$$

$\tau_,$ being treated as a constant in effecting the 1st integration, but being afterwards made equal to $x+\tau$; if then $\Phi_{x,0}=0$ for all values of x, we have $\Phi_{x,\tau} = \int_0^\tau F_{\tau,-\tau,\tau}d\tau$, & the same expression holds for the case $x=0$, except that then $\tau_,$ is to be made equal to τ after the integration; hence, generally,†

* [The results can be verified by taking the integrals

$$\int\frac{e^{ipv}-e^{iv}}{1-p^2}\,dv \quad\text{and}\quad \int\frac{e^{ipv}-pe^{iv}}{p\,(1-p^2)}\,dp,\quad (p>0)$$

over suitable contours.]

† [Changing the notation, we have to show that

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \int_0^\tau du\,\beta_{\tau-u}\gamma'_u,\quad \beta_{\tau-u} = \Sigma\frac{(\tau-u)^n}{n!}\beta_0^{(n)}.$$

Then making use of $\int_0^\tau(\tau-u)^n\gamma'_u\,du = n!\left(\int_0^\tau du\right)^{n+1}\gamma'_u$, we verify the required form. It may be noticed that $\frac{d^2\gamma_\tau}{d\tau^2} = 2J_0(\tau\sqrt{2})$.]

$$(\nabla_\tau^{-2}-1)^{-\frac{1}{2}}\beta_\tau = \int_0^\tau d\tau\,\beta_{\tau,-\tau}\gamma'_\tau$$
$$\tau,=\tau$$

$$= \frac{\sqrt{2}}{\pi}\int_0^\tau d\tau \int_0^{\frac{\pi}{2}} d\theta \sec\theta \sin(\tau\sqrt{2}\cos\theta)\,\beta_{\tau,-\tau}$$
$$\tau,=\tau$$

$$= \frac{\sqrt{2}}{\pi}\int_0^\tau d\tau \int_0^{\sqrt{2}} \frac{dn}{n}\frac{\sin n\tau}{\sqrt{1-\frac{1}{2}n^2}}\beta_{\tau,-\tau}.$$
$$\tau,=\tau$$

[*The operator I_τ.**]

Can we find any analogous expressions for $\eta_{1,\tau}$, $\eta_{2,\tau}$, &c., in so far as they depend on β_τ? For if so, we can easily get afterwards expressions for the parts which depend on α_τ.

Already we have found the law of the developments, according to ascending powers of ∇_τ; but it will be useful to investigate developments proceeding according to ascending powers of $\left(\int_0^\tau d\tau\right)^2$ instead. For this purpose it is convenient to employ, instead of the equations (11) of page 532, the following;

$$\eta_{0,\tau} = I_\tau^2(\beta_\tau + \eta_{1,\tau} - \eta_{0,\tau}); \quad \eta_{1,\tau} = \tfrac{1}{2}I_\tau^2(\eta_{0,\tau} - 2\eta_{1,\tau} + \eta_{2,\tau}); \quad \text{\&c.};$$

in which $I_\tau = \int_0^\tau d\tau$. We may for the present omit the τ's & write simply

$$\eta_0 = I^2(\beta + \eta_1 - \eta_0); \quad \eta_1 = \tfrac{1}{2}I^2(\eta_0 - 2\eta_1 + \eta_2); \ \text{\&c.}$$

And it may be useful to investigate first from the equations

$$\eta_1 = \tfrac{1}{2}I^2(\eta_0 - 2\eta_1 + \eta_2), \quad \eta_2 = \tfrac{1}{2}I^2(\eta_1 - 2\eta_2 + \eta_3), \ \text{\&c.},$$

which are all similar to each other, developments of the forms

$$\eta_n = \Sigma_{(k)0}^{\infty} e_{n,k} I^{2(n+k)}\eta_0;$$

or, more fully, $\quad \eta_1 = e_{1,0}I^2\eta_0 + e_{1,1}I^4\eta_0 + \text{\&c.}, \quad \eta_2 = e_{2,0}I^4\eta_0 + \text{\&c.}, \ \text{\&c.}$

$$e_{1,0} = \tfrac{1}{2}; \quad e_{1,1} + e_{1,0} = 0; \quad e_{1,2} + e_{1,1} = \tfrac{1}{2}e_{2,0}; \ \text{\&c.}$$

$$e_{2,0} = \tfrac{1}{2}e_{1,0}; \quad e_{2,1} + e_{2,0} = \tfrac{1}{2}e_{1,1}; \ \text{\&c.}; \ \text{\&c.}$$

In general

$$e_{n,k} + e_{n,k-1} = \tfrac{1}{2}(e_{n-1,k} + e_{n+1,k-2});$$

negative indices being considered to make the resulting coefficient vanish, & the only value of $e_{0,k}$ which does not vanish being $e_{0,0} = 1$.

Now make $e_{n,k} = f_{n+k,k}$; we shall have

$$f_{n+k,k} + f_{n+k-1,k-1} = \tfrac{1}{2}(f_{n+k-1,k} + f_{n+k-1,k-2}),$$

so that

$$f_{m+1,k} = \tfrac{1}{2}(f_{m,k} - 2f_{m,k-1} + f_{m,k-2}).$$

* [This is Heaviside's operator. Cf. Jeffreys, *Operational Methods in Mathematical Physics* (1927); Poole, *Theory of Linear Differential Equations* (1936), Chap. II.]

Hence

$$\eta_n = \left(\frac{1+I^2-\sqrt{1+2I^2}}{I^2}\right)^n \eta_0 = \left(\frac{I^2}{1+I^2+\sqrt{1+2I^2}}\right)^n \eta_0;$$

and $$\eta_0 = \frac{I^2\beta}{\sqrt{1+2I^2}}.$$

As a verification, these last expressions give

$$\frac{\eta_{n+1}-2\eta_n+\eta_{n-1}}{\eta_n} = \frac{1+I^2-\sqrt{1+2I^2}}{I^2} + \frac{1+I^2+\sqrt{1+2I^2}}{I^2} - 2 = \frac{2}{I^2};$$

$\therefore$ $$\eta_n = \frac{I^2}{2}(\eta_{n+1}-2\eta_n+\eta_{n-1});$$

also $$(1+I^2)\eta_0 - I^2\eta_1 = \sqrt{1+2I^2}\,\eta_0 = I^2\beta.$$

If we now make $$\frac{I^2}{1+I^2} = \nabla,$$

& $\because$ $$1-\nabla = \frac{1}{1+I^2},\quad \frac{1+I^2}{I^2} = \nabla^{-1},\quad \frac{\sqrt{1+2I^2}}{I^2} = (\nabla^{-2}-1)^{\frac{1}{2}},$$

we shall have

$$\eta_0 = (\nabla^{-2}-1)^{-\frac{1}{2}}\beta, \quad \text{and} \quad \eta_n = \{\nabla^{-1}-(\nabla^{-2}-1)^{\frac{1}{2}}\}^n(\nabla^{-2}-1)^{-\frac{1}{2}}\beta.$$

If, by a further symbolic transformation, we were to make $\nabla = \sin 2\delta$, *we should have the expression*

$$\eta_n = (\tan\delta)^n . \tan 2\delta . \beta.$$

[tan $\delta_\tau . \phi_\tau$ as an integral.]

We have found that the operation here called $\tan 2\delta$, or more fully $\tan 2\delta_\tau$, is such that

$$\tan 2\delta_\tau . \beta_\tau = \frac{\sqrt{2}}{\pi}\int_0^\tau d\tau \int_0^{\frac{\pi}{2}} d\theta \sec\theta \sin(\tau\sqrt{2}\cos\theta)\,\beta_{\tau,-\tau}.$$
$$\tau,=\tau$$

Can we find any analogous expression for the effect of the operation $\tan\delta_\tau^n$, or even $\tan\delta_\tau$?

$$\tan\delta_\tau = \nabla_\tau^{-1} - (\nabla_\tau^{-2}-1)^{\frac{1}{2}} = \nabla_\tau^{-1}\{1-\sqrt{1-\nabla_\tau^2}\}$$
$$= \frac{1+I_\tau^2-\sqrt{1+2I_\tau^2}}{I_\tau^2} = -\Sigma_{(k)2}^\infty [\tfrac{1}{2}]^k [0]^{-k} 2^k I_\tau^{2k-2}$$
$$= \Sigma_{(k)0}^\infty [-\tfrac{3}{2}]^k [0]^{-(k+2)} 2^k I_\tau^{2k+2} = \Sigma_{(k)0}^\infty (-1)^k \frac{1.3.5.7\ldots(2k+1)}{2.3.4.5\ldots(k+2)}\left(\int_0^\tau d\tau\right)^{2k+2}.$$

Let this operation be performed on τ^i, and because

$$\left(\int_0^\tau d\tau\right)^{2k+2}\tau^i = \frac{\tau^{2k+2+i}}{[2k+2+i]^{2k+2}},$$

we have

$$\tan\delta_\tau\,.\,\tau^i=\left(\int_0^\tau d\tau\right)^i\Sigma_{(k)0}^{\infty}(-1)^k\frac{1\,.\,3\ldots(2k+1)}{2\,.\,3\ldots(k+2)}\frac{\tau^{2k+2}[i]^i}{[2k+2]^{2k+2}}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\Sigma_{(k)0}^{\infty}(-1)^k\,2^{-(k+1)}[0]^{-(k+1)}[0]^{-(k+2)}\tau^{2k+2}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\frac{1}{\tau^2}\int_0^{\tau^2}\left\{1-\Sigma_{(k)0}^{\infty}\left(-\frac{t}{2}\right)^k([0]^{-k})^2\right\}dt$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\left\{1-\frac{1}{\tau^2}\int_0^{\tau^2}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(\sqrt{2t}\cos\theta)\,d\theta\right)dt\right\}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^i\left\{1-\frac{\sqrt{2}}{\tau}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta)+\frac{1}{\tau^2}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta^2\,\text{vers}\,(\tau\sqrt{2}\cos\theta)\right\}$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^{i+1}\frac{4}{\pi\tau^3}\int_0^{\frac{\pi}{2}}d\theta\{\tau\sqrt{2}\cos\theta\sin(\tau\sqrt{2}\cos\theta)$$
$$-\tau^2\cos(\tau\sqrt{2}\cos\theta)\cos\theta^2-\text{vers}\,(\tau\sqrt{2}\cos\theta)\}\sec\theta^2$$

$$=[i]^i\left(\int_0^\tau d\tau\right)^{i+1}\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^{\frac{\pi}{2}}d\theta\,\tau^2\sin(\tau\sqrt{2}\cos\theta)\cos\theta.$$

Hence

$$\tan\delta_\tau\,.\,\tau^i=\int_{0\atop \tau,=\tau}^{\tau}d\tau\,(\tau,-\tau)^i\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^{\frac{\pi}{2}}d\theta\,\tau^2\cos\theta\sin(\tau\sqrt{2}\cos\theta),$$

and generally

$$\tan\delta_\tau\,.\,\phi_\tau=\int_{0\atop \tau,=\tau}^{\tau}d\tau\,\phi_{\tau,-\tau}\frac{4\sqrt{2}}{\pi\tau^3}\int_0^\tau d\tau\int_0^{\frac{\pi}{2}}d\theta\,\tau^2\cos\theta\sin(\tau\sqrt{2}\cos\theta)$$

$$=\frac{4}{\pi}\int_{0\atop \tau=\tau,}^{\tau}d\tau\,\phi_{\tau,-\tau}\int_0^{\frac{\pi}{2}}d\theta\{\tau\sqrt{2}\cos\theta\sin(\tau\sqrt{2}\cos\theta)$$
$$-\tau^2\cos\theta^2\cos(\tau\sqrt{2}\cos\theta)-\text{vers}\,(\tau\sqrt{2}\cos\theta)\}\frac{\sec\theta^2}{\tau^3}$$

$$=\frac{4\sqrt{2}}{\pi}\int_{0\atop \tau,=\tau}^{\tau}d\tau\,\phi_{\tau,-\tau}\tau^{-3}\int_0^{\sqrt{2}}dn\{n\tau\sin n\tau-1+(1-\tfrac{1}{2}n^2\tau^2)\cos n\tau\}\,n^{-2}(1-\tfrac{1}{2}n^2)^{-\frac{1}{2}}.$$

[$\eta_{x,t}$ *expressed by means of operator* D_t.]

(Jan. 22nd.)

It will be interesting to consider the simplest forms to which the repetition of this operation, & its combination with the operation $\tan 2\delta_\tau$, conduct; but at present I prefer to resume the problem in another way, as follows.

We have the system of equations, $\left(\text{in which } D=D_\tau=\dfrac{d}{d\tau}\right)$,

$$\eta_1=-\beta+(1+D^2)\eta_0;\quad \eta_2=2(1+D^2)\eta_1-\eta_0;\quad \eta_3=2(1+D^2)\eta_2-\eta_1;\quad \&c.;$$

so that the series

$$\eta_0+\eta_1 x+\eta_2 x^2+\ldots+\eta_n x^n+\ldots$$

being multiplied by $1-2(1+D^2)x+x^2$ must reduce itself to the 2 terms

$$\eta_0+\{\eta_1-2(1+D^2)\eta_0\}x=\eta_0-\{\beta+(1+D^2)\eta_0\}x;$$

if then we develope the fraction

$$\frac{\eta_0-\{\beta+(1+D^2)\eta_0\}x}{(1-x)^2-2D^2x}$$

according to ascending powers of x, the coefficient of x^n in this development will be the function η_n.

The coefficient of x^n in the development of $\dfrac{1}{(1-x)^2-2D^2x}$ is

$$\text{coeff}^{\text{t}} \text{ of } x^n \text{ in } (1-x)^{-2}+2D^2\times\text{coeff}^{\text{t}} \text{ of } x^{n-1} \text{ in } (1-x)^{-4}$$
$$+(2D^2)^2\times\text{coeff}^{\text{t}} \text{ of } x^{n-2} \text{ in } (1-x)^{-6}+\&c.$$
$$=(n+1)+2D^2\frac{[n+2]^3}{[3]^3}+(2D^2)^2\frac{[n+3]^5}{[5]^5}+\&c.$$

Hence

$$\eta_n-\eta_0=\frac{n}{2}\left(\frac{[n]^1}{[1]^1}\frac{2D^2}{1}+\frac{[n+1]^3}{[3]^3}\frac{(2D^2)^2}{2}+\frac{[n+2]^5}{[5]^5}\frac{(2D^2)^3}{3}+\ldots\right)\eta_0$$
$$-\left(\frac{[n]^1}{[1]^1}+\frac{[n+1]^3}{[3]^3}2D^2+\frac{[n+2]^5}{[5]^5}(2D^2)^2+\ldots\right)\beta.$$

Now, in so far as η_0 depends on β alone, we have

$$\eta_0=(\nabla^{-2}-1)^{-\frac{1}{2}}\beta=\frac{I^2\beta}{\sqrt{1+2I^2}}=\tan 2\delta\,.\,\beta=\int_{0\atop \tau,=\tau}^{\tau}d\tau\,\beta_{\tau,-\tau}\gamma'_\tau,$$

in which

$$\gamma'_\tau=\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta).$$

Hence

$$D\eta_0=\beta_0\gamma'_\tau+\int_{0\atop \tau,=\tau}^{\tau}d\tau\,\beta'_{\tau,-\tau}\gamma'_\tau;\quad D^2\eta_0=\beta_0\gamma''_\tau+\beta'_0\gamma'_\tau+\int_{0\atop \tau,=\tau}^{\tau}d\tau\,\beta''_{\tau,-\tau}\gamma'_\tau;\quad \&c.$$

Let us suppose, which we shall find to be permitted in the present investigation, that β_τ is a linear function of τ, & $\because$ that β''_τ, β'''_τ, &c. vanish; $\because$ also $D^2\beta$, $D^4\beta$, &c. vanish. [We finally obtain]

$$\eta_{n,\tau}=(\beta_0 D_\tau+\beta'_0)\left\{\gamma^{\backprime}_\tau+\frac{n}{2}\left(\frac{[n]^1}{[1]^1}2D^2_\tau+\frac{[n+1]^3}{[3]^3}\frac{(2D^2_\tau)^2}{2}+\frac{[n+2]^5}{[5]^5}\frac{(2D^2_\tau)^3}{3}+\&c.\right)\gamma^{\backprime}_\tau-n\tau\right\},$$

in which

$$\gamma^{\backprime}_\tau=\int_0^\tau\gamma_\tau d\tau=\frac{1}{\pi\sqrt{2}}\int_0^{\frac{\pi}{2}}d\theta\sec\theta^3\{\tau\sqrt{2}\cos\theta-\sin(\tau\sqrt{2}\cos\theta)\}.$$

The properties asserted of this set of functions $\eta_{n,\tau}$ are: 1st, that they vanish, and that their 1st differential coefficients relatively to τ vanish, when $\tau=0$; & 2nd, that they satisfy the system of differential equations

$$(1+D_\tau^2)\,\eta_{0,\tau}=\eta_{1,\tau}+\beta_0+\beta_0'\tau, \quad \& \quad (1+D_\tau^2)\,\eta_{n,\tau}=\tfrac{1}{2}(\eta_{n+1,\tau}+\eta_{n-1,\tau}), \quad \text{if } n>0.$$

Reciprocally, we shall have proved the justice of the expression for $\eta_{n,\tau}$ if we prove that it possesses these properties. It is therefore important to make this verification. [The verification follows and incidentally involves the identity

$$\Sigma_{(k)0}^{n-1}\frac{(-1)^k}{k+1}([0]^{-k})^2\frac{[n+k]^{2k+1}}{2k+1}=1.]^*$$

Suppose now that, when $n>0$, $\eta_{n,\tau}=\eta_{-n,\tau}=$ transversal vibration or displacement of the n^{th} particle before or behind the particle corresponding to $n=0$; but that this last mentioned particle has its displacement represented by $\eta_{0,\tau}-\beta_0-\tau\beta_0'$; then to the expressions lately given we are to add those which would correspond to the introduction of the terms $-\frac{1}{2}\beta_0-\frac{1}{2}\tau\beta_0'$ in the expressions for $D_\tau^2\eta_{1,\tau}$ and $D_\tau^2\eta_{-1,\tau}$; & this again comes to changing, in the integrals, $\eta_{0,\tau}$ to $\eta_{0,\tau}-\eta_{1,\tau}$; $\eta_{1,\tau}$ to $\eta_{1,\tau}-\frac{1}{2}(\eta_{0,\tau}+\eta_{2,\tau})$, &c.; or finally to replacing the displacements at the time τ by the negatives of the 2nd differential coefficients, relatively to τ,† of the expressions lately given: so that these displacements are

$$y_{0,\tau}=y_{0,0}\gamma_\tau''+y_{0,0}'\gamma_\tau'; \ \&c.;$$

or, more generally,

$$\boxed{y_{x,\tau}=(y_{0,0}D_\tau+y_{0,0}')\left\{1+\frac{x}{2}\left(\frac{[x]^1}{[1]^1}2D_\tau^2+\frac{[x+1]^3}{[3]^3}\frac{(2D_\tau^2)^2}{2}+\frac{[x+2]^5}{[5]^5}\frac{(2D_\tau^2)^3}{3}+\dots\right)\right\}\gamma_\tau',}$$

in which, as before,

$$\gamma_\tau'=\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta),$$

and $y_{0,0}$, $y_{0,0}'$ are the initial displacements & velocities of the particle 0.

* [We have

$$\frac{(-1)^k([0]^{-k})^2}{k+1}\frac{[n+k]^{2k+1}}{2k+1}=\frac{(-1)^k[2k]^{2k}}{[k]^k[k+1]^{k+1}}\frac{[n+k]^{2k+1}}{[2k+1]^{2k+1}}=\frac{[\frac{1}{2}]^{k+1}2^{2k+1}}{[k+1]^{k+1}}\frac{[n+k]^{2k+1}}{[2k+1]^{2k+1}}.$$

Hence

$$\Sigma_{(k)0}^{n-1}\frac{(-1)^k([0]^{-k})^2}{k+1}\frac{[n+k]^{2k+1}}{2k+1}=\text{coefficient of } x \text{ in } \Sigma_{(k)0}^{n-1}\frac{[\frac{1}{2}]^{k+1}2^{2k+1}}{[k+1]^{k+1}}\frac{(1+x)^{n+k}}{x^{2k}}$$

$$= \quad ,, \quad ,, \ ,, \ ,, \ \tfrac{1}{2}x^2(1+x)^{n-1}\Sigma_{(k)0}^{\infty}\frac{[\frac{1}{2}]^{k+1}}{[k+1]^{k+1}}\left\{\frac{4(1+x)}{x^2}\right\}^{k+1}$$

$$= \quad ,, \quad ,, \ ,, \ ,, \ \tfrac{1}{2}x^2(1+x)^{n-1}\left\{1+\frac{4(1+x)}{x^2}\right\}^{\frac{1}{2}}$$

$$= \quad ,, \quad ,, \ ,, \ ,, \ \tfrac{1}{2}x(1+x)^{n-1}(x+2)$$

$$=1.]$$

† [$y_{n,\tau}=-\frac{d^2\eta_{n,\tau}}{d\tau^2}$. n is also replaced by x. The transformation gives the solution of a new problem as described hcre.]

For example, the displacement of this very particle at the time τ is, *rigorously*,

$$y_{0,\tau}=y_{0,0}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta\cos(\tau\sqrt{2}\cos\theta)+y'_{0,0}\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta).$$

As τ increases without limit this displacement tends to become $=\sqrt{\tfrac{1}{2}}\,y'_{0,0}$;* it bears therefore a constant ratio $\sqrt{\tfrac{1}{2}}$ to the initial velocity $y'_{0,0}$. This is a very curious result. The ratio $\sqrt{\tfrac{1}{2}}$ depends no doubt upon the assumed unit of time, & on the energy of attraction between 2 adjacent particles.

The expression for $y_{x,\tau}$ ought to satisfy *generally* the equation in mixed differences

$$y''_{x,\tau}=\tfrac{1}{2}(y_{x+1,\tau}-2y_{x,\tau}+y_{x-1,\tau}).$$

It ought also to give $y_{x,0}=0$ and $y'_{x,0}=0$, when $x>$ or <0; but $=y_{0,0}$ and $y'_{0,0}$ respectively, when $x=0$. And so it does.

(Jan. 23rd, 1839.)

The last expression for $y_{x,\tau}$ may be thus written:

$$y_{x,\tau}=(y_{0,0}D_\tau+y'_{0,0})\left\{1+\frac{1}{2}\frac{x^2}{1}2D_\tau^2+\frac{1}{2}\frac{x^2(x^2-1)}{1.2.3}\frac{(2D_\tau)^2}{2}+\frac{1}{2}\frac{x^2(x^2-1)(x^2-4)}{1.2.3.4.5}\frac{(2D_\tau^2)^3}{3}+\dots\right\}\frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta);$$

and from it we may deduce the more general one:

$$y_{x,\tau}=\frac{1}{\pi\sqrt{2}}\Sigma_{(i)-\infty}^{\infty}(y_{i,0}D_\tau+y'_{i,0})\left\{2+\frac{(x-i)^2}{1}\frac{(2D_\tau^2)}{1}+\frac{(x-i)^2\{(x-i)^2-1\}}{1.2.3}\frac{(2D_\tau^2)^2}{2}+\frac{(x-i)^2\{(x-i)^2-1\}\{(x-i)^2-4\}}{1.2.3.4.5}\frac{(2D_\tau^2)^3}{3}+\dots\right\}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(\tau\sqrt{2}\cos\theta);$$

in which $D_\tau=\dfrac{d}{d\tau}$, and the function $y_{x,\tau}$ is obliged to satisfy generally the differential equation

$$D_\tau^2 y_{x,\tau}+y_{x,\tau}=\tfrac{1}{2}(y_{x+1,\tau}+y_{x-1,\tau}),$$

and also to become $y_{i,0}$ when $\tau=0$, & to give $D_\tau y_{x,\tau}=y'_{i,0}$ when $\tau=0$.

Restoring ∵ $ta\sqrt{2}$ for τ, and ∵ putting $\dfrac{1}{a\sqrt{2}}D_t$ for D_τ; changing also $y_{i,0}$ to $\eta_{i,0}$ and $y'_{i,0}$ to $\dfrac{1}{a\sqrt{2}}\eta'_{i,0}$, we get the expression

$$\eta_{x,t}=\frac{1}{2a\pi}\Sigma_{(i)-\infty}^{\infty}(\eta_{i,0}D_t+\eta'_{i,0})\left\{2+\frac{(x-i)^2}{1}\frac{D_t^2}{1a^2}+\frac{(x-i)^2\{(x-i)^2-1\}}{1.2.3}\frac{D_t^4}{2a^4}+\frac{(x-i)^2\{(x-i)^2-1\}\{(x-i)^2-4\}}{1.2.3.4.5}\frac{D_t^6}{3a^6}+\dots\right\}\int_0^{\frac{\pi}{2}} d\theta\sec\theta\sin(2at\cos\theta),$$

* [The first integral is $J_0(\tau\sqrt{2})$. The asymptotic form for the second can be obtained by considering the integral $\int e^{izt}(1-t^2)^{-\frac{1}{2}}t^{-1}\,dt$ over a suitable contour.]

as the complete integral of the equation (1) of page 527,

$$\eta''_{x,t}=a^2(\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t}).$$

The expression may be written more concisely thus:*

$$\boxed{\eta_{x,t}=\frac{1}{2a\pi}\Sigma^{\infty}_{(i)-\infty}(\eta_{i,0}D_t+\eta'_{i,0})\left\{2+(x-i)\Sigma^{\infty}_{(k)0}\frac{[x-i+k]^{2k+1}}{[2k+1]^{2k+1}}\frac{D_\tau^{2k+2}}{(k+1)a^{2k+2}}\right\}\int_0^{\frac{\pi}{2}}\frac{d\theta\sin(2at\cos\theta)}{\cos\theta}.}$$

In it $\eta_{i,0}$ and $\eta'_{i,0}$ may be considered as the 2 arbitrary functions, and if we denote them by F_i and f_i we may write

$$\eta_{x,t}=\frac{1}{a\pi}\Sigma^{\infty}_{(i)-\infty}(F_iD_t+f_i)\left\{1+\frac{x-i}{2}\Sigma^{\infty}_{(k)0}\frac{[x-i+k]^{2k+1}}{[2k+1]^{2k+1}}\frac{D_\tau^{2k+2}}{(k+1)a^{2k+2}}\right\}\int_0^{\frac{\pi}{2}}\frac{\sin(2at\cos\theta)\,d\theta}{\cos\theta}.$$

[$\eta_{x,t}$ *expressed as an integral.*]

(Jan. 25th, 1839.)

Employing the very simple expression†

$$\eta_{x,t}=F_x+\frac{a^2t^2}{1.2}\Delta_x^2F_{x-1}+\frac{a^4t^4}{1.2.3.4}\Delta_x^4F_{x-2}+\dots$$
$$+tf_x+\frac{a^2t^3}{1.2.3}\Delta_x^2f_{x-1}+\frac{a^4t^5}{1.2.3.4.5}\Delta_x^4f_{x-2}+\dots$$

for the integral of the equation in mixed differences

$$D_t^2\eta_{x,t}=a^2\Delta_x^2\eta_{x-1,t},$$

F_x and f_x being the initial values of $\eta_{x,t}$ and of $D_t\eta_{x,t}$; or still more simply employing the expression

$$\eta_{x,t}=F_x+\frac{t^2}{1.2}\Delta_x^2F_{x-1}+\frac{t^4}{1.2.3.4}\Delta_x^4F_{x-2}+\dots$$
$$+tf_x+\frac{t^3}{1.2.3}\Delta_x^2f_{x-1}+\frac{t^5}{1.2.3.4.5}\Delta_x^4f_{x-2}+\dots$$

as the integral of the equation $D_t^2\eta_{x,t}=\Delta_x^2\eta_{x-1,t}$; let us suppose that f_x is $=0$ for all values of x, & that F_x also vanishes except for negative and odd values, for which we shall suppose it to be constantly $=1$; and let us seek the corresponding form of the function $\eta_{0,t}$, with a view to show that it reduces itself to the form $\frac{1}{2}(\sin t)^2$.

We are ∵ to have, for all values of t,

$$\tfrac{1}{2}(\sin t)^2=F_0+\frac{t^2}{1.2}\Delta_0^2F_{0-1}+\frac{t^4}{1.2.3.4}\Delta_0^4F_{0-2}+\&c.,$$

* [Hamilton verifies this expression and in the verification the following results appear

$$\tfrac{1}{2}(\sin at)^2=\Sigma^{\infty}_{(i)0}\left\{1+\frac{2i+1}{2}\Sigma^{2i}_{(k)0}\frac{[2i+k+1]^{2k+1}}{[2k+1]^{2k+1}}\frac{D^{2k+2}}{(k+1)a^{2k+2}}\right\}\Sigma^{\infty}_{(r)0}(-a^2t^2)^r([0]^{-r})^2;$$

and

$$0=\Sigma^{\infty}_{(i)0}\left\{1-\frac{(2i+1)^2}{1^2}+\frac{(2i+1)^2\{(2i+1)^2-1^2\}}{1^2.2^2}-\dots\right\}.]$$

† [This expression was arrived at in the course of the verification referred to in the previous note.]

in which $\Delta_0^2 F_{0-1} = F_{-1} - 2F_0 + F_1$,

$$\Delta_0^4 F_{0-2} = F_{-2} - 4F_{-1} + 6F_0 - 4F_1 + F_2, \text{ \&c.},$$

& $F_0 = 0$, $F_1 = 0$, $F_2 = 0$, ...; also $F_{-2} = 0$, $F_{-4} = 0$, ...; but $F_{-1} = F_{-3} = \text{\&c.} = 1$.

So that we ought to have the equations (because $\frac{1}{2}\sin t^2 = \frac{1}{4}(1 - \cos 2t)$),

$$+1 = -\Delta_0^2 F_{0-1} = F_{-1}, \quad -4 = \Delta_0^4 F_{0-2} = -4F_{-1},$$

$$+4^2 = \Delta_0^6 F_{0-3} = F_{-3} + \frac{6\,.\,5}{1\,.\,2} F_{-1},$$

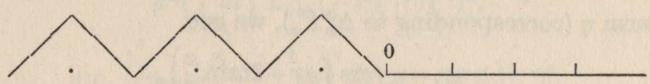
$$-4^3 = \Delta_0^8 F_{0-4} = -8F_{-3} - \frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3} F_{-1}, \text{ \&c.};$$

which are true. In general, we ought to have

$$4^{2l} = \Delta_0^{4l+2} F_{0-2l-1} = 1 + \frac{[4l+2]^2}{[2]^2} + \frac{[4l+2]^4}{[4]^4} + \ldots,$$

$$4^{2l+1} = (-1)^l \Delta_0^{4l+4} F_{0-2l-2} = \frac{[4l+4]^1}{[1]^1} + \frac{[4l+4]^3}{[3]^3} + \ldots,$$

& accordingly these series are the quarters of the developments of $(1+1)^{4l+2}$, $(1+1)^{4l+4}$.



(Jan. 26th, 1839.)

Had we supposed that $F_x = 1$ for *all* odd values of x, (positive as well as negative), being still $= 0$ for all even values of x, & f_x being still $= 0$ for *all* values of that variable, we should have just doubled the last found expression for $\eta_{0,t}$, & thereby found $\eta_{0,t} = \sin t^2$.

I think that we ought to have found $\eta_{0,t} = \cos t^2$, if we had supposed $f_x = 0$ for all values of x, & $F_x = 0$ for all odd but $= 1$ for all even values, the value 0 included.

In general, one integral of the equation in mixed differences is

$$\eta_{x,t} = \eta \operatorname{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right),$$

η & μ being arbitrary constants. In fact this expression gives

$$\Delta_x^2 \eta_{x-1,t} = \eta \left\{ 2\cos\left(\mu x - 2t\sin\frac{\mu}{2}\right) - \cos\left(\mu x + \mu - 2t\sin\frac{\mu}{2}\right) - \cos\left(\mu x - \mu - 2t\sin\frac{\mu}{2}\right) \right\}$$

$$= 2\eta \cos\left(\mu x - 2t\sin\frac{\mu}{2}\right) \operatorname{vers}\mu = D_t^2 \eta_{x,t}.$$

We ought ∵ to have $\eta_{x,t} =$ the above cited expression for *all* values of t, if it be so for two values infinitely near each other; ∵ if, for all values of x, we have

$$F_x = \eta \operatorname{vers} \mu x, \quad f_x = -2\eta \sin \mu x \sin\frac{\mu}{2}.$$

These conditions give

$$F_{x+i} = \eta - \eta \cos \mu x \cos \mu i + \eta \sin \mu x \sin \mu i,$$

$$f_{x+i} = -2\eta \sin\frac{\mu}{2} \sin \mu x \cos \mu i - 2\eta \sin\frac{\mu}{2} \cos \mu x \sin \mu i;$$

$$\because \quad \Delta_x^2 F_{x-1} = -\eta\cos\mu x\{\cos\mu - 2\cos 0 + \cos(-\mu)\} + \eta\sin\mu x\{\sin\mu - 2\sin 0 + \sin(-\mu)\}$$

$$= 4\eta\cos\mu x\left(\sin\frac{\mu}{2}\right)^2,$$

$$\Delta_x^4 F_{x-2} = -\eta\cos\mu x\{\cos 2\mu - 4\cos\mu + 6 - 4\cos(-\mu) + \cos(-2\mu)\}$$

$$= -\eta\cos\mu x\,(e^{2\mu\sqrt{-1}} - 4e^{\mu\sqrt{-1}} + 6 - 4e^{-\mu\sqrt{-1}} + e^{-2\mu\sqrt{-1}})$$

$$= -\eta\cos\mu x\,(e^{\frac{\mu}{2}\sqrt{-1}} - e^{-\frac{\mu}{2}\sqrt{-1}})^4 = -2^4\eta\cos\mu x\left(\sin\frac{\mu}{2}\right)^4,$$

$$\Delta_x^{2r} F_{x-r} = (-1)^{r+1}\,2^{2r}\,\eta\cos\mu x\left(\sin\frac{\mu}{2}\right)^{2r}, \quad (r>0),$$

$$\Delta_x^{2r} f_{x-r} = (-1)^{r+1}\,2^{2r+1}\,\eta\sin\mu x\left(\sin\frac{\mu}{2}\right)^{2r+1};$$

$$\because \quad \frac{t^{2r}}{1.2.3\ldots(2r)}\Delta_x^{2r}F_{x-r} = -\eta\cos\mu x\,(-1)^r\left(2t\sin\frac{\mu}{2}\right)^{2r}[0]^{-2r}, \quad (r>0),$$

$$\frac{t^{2r+1}}{1.2.3\ldots(2r+1)}\Delta_x^{2r}f_{x-r} = -\eta\sin\mu x\,(-1)^r\left(2t\sin\frac{\mu}{2}\right)^{2r+1}[0]^{-(2r+1)};$$

$\because$ summing these 2 last expressions from $r=0$ to $r=\infty$, then adding them to each other & to the additional term η (corresponding to $\Delta_x^0 F_x$), we get

$$\eta_{x,t} = \eta\,\mathrm{vers}\left(\mu x - 2t\sin\frac{\mu}{2}\right),$$

as I expected.

Resuming now the original problem of this book, let the expressions on the preceding page for F_x and f_x hold only for negative values of x, & for the value $x=0$; the functions F_x & f_x vanishing for $x=0$ or >0. And let me try to calculate the function $\eta_{0,t}$ by calculating the differences

$$\Delta_0^{2r}F_{0-r}, \quad \Delta_0^{2r}f_{0-r}. \quad \text{(We may suppose for simplicity } \eta=1.)$$

$$\Delta_0^{2r}F_{0-r} = F_{-r} - \frac{[2r]^1}{[1]^1}F_{-r+1} + \frac{[2r]^2}{[2]^2}F_{-r+2} - \ldots$$

$$= 1 - \frac{[2r]^1}{[1]^1} + \frac{[2r]^2}{[2]^2} - \ldots + \frac{(-1)^r}{2}\frac{[2r]^r}{[r]^r}$$

$$-1\cos r\mu + \frac{[2r]^1}{[1]^1}\cos(r\mu-\mu) - \frac{[2r]^2}{[2]^2}\cos(r\mu-2\mu) + \ldots - \frac{(-1)^r}{2}\frac{[2r]^r}{[r]^r}$$

$$= -\tfrac{1}{2}(e^{\frac{\mu}{2}\sqrt{-1}} - e^{-\frac{\mu}{2}\sqrt{-1}})^{2r} = (-1)^{r+1}\,2^{2r-1}\left(\sin\frac{\mu}{2}\right)^{2r}, \quad r>0;$$

& the part of $\eta_{0,t}$ corresponding to F (that is to the initial displacements) is simply

$$\tfrac{1}{2}\,\mathrm{vers}\left(2t\sin\frac{\mu}{2}\right).$$

The question then which remains to be solved is this: *How nearly will the initial velocities of the particles behind the origin produce the remaining half of this versed sine in the complete expression of the displacement $\eta_{0,t}$?*

$$\Delta_0^{2r}f_{0-r} = 2\sin\frac{\mu}{2}\left(\sin r\mu - \frac{[2r]^1}{[1]^1}\sin(r\mu-\mu) + \ldots + \frac{(-1)^{r-1}[2r]^{r-1}}{[r-1]^{r-1}}\sin\mu\right).$$

If we seek only the limit to which $\frac{1}{\mu^2}\Delta_0^{2r} f_{0-r}$ tends, as μ tends to 0, we have, for this limit, the expression

$$r-\frac{[2r]^1}{[1]^1}(r-1)+\frac{[2r]^2}{[2]^2}(r-2)-\ldots+\frac{(-1)^{r-1}[2r]^{r-1}}{[r-1]^{r-1}};$$

of which the values are

$$1;\quad 2-4=-2;\quad 3-6.2+15=+6;$$

$$4-8.3+28.2-56=-20;\quad 5-10.4+45.3-120.2+210=+70;$$

& the general expression seems to be

$$4^{r-1}[-\tfrac{1}{2}]^{r-1}[0]^{-(r-1)}=(-1)^{r-1}[2r-2]^{r-1}[0]^{-(r-1)}.$$

[A proof by induction of this expression is given.]

We have ∵ at present, by neglecting μ^4,

$$\Delta_0^{2r} f_{0-r}=\mu^2(-1)^{r-1}[2r-2]^{r-1}[0]^{-(r-1)};$$

which is to be multiplied by $t^{2r+1}[0]^{-(2r+1)}$ and summed from $r=1$ to $r=\infty$; & thus we get, by changing r to $r+1$ & summing from 0 to ∞,

$$\mu^2\left(\int_0^t dt\right)^3 \Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2;$$

that is,

$$\lim_{\mu=0}\frac{1}{\mu^2}\eta_{0,t}=\frac{t^2}{4}+\left(\int_0^t dt\right)^3 \Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2,$$

a result which agrees with former investigations in this book, & in which the part added to $\frac{t^2}{4}$ tends to become $=\frac{t^2}{4}$ as t increases indefinitely; or rather, perhaps, tends to become $=\frac{t^2}{4}-\frac{1}{16}$.*

Without neglecting any power of μ

$$\Delta_0^{2r+2} f_{0-(r+1)}=2\sin\frac{\mu}{2}\left\{\sin(r\mu+\mu)-\frac{[2r+2]^1}{[1]^1}\sin r\mu+\ldots\right\}$$

$$=2\sin\frac{\mu}{2}\{\sin(r\mu+\mu)-([2r]^1[0]^{-1}+2)\sin r\mu+([2r]^2[0]^{-2}+2[2r]^1[0]^{-1}+1)\sin(r\mu-\mu)$$

$$-([2r]^3[0]^{-3}+2[2r]^2[0]^{-2}+[2r]^1[0]^{-1})\sin(r\mu-2\mu)+\ldots$$

$$+(-1)^{r-1}([2r]^{r-1}[0]^{-(r-1)}+2[2r]^{r-2}[0]^{-(r-2)}+[2r]^{r-3}[0]^{-(r-3)})\sin 2\mu$$

$$+(-1)^r([2r]^r[0]^{-r}+2[2r]^{r-1}[0]^{-(r-1)}+[2r]^{r-2}[0]^{-(r-2)})\sin\mu\}$$

$$=2(-1)^r[2r]^r[0]^{-r}\sin\mu\sin\frac{\mu}{2}-\left(2\sin\frac{\mu}{2}\right)^3\{\sin r\mu-[2r]^1[0]^{-1}\sin(r\mu-\mu)$$

$$+[2r]^2[0]^{-2}\sin(r\mu-2\mu)-\ldots+(-1)^{r-1}[2r]^{r-1}[0]^{-(r-1)}\sin\mu\}$$

$$=-\left(2\sin\frac{\mu}{2}\right)^2\Delta_0^{2r} f_{0-r}+2(-1)^r[2r]^r[0]^{-r}\sin\mu\sin\frac{\mu}{2}.$$

If then we consider $\Delta_0^{2r} f_{0-r}$ as a function of r, & denote it by $R_r\left(2\sin\frac{\mu}{2}\right)^{2r}$,

$$R_{r+1}+R_r=\cos\frac{\mu}{2}(-1)^r[2r]^r[0]^{-r}\left(2\sin\frac{\mu}{2}\right)^{-2r},$$

* [See p. 539.]

which is an ordinary equation in differences of the 1st order and 1st degree, with constant coefficients, but with a variable term. It gives

$$-(R_r+R_{r-1})=\cos\frac{\mu}{2}(-1)^r[2r-2]^{r-1}[0]^{-(r-1)}\left(2\sin\frac{\mu}{2}\right)^{-2(r-1)},$$

& also $R_0=0$,

$$\because\quad \Delta_0^{2r+2}f_{0-(r+1)}=\left(2\sin\frac{\mu}{2}\right)^2\cos\frac{\mu}{2}(-1)^r\left\{[2r]^r[0]^{-r}+[2r-2]^{r-1}[0]^{-(r-1)}\left(2\sin\frac{\mu}{2}\right)^2\right.$$
$$\left.+[2r-4]^{r-2}[0]^{-(r-2)}\left(2\sin\frac{\mu}{2}\right)^4+\ldots+[2]^1[0]^{-1}\left(2\sin\frac{\mu}{2}\right)^{2r-2}+\left(2\sin\frac{\mu}{2}\right)^{2r}\right\}.$$

Multiplying this by $t^{2r+3}[0]^{-(2r+3)}$ & summing relatively to r from 0 to ∞, & then adding the result to

$$\tfrac{1}{2}\operatorname{vers}\left(2t\sin\frac{\mu}{2}\right),$$

we get

$$\eta_{0,t}=\tfrac{1}{2}\operatorname{vers}\left(2t\sin\frac{\mu}{2}\right)$$
$$+\cos\frac{\mu}{2}\left(2\sin\frac{\mu}{2}\right)^2\left(\int_0^t dt\right)^3\left\{1-\left(2\sin\frac{\mu}{2}\right)^2\left(\int_0^t dt\right)^2+\left(2\sin\frac{\mu}{2}\right)^4\left(\int_0^t dt\right)^4-\ldots\right\}\Sigma_{(r)0}^{\infty}(-t^2)^r([0]^{-r})^2;$$

a result agreeing with those obtained at an earlier stage of these investigations.*

(Jan. 31st, 1839.)

[Hamilton identifies this result with]

$$\eta_{0,v/m\sqrt{2}}-\tfrac{1}{2}\operatorname{vers}v=\frac{1}{\pi}\int_0^{\frac{\sqrt{2}}{m}}\frac{\sin pv-p\sin v}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,\frac{dp}{p},$$

$v=mt\sqrt{2}$, $m=\sqrt{2}\sin\dfrac{\mu}{2}$.

The 2nd member of this equation is rigorously an expression for the part of $\eta_{0,t}$ which depends on the initial velocities; & it may be still more simply written thus:

$$\frac{1}{\pi}\int_0^{\frac{\sqrt{2}}{m}}\frac{\sin pv}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}\,\frac{dp}{p};$$

because we have, exactly,

$$0=\int_0^{\frac{\sqrt{2}}{m}}\frac{dp}{1-p^2}\sqrt{\frac{1-\frac{1}{2}m^2}{1-\frac{1}{2}m^2p^2}}.\quad\text{[Cauchy's Principal Value.]}$$

At the same time we see that we may write

$$\eta_{0,t}-\tfrac{1}{2}\operatorname{vers}v=\frac{1}{\pi}\int_0^{\infty}\frac{\sin pv}{p}\,\frac{d\varpi}{1-\varpi^2},$$

if $\varpi^{-2}(1-\frac{1}{2}m^2)=p^{-2}-\frac{1}{2}m^2$, or $p^{-2}=\frac{1}{2}m^2+(1-\frac{1}{2}m^2)\varpi^{-2}$,

that is, $p=\varpi\{1-\frac{1}{2}m^2(1-\varpi^2)\}^{-\frac{1}{2}}$.

* [See p. 540.]

Or we may write

$$\boxed{\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \frac{1}{\pi}\int_0^1 \left(\frac{\sin pv}{p} - \frac{\sin qv}{q}\right)\frac{d\varpi}{1-\varpi^2},}$$

in which

$$p = \varpi\{1 - \tfrac{1}{2}m^2(1-\varpi^2)\}^{-\frac{1}{2}}, \quad q = \varpi^{-1}\{1 - \tfrac{1}{2}m^2(1-\varpi^{-2})\}^{-\frac{1}{2}}.$$

If $v = mt\sqrt{2} = 2t\sin\frac{\mu}{2}$ be large and if m be $<\sqrt{2}$, we need only retain those parts of the integral which have small divisors; but p increases from 0 to 1, & q decreases from $\frac{\sqrt{2}}{m}$ to 1; also when ϖ is nearly 0, we have nearly $\varpi = \sqrt{1-\frac{1}{2}m^2}\,p$; and when ϖ is nearly 1, we have nearly

$$p = 1 - (1-\tfrac{1}{2}m^2)(1-\varpi), \quad q = 1 + (1-\tfrac{1}{2}m^2)(1-\varpi);$$

also $$\frac{\sin pv}{p} - \frac{\sin qv}{q} = \sin pv - \sin qv = -2\cos v \sin\{v(1-\tfrac{1}{2}m^2)(1-\varpi)\} \text{ nearly};$$

$\because$, v being large, we have nearly

$$\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \frac{1}{\pi}\left\{\sqrt{1-\tfrac{1}{2}m^2}\int_0^\infty \frac{\sin pv}{p}\,dp - \cos v\int_0^\infty \frac{\sin p'v}{p'}\,dp'\right\},$$

if $p' = (1-\frac{1}{2}m^2)(1-\varpi)$; $\because$ finally, we have the following approximate expression for the part of $\eta_{0,t}$ which depends on the initial velocities:

$$\eta_{0,t} - \tfrac{1}{2}\operatorname{vers} v = \tfrac{1}{2}\{\sqrt{1-\tfrac{1}{2}m^2} - \cos v\};$$

in which $v = m\tau = 2t\sin\frac{\mu}{2}$ = a large positive number.

The whole expression for the displacement of the particle 0 at the time t is therefore

$$\eta_{0,t} = \left(\cos\frac{\mu}{4}\right)^2 - \cos\left(2t\sin\frac{\mu}{2}\right) = \operatorname{vers}\left(2t\sin\frac{\mu}{2}\right) - \left(\sin\frac{\mu}{4}\right)^2,$$

if $t\sin\frac{\mu}{2}$ be large.

[By a lengthy process of summation the following result is arrived at.]

The expression

$$\boxed{\eta_{x,t} = \tfrac{1}{2}\operatorname{vers}\left(\mu x - 2t\sin\frac{\mu}{2}\right) + \frac{\operatorname{vers}\mu}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos\frac{\mu}{2}\cos 2x\theta\sin(2t\sin\theta) - \cos\theta\sin 2x\theta\cos(2t\sin\theta)}{\sin\theta(\cos 2\theta - \cos\mu)}\,d\theta}$$

contains, *for all values of x and t*, the *rigorous solution of the problem* proposed at the commencement of this manuscript; namely to find a function $\eta_{x,t}$ such that it shall satisfy generally the equation in mixed differences

$$D_t^2\eta_{x,t} = \Delta_x^2\eta_{x-1,t},$$

and also the initial conditions

$$\eta_{x,0} = \operatorname{vers}\mu x \quad \text{or} \quad = 0,$$

according as $x \not> 0$ or > 0, and

$$D_0 \eta_{x,0} = -2 \sin \frac{\mu}{2} \sin \mu x \quad \text{or} \quad = 0,$$

according as $x \not> 0$ or > 0.

For a complete *à posteriori* proof of the correctness of the solution, it is sufficient to prove that*

1st: $$\frac{\text{vers}\,\mu}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin 2x\theta \, d\theta}{\sin\theta (\cos 2\theta - \cos\mu)} = \pm \tfrac{1}{2} \text{vers}\,\mu x, \text{ according as } x \text{ is } \gtrless 0;$$

2nd: $$\frac{\sin\mu}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos 2x\theta \, d\theta}{\cos 2\theta - \cos\mu} = \pm \tfrac{1}{2} \sin \mu x, \text{ according as } x \text{ is } \gtrless 0;$$

3rd: the latter integral, like the former, vanishes with x.

And whatever x may be, we have, when t is large and positive,

$$\eta_{x,t} = \text{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right) - \left(\sin\frac{\mu}{4}\right)^2;\dagger$$

but when t is large and negative, $\eta_{x,t} = \left(\sin\frac{\mu}{4}\right)^2$.

[*Discussion of asymptotic value of this integral.*]

(Feb. 5th.)

The expression for $\eta_{x,t}$ on the previous page can be written

$$\eta_{x,t} = \tfrac{1}{2} \text{vers}\left(\mu x - 2t \sin\frac{\mu}{2}\right) - \frac{\text{vers}\,\mu}{4\pi} \int_0^{\pi} \frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta \left(\cos\theta - \cos\frac{\mu}{2}\right)} d\theta,$$

or this more elegant one $\left(\text{in which } \nu = \frac{\mu}{2}\right)$,

$$\boxed{\eta_{x,t} = \sin(x\nu - t\sin\nu)^2 - \frac{\sin\nu^2}{2\pi} \int_0^{\pi} \frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta(\cos\theta - \cos\nu)} d\theta.}$$

This is a *rigorous* form for that particular integral of the equation

$$\left(D_t^2 - \frac{\Delta_x^2}{1+\Delta_x}\right) \eta_{x,t} = 0,$$

which gives $\eta_{x,0} = 2(\sin x\nu)^2$ or $= 0$, and $D_0 \eta_{x,0} = -2 \sin\nu \sin 2x\nu$ or $= 0$, according as $x \not> 0$ or > 0.

* [The verification here suggested is similar to that on pages 469–472.]

† [The integral may be written

$$\int_0^{\frac{\pi}{2}} \frac{\left(\cos\frac{\mu}{2} - \cos\theta\right) \cos 2x\theta \sin(2t\sin\theta)\, d\theta}{\sin\theta(\cos 2\theta - \cos\mu)} + \int_0^{\frac{\pi}{2}} \frac{\cos\theta \sin(2t\sin\theta - 2x\theta)\, d\theta}{\sin\theta(\cos 2\theta - \cos\mu)}.$$

The value of the first part when $t \to \infty$ is $\dfrac{-\pi\left(\sin\frac{\mu}{4}\right)^2}{\text{vers}\,\mu}$, the asymptotic value of the second can be obtained by taking the complex integral $\displaystyle\int \frac{e^{2itu - 2ix\sin^{-1}u}\, du}{u(u^2 - \sin^2\frac{1}{2}\mu)}$ about the contour $(\sin\frac{1}{2}\mu, 0)$, $(-\sin\frac{1}{2}\mu, 0)$, $(\sin\frac{1}{2}\mu, i\infty)$, $(-\sin\frac{1}{2}\mu, i\infty)$ with indentations at $u = \sin\frac{1}{2}\mu, 0, -\sin\frac{1}{2}\mu$.]

We shall suppose that ν is >0 but $<\frac{\pi}{2}$; and that t and x are large positive numbers, x being also integer. And on these suppositions we shall seek the *approximate* value of the definite integral in $\eta_{x,t}$.

Because x and t are large, the arc $2x\theta - 2t\sin\theta$ will in general vary much & rapidly while θ varies from 0 to π; & ∵ the sine of this arc will fluctuate often & rapidly between the finite limits ± 1, while θ varies so. Hence we need only consider those particular values of θ for which this sine varies less rapidly than usual, or is divided by an unusually small quantity. But the rate of variation of the arc is expressed by $2(x - t\cos\theta)$; if then we at first suppose that x is $>t$, (corresponding to the case of *darkness*,) we may omit the consideration of the case of slow variation, & may confine ourselves to the cases of $\theta =$ nearly 0 or ν or π.

When θ is nearly 0, x being $>t$, we get $-\frac{\sin\nu^2}{4\,\mathrm{vers}\,\nu} = -\frac{1}{4}(1+\cos\nu)$ as the corresponding part of the integral. When θ is nearly π, we get the part $-\frac{\sin\nu^2}{4(1+\cos\nu)} = -\frac{1}{4}(1-\cos\nu)$; so that the sum of those two parts, corresponding to the divisor $\sin\theta$, is $-\frac{1}{2}$. When θ is nearly ν, the arc $2x\nu - 2t\sin\nu + 2x(\theta-\nu) - 2t(\sin\theta - \sin\nu) =$ nearly

$$2x\nu - 2t\sin\nu + 2(x - t\cos\nu)(\theta-\nu);$$

also the small divisor $\cos\theta - \cos\nu =$ nearly $-\sin\nu(\theta-\nu)$; & the corresponding part of the integral is $\frac{1}{2}\cos(2x\nu - 2t\sin\nu)$. Thus the whole integral is nearly

$$= -\tfrac{1}{2}\,\mathrm{vers}\,(2x\nu - 2t\sin\nu) = -\sin(x\nu - t\sin\nu)^2;$$

and $\eta_{x,t} = 0$, if x be large and positive, & $>t$.

Next let $x<t$, but let t be large and positive, & let x be also >0. The parts corresponding to $\theta =$ nearly 0 & π are now $\frac{1}{4}(\pm 1 + \cos\nu)$, & their sum is $\frac{1}{2}\cos\nu$. The part corresponding to $\theta =$ nearly ν is, if $x \lesseqgtr t\cos\nu$, $\mp\frac{1}{2}\cos(2x\nu - 2t\sin\nu)$. What is the part corresponding to $\cos\theta =$ nearly $\frac{x}{t}$? It is insensible.

(Feb. 6th.)

But it is important to observe that the foregoing calculations suppose that x differs from t & from $t\cos\nu$ by *large* quantities, positive or negative. When x is *nearly equal* to t or to $t\cos\nu$, we must employ some new considerations.

If $x = t\cos\nu$, then *rigorously*

$$\eta_{x,t} = \eta_{t\cos\nu,t} = \sin t(\nu\cos\nu - \sin\nu)^2 - \frac{\sin\nu^2}{2\pi}\int_0^\pi \frac{\sin 2t(\theta\cos\nu - \sin\theta)}{\sin\theta(\cos\theta - \cos\nu)}d\theta;$$

& when θ is nearly $=\nu$, we have

$$\theta\cos\nu - \sin\theta = \nu\cos\nu - \sin\nu + (\theta-\nu)^2\frac{\sin\nu}{2} \text{ nearly};$$

$$\because \quad 2t(\theta\cos\nu-\sin\theta)=2t(\nu\cos\nu-\sin\nu)+t\sin\nu\,(\theta-\nu)^2 \text{ nearly};$$

$\theta-\nu=\sqrt{\dfrac{\delta\phi}{t\sin\nu}}$ nearly, if $\phi=2t(\nu\cos\nu-\sin\nu)$ and $\phi+\delta\phi=2t(\theta\cos\nu-\sin\theta)$; also

$$\cos\theta-\cos\nu=-\sqrt{\frac{\sin\nu\,\delta\phi}{t}} \text{ nearly, and } d\theta=\frac{d\delta\phi}{2\sqrt{t\sin\nu\,\delta\phi}};$$

$$\therefore \quad \frac{d\theta}{\cos\theta-\cos\nu}=-\frac{d\delta\phi}{2\sin\nu\,\delta\phi};$$

but to a given positive value of $\delta\phi$ correspond two opposite values of $\theta-\nu$; $\because$ the corresponding part of $\eta_{t\cos\nu,t}$ is 0; whereas the corresponding part of $\eta_{x,t}$ is $\mp\frac{1}{2}\cos 2(x\nu-t\sin\nu)$, if x be considerably less or considerably more than $t\cos\nu$. And if we take in the connected periodic part outside the integral sign, we have

$$-\cos(2x\nu-2t\sin\nu), \quad -\tfrac{1}{2}\cos(2t\nu\cos\nu-2t\sin\nu), \quad \text{or } 0,$$

according as x is sensibly less than, or exactly equal to, or sensibly greater than $t\cos\nu$.

It seems then that we may regard the *velocity of propagation* as being $=\cos\nu$, and *not* as being $=\dfrac{\sin\nu}{\nu}$.

This distinction between the velocity of propagation of a wave, and the ratio of the space-period to the time-period of its vibrations, appears to me to be entirely new; and to be one of the most curious results hitherto obtained, by introducing the consideration of finite intervals.

Let $\nu=\dfrac{\pi}{n}$, n being an integer number, expressing the number of molecular intervals contained in the length of the wave. And let only i such lengths behind the origin of x be disturbed at the origin of t. We meet these suppositions by changing x in the recent $\eta_{x,t}$ to $x+in$, & subtracting the new expression from the old. *We therefore obtain*

$$\boxed{y_{x,t}=\frac{1}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin in\theta}{\sin\theta}\,\frac{\cos(2x\theta+in\theta-2t\sin\theta)}{\cos\theta-\cos\dfrac{\pi}{n}}\,d\theta;}$$

as the rigorous expression for a function which satisfies the equation in mixed differences

$$\left(D_t^2-\frac{\Delta_x^2}{1+\Delta_x}\right)y_{x,t}=0,$$

and also the initial conditions

$$y_{x,0}=\operatorname{vers}\frac{2x\pi}{n}, \text{ or } =0,$$

and $$D_0 y_{x,0}=-2\sin\frac{\pi}{n}\sin\frac{2x\pi}{n}, \text{ or } =0,$$

according as x does or does not satisfy the conditions

$$x \not> 0,\ x \not< -in.$$

[Here follows a verification.]

Let us consider the cases which correspond to x and t being both large and positive, & inquire, for these cases, the *approximate* values of $y_{x,t}$.

In this research it is useful to remember that we may write, *rigorously*,

$$y_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin(2x\theta+2in\theta-2t\sin\theta)}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}d\theta-\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin(2x\theta-2t\sin\theta)}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}d\theta.$$

It is ∵ sufficient to study the latter of these 2 integrals; which was indeed considered in part before, (see pages 554, 555,) but shall now be considered anew. For simplicity we shall write the above equation in the form

$$y_{x,t}=z_{x+in,t}-z_{x,t}.$$

Now, because x and t are large, $\sin(2x\theta-2t\sin\theta)$ fluctuates often and rapidly between the finite limits ± 1, while θ varies from 0 to π; & these rapid & repeated fluctuations destroy sensibly all those parts of the integral $z_{x,t}$ which are not rendered sensible by having small divisors. We need therefore attend only, in the calculation of $z_{x,t}$, to those values of θ which are nearly 0 or π or $\frac{\pi}{n}$. And the chief theorem for all these values is that

$$\int_0^\infty \frac{\sin\psi\, d\psi}{\psi}=\frac{\pi}{2}.$$

When θ is nearly 0, we have

$$\frac{\sin\theta}{2x\theta-2t\sin\theta}=\frac{1}{2(x-t)}\text{ nearly};$$

this relation becomes more and more exact as x and t increase, while the denominator of the fraction remains constant; because, under these conditions, θ diminishes more and more & tends to 0. At the same time

$$\cos\theta-\cos\frac{\pi}{n}\text{ tends to }\operatorname{vers}\frac{\pi}{n};\text{ and }\frac{d\theta}{d\psi}\text{ tends to }\frac{1}{2(x-t)},\text{ if }\psi=2x\theta-2t\sin\theta.$$

Thus the corresponding part of the element of $z_{x,t}$ tends to this limit:

$$\frac{1}{2\pi}\left(1+\cos\frac{\pi}{n}\right)\frac{\sin\psi\, d\psi}{\psi},$$

at least if $x-t$ be considerably $>$ or <0.

It must also be observed that while θ increases from 0 to $+$, that is to a small positive quantity, ψ is to be considered as increasing or diminishing from 0 to a very large positive or negative quantity, according as $x-t$ is $>$ or <0; $x-t$ being supposed to be always large.

In the first of these 2 last mentioned cases, we get ∵, for the corresponding part of the integral $z_{x,t}$,

$$+\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right);\qquad (x>t);$$

and, in the 2nd case,

$$-\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right);\qquad (x<t);$$

which may be explained by saying that while $\frac{d\psi}{\psi}$ is in both cases positive, the factor $\sin\psi$ is in the first case positive, but in the second case negative.

If x were exactly $=t$, we should have, rigorously,

$$z_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin 2t(\theta-\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)};$$

&, nearly, for the part which corresponds to θ nearly $=0$,

$$\psi=2t(\theta-\sin\theta)=\frac{t\theta^3}{3},\quad d\psi=t\theta^2 d\theta,\quad \frac{d\theta}{\sin\theta}=\frac{d\theta}{\theta}=\frac{d\psi}{3\psi};$$

and the part would be $\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$. Such are the parts which correspond to values of θ a little greater than 0.

Supposing next that θ is only a little less than π; we have, nearly,

$$\frac{2x(\pi-\theta)+2t\sin(\pi-\theta)}{\sin(\pi-\theta)}=2(x+t),$$

& this relation becomes more and more exact as $x+t$ increases, the arc

$$\psi=2x(\pi-\theta)+2t\sin(\pi-\theta)$$

remaining constant; because under these conditions the arc $\pi-\theta$ tends indefinitely to 0.

Also $-\frac{d\theta}{d\psi}$ tends to be $=\frac{+1}{2(x+t)}$, being rigorously $=\frac{+1}{2(x-t\cos\theta)}$; therefore, since $x+t$ is large & >0, the part of $z_{x,t}$ which is now under consideration is $+\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)$.

Thus the sum of these 2 first parts of the integral $z_{x,t}$ is

$+\frac{1}{2}$, if x be much $>t$;

$-\frac{1}{2}\cos\frac{\pi}{n}$, if x be much $<t$;

and $\frac{1}{6}\left(2-\cos\frac{\pi}{n}\right)$, if x be $=t$;

t and x being both large & positive.

Supposing finally that θ is nearly $=\frac{\pi}{n}$, & putting

$$\psi=2x\theta-2t\sin\theta-\phi,\quad \phi=\frac{2x\pi}{n}-2t\sin\frac{\pi}{n},$$

we have nearly $\psi=2\left(\theta-\frac{\pi}{n}\right)\left(x-t\cos\frac{\pi}{n}\right)+t\sin\frac{\pi}{n}\left(\theta-\frac{\pi}{n}\right)^2;$

and if $x-t\cos\frac{\pi}{n}$ be large, we may neglect the 2nd term.

In this manner we get, for the 3rd part of $z_{x,t}$,

$$\mp\frac{\cos\phi}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\psi\,d\psi}{\psi}=\mp\frac{\cos\phi}{2}=\mp\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

according as x is much $>$ or much $<$ $t\cos\frac{\pi}{n}$. If $x=t\cos\frac{\pi}{n}$, then

$$\theta-\frac{\pi}{n}=t^{-\frac{1}{2}}\sqrt{\frac{\psi}{\sin\frac{\pi}{n}}};\quad \frac{d\theta}{\theta-\frac{\pi}{n}}=\frac{d\psi}{2\psi};\quad \phi=2t\left(\frac{\pi}{n}\cos\frac{\pi}{n}-\sin\frac{\pi}{n}\right);$$

to each value of $\phi+\psi=2x\theta-2t\sin\theta$ correspond now two values of $\theta-\frac{\pi}{n}$ which are nearly equal and opposite; ∵ the corresponding part of $z_{x,t}$ is insensible.*

If then $\cos\frac{\pi}{n}$ be sensibly less than 1, so that $t\cos\frac{\pi}{n}$ may be regarded as considerably less than t, when t is large and positive; if x be also large and positive; we have the following approximate values for the integral $z_{x,t}$:

$$z_{x,t}=\ -\tfrac{1}{2}\cos\frac{\pi}{n}+\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),\qquad \text{if } x \text{ be much } <t\cos\frac{\pi}{n};$$

$$-\tfrac{1}{2}\cos\frac{\pi}{n},\qquad \text{if } x=t\cos\frac{\pi}{n};$$

$$-\tfrac{1}{2}\cos\frac{\pi}{n}-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),\qquad \text{if } x \text{ be much } >t\cos\frac{\pi}{n} \text{ but } <t;$$

$$\tfrac{1}{2}-\tfrac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),\quad \text{if } x=t;$$

$$\tfrac{1}{2}-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),\qquad \text{if } x \text{ be much } >t;$$

and consequently, under the same 5 different suppositions, the function

$$\eta_{x,t}=\tfrac{1}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-z_{x,t}$$

takes the 5 following values:

$$\left(\cos\frac{\pi}{2n}\right)^2-\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)=\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-\left(\sin\frac{\pi}{2n}\right)^2;$$

$$\left(\cos\frac{\pi}{2n}\right)^2-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right);\quad \left(\cos\frac{\pi}{2n}\right)^2;\quad \tfrac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2;\quad 0.$$

What would happen if $x-t\cos\frac{\pi}{n}$ were of the same order as $t^{\frac{1}{2}}$? Let $x-t\cos\frac{\pi}{n}=\xi\sqrt{t}$; then, by the last page,

$$\psi=2\xi t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)+\sin\frac{\pi}{n}.t\left(\theta-\frac{\pi}{n}\right)^2;$$

∵ if ξ be positive and a good deal larger than $\sqrt{\tfrac{1}{4}\sin\frac{\pi}{n}}$, we shall have, nearly,

$$t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)=\frac{\psi}{2\xi}\left(1-\frac{\psi}{4\xi^2}\sin\frac{\pi}{n}\right);$$

or, more accurately,

$$\psi\sin\frac{\pi}{n}+\xi^2=\left\{\xi+\sin\frac{\pi}{n}.t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)\right\}^2,$$

$$∵\qquad t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)\sin\frac{\pi}{n}=\xi\left\{-1+\sqrt{1+\frac{\psi}{\xi^2}\sin\frac{\pi}{n}}\right\},$$

* [See p. 555.]

that is,

$$t\left(\theta-\frac{\pi}{n}\right)\sin\frac{\pi}{n}=\left(x-t\cos\frac{\pi}{n}\right)\left\{-1+\sqrt{1+\frac{t\psi\sin\frac{\pi}{n}}{\left(x-t\cos\frac{\pi}{n}\right)^2}}\right\};$$

$$\theta-\frac{\pi}{n}=\frac{1}{t\sin\frac{\pi}{n}}\left\{-\left(x-t\cos\frac{\pi}{n}\right)+\sqrt{\left(x-t\cos\frac{\pi}{n}\right)^2+t\sin\frac{\pi}{n}.\psi}\right\}$$

$$=\text{(by hypothesis) a quantity of the same order as }\frac{1}{\sqrt{t}};$$

we may ∵ neglect its square in the denominator of the element of $z_{x,t}$; & thus we have to calculate

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\phi+\psi)\,d\,.\,t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)}{t^{\frac{1}{2}}\left(\theta-\frac{\pi}{n}\right)}$$

$$=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\left(\phi+2\xi\theta'+\sin\frac{\pi}{n}\theta'^2\right)d\theta'}{\theta'}=-\frac{1}{\pi}\int_0^{\infty}\frac{\cos\left(\phi+\theta'^2\sin\frac{\pi}{n}\right)\sin 2\xi\theta'\,d\theta'}{\theta'};$$

which tends to $-\frac{\cos\phi}{2}$, when ξ, being positive, tends to become infinitely greater than $\sin\frac{\pi}{n}$.

Thus by taking $\frac{x}{t}=\cos\frac{\pi}{n}+\frac{\xi}{\sqrt{t}}$, in which ξ is a large positive constant number, we shall have with a given, and great, degree of accuracy:

$$z_{x,t}=-\tfrac{1}{2}\cos\frac{\pi}{n}-\tfrac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

$$\eta_{x,t}=\left(\cos\frac{\pi}{n}\right)^2;$$

so that we are justified in considering $\cos\frac{\pi}{n}$ as the *velocity of propagation.*

If ξ were much less than $\frac{1}{2}\sqrt{\sin\frac{\pi}{n}}$, or more precisely if $\theta'^2\sin\frac{\pi}{n}$ increased much more rapidly than $(2\xi\theta')^2$, so that the ratio $\frac{1}{4}\xi^{-2}\sin\frac{\pi}{n}=s'^{}s^{-2}$ is large, s being a number such that $\int_0^s\theta^{-1}\sin\theta\,d\theta$ is nearly $=\frac{\pi}{2}$; then, for the integral

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\cos(\phi+s^{-2}s'\psi'^2)\sin\psi'd\psi'}{\psi'},$$

in which $\psi'=2\xi\theta'$, we cannot correctly substitute

$$-\frac{1}{\pi}\int_0^s\frac{\cos\phi\sin\psi'}{\psi'}d\psi'=-\frac{\cos\phi}{2},$$

but are rather to substitute, as the opposite limit,

$$-\frac{1}{\pi}\int_0^\infty \cos\left(\phi+\frac{s^\backprime}{s^2}\psi^{\backprime 2}\right)d\psi^\backprime;$$

in which we are to remember that $s^\backprime s^{-2}$ is now large. It is remarkable that this last limit, if it do not vanish, must be proportional to $s\,(s^\backprime)^{-\frac{1}{2}}$; for it may be written thus:

$$-\frac{1}{\pi}\frac{s}{\sqrt{s^\backprime}}\int_0^\infty \cos\left(\phi+\psi^{\backprime\backprime 2}\right)d\psi^{\backprime\backprime};$$

in which $\psi^{\backprime\backprime}=\frac{\sqrt{s^\backprime}}{s}\psi^\backprime$. Also $\frac{s}{\sqrt{s^\backprime}}=\frac{2\xi}{\sin\frac{\pi}{n}}$, which we have supposed to be small; it is

$$=\frac{2\left(x-t\cos\frac{\pi}{n}\right)}{\sqrt{t\sin\frac{\pi}{n}}}.$$

Cauchy gives in his Memoir on Definite and Singular Integrals (*Mém. Sav. Etr.*, Tome I)

$$\int_0^\infty \cos(\psi^2)\,d\psi=\int_0^\infty \sin(\psi^2)\,d\psi=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}.$$

We have then $$-\frac{1}{\pi}\frac{s}{\sqrt{s^\backprime}}\int_0^\infty \cos\left(\phi+\psi^{\backprime\backprime 2}\right)d\psi^{\backprime\backprime}=\frac{x-t\cos\frac{\pi}{n}}{\sqrt{\pi t\sin\frac{\pi}{n}}}\sin\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}-\frac{\pi}{4}\right).$$

There is something here analogous to a *change of phase*.*

(Feb. 7th.)

Admitting that $$\int_0^\infty \cos(\psi^2)\,d\psi=\int_0^\infty \sin(\psi^2)\,d\psi=\frac{1}{2}\sqrt{\frac{\pi}{2}},$$

we have, as Cauchy remarks,

$$\frac{1}{2}\sqrt{\frac{\pi}{2}}=\frac{1}{2}\int_{-\infty}^\infty \cos(\psi^2)\,d\psi=\frac{1}{4}\int_{-\infty}^\infty \cos(\overline{\psi+\varpi}^2)\,d\psi+\frac{1}{4}\int_{-\infty}^\infty \cos(\overline{\psi-\varpi}^2)\,d\psi$$

$$=\frac{1}{4}\int_{-\infty}^\infty \{\cos(\psi^2+\varpi^2+2\varpi\psi)+\cos(\psi^2+\varpi^2-2\varpi\psi)\}\,d\psi$$

$$=\frac{1}{2}\int_{-\infty}^\infty \cos(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi=\int_0^\infty \cos(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi;$$

& for the same reason $$\frac{1}{2}\sqrt{\frac{\pi}{2}}=\int_0^\infty \sin(\psi^2+\varpi^2)\cos(2\varpi\psi)\,d\psi;$$

$\therefore$ $$\int_0^\infty \cos(\psi^2)\cos(2\varpi\psi)\,d\psi=\frac{\sqrt{\pi}}{2}\cos\left(\varpi^2-\frac{\pi}{4}\right);$$

$$\int_0^\infty \sin(\psi^2)\cos(2\varpi\psi)\,d\psi=\frac{\sqrt{\pi}}{2}\cos\left(\varpi^2+\frac{\pi}{4}\right).$$

* [Cf. Kelvin, *Proc. Roy. Soc.* Vol. XLII (1887), p. 80.]

Hence I deduce, by integration relatively to ϖ,

$$\int_0^\infty \frac{\cos(\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \sqrt{\pi}\int_0^\varpi \cos\left(\varpi^2 - \frac{\pi}{4}\right)d\varpi;$$

$$\int_0^\infty \frac{\sin(\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \sqrt{\pi}\int_0^\varpi \cos\left(\varpi^2 + \frac{\pi}{4}\right)d\varpi.$$

Hence $$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\cos(\phi+\psi^2)\sin(2\varpi\psi)}{\psi}\,d\psi = \int_0^\varpi \cos\left(\phi + \frac{\pi}{4} - \varpi^2\right)d\varpi;$$

which, when ϖ is very small, is nearly $=\varpi\cos\left(\phi+\frac{\pi}{4}\right)$; but, when ϖ is very large & positive, is nearly

$$=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\cos\left(\phi+\frac{\pi}{4}\right)+\sin\left(\phi+\frac{\pi}{4}\right)\right\} = \tfrac{1}{2}\sqrt{\pi}\cos\phi;$$

& these two extreme values agree with those lately obtained.

To calculate $\because$

$$-\frac{1}{\pi}\int_0^\infty \frac{\cos(\phi + s^{\backprime}s^{-2}\psi^{\backprime 2})\sin\psi^{\backprime}d\psi^{\backprime}}{\psi^{\backprime}},$$

we may put, as on the last page,

$$\psi^{\backprime} = \frac{s}{\sqrt{s^{\backprime}}}\psi^{\backprime\backprime} = 2\varpi\psi^{\backprime\backprime},$$

& we get

$$-\frac{1}{\pi}\int_0^\infty \frac{\cos(\phi + s^{\backprime}s^{-2}\psi^{\backprime 2})}{\psi^{\backprime}}\sin\psi^{\backprime}d\psi^{\backprime} = -\frac{1}{\sqrt{\pi}}\int_0^{\frac{s}{2\sqrt{s^{\backprime}}}} \cos\left(\phi+\frac{\pi}{4}-\varpi^2\right)d\varpi$$

$$= -\frac{1}{\sqrt{\pi}}\int_0^{\frac{x - t\cos\frac{\pi}{n}}{\sqrt{t\sin\frac{\pi}{n}}}} \cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n} + \frac{\pi}{4} - \varpi^2\right)d\varpi.$$

We may even extend this expression to *all values of* x, t being large and positive; for as soon as $x - t\cos\frac{\pi}{n}$ is of the same order as t, whether positive or negative, it becomes sensibly

$$= \mp\frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n} + \frac{\pi}{4} - \varpi^2\right)d\varpi = \mp\tfrac{1}{2}\cos\left(\frac{2x\pi}{n} - 2t\sin\frac{\pi}{n}\right).*$$

* [It is to be observed that the fundamental integral, $z_{x,t}$, written as a usual integral, is $\left(\text{putting } \alpha = \frac{\pi}{n}\right)$

$$\frac{1}{2\pi}(\sin\alpha)^2\int_0^\pi \left\{\frac{\sin(2x\theta - 2t\sin\theta)}{\sin\theta} - \frac{\sin(2x\alpha - 2t\sin\alpha)}{\sin\alpha}\right\}\frac{d\theta}{\cos\theta - \cos\alpha}.$$

For values of θ nearly equal to α this becomes

$$\frac{\cos(2x\alpha - 2t\sin\alpha)}{2\pi}\int_0^\pi \frac{\sin\{x(\theta-\alpha) - t(\sin\theta - \sin\alpha)\}\,d\theta}{\sin\frac{1}{2}(\theta-\alpha)}.]$$

[*Airy's integral.*]

Can we find any analogous general expression for the part of the same integral $[z_{x,t}]$, which corresponds to values of θ nearly $=0$?

For such values, $\psi = 2x\theta - 2t\sin\theta = 2(x-t)\theta + \frac{t}{3}\theta^3 \pm$ terms which may be neglected in comparison with these, even if $\frac{x-t}{t}$ be small; because θ is to be at largest of the same order as $\sqrt[3]{\frac{s}{t}}$, in which s is a large but constant number, independent of t, & such that $\int_0^s \frac{\sin\psi}{\psi}d\psi$ is sufficiently near to $\frac{\pi}{2}$ for the purposes of our approximation. For the same reason, we may treat the denominator $\sin\theta\left(\cos\theta - \cos\frac{\pi}{n}\right)$ as being $=\theta \operatorname{vers}\frac{\pi}{n}$; & ∵ shall have, for the corresponding part of $z_{x,t}$, the expression

$$\frac{1}{2\pi}\left(1+\cos\frac{\pi}{n}\right)\int_0^\infty \frac{\sin(2\xi_{,}\theta_{,} + \frac{1}{3}\theta_{,}^3)\,d\theta_{,}}{\theta_{,}},$$

in which $\theta_{,} = \theta\sqrt[3]{t}$, and $\xi_{,} = \frac{x-t}{\sqrt[3]{t}}$.

If $\xi_{,}$ be very large, positive or negative, this expression is evidently $=\pm\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)$. If on the other hand $\xi_{,}$ be very small, the expression becomes $\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$. But according to what law does its value vary for moderate values of $\xi_{,}$?

We have, accurately,

$$\int_0^\infty \frac{\sin(2\xi_{,}\theta_{,} + \frac{1}{3}\theta_{,}^3)\,d\theta_{,}}{2\theta_{,}} = \frac{\pi}{12} + \int_0^{\xi_{,}} d\xi_{,}\int_0^\infty d\theta_{,}\cos(2\xi_{,}\theta_{,} + \frac{1}{3}\theta_{,}^3);$$

so that it is desirable to try whether we can obtain a finite expression for this last integral, $\int_0^\infty \cos(2\xi\theta + \frac{1}{3}\theta^3)\,d\theta$.*

Can we calculate $\int_0^\infty e^{-\alpha^2(\theta^3+6\xi\theta)}d\theta$, when α is real, or $\int_0^\infty e^{-\alpha^2x^3}\cos rx\,dx$, which bears a strong affinity to the integral discussed by Laplace, *Calc. Prob.*, page 97, $\int e^{-\alpha^2x^2}\cos rx\,dx$?

[The calculation is reduced to the solution of a Riccati equation but is not carried further.]

[*Detailed description of wave motion.*]

(Feb. 8th.)

Waiving for the present the discussion begun on this page, let me resume the former discussion of $z_{x,t}$.

* [Airy's integral. Cf. Watson, *Theory of Bessel Functions*, pp. 188–190.]

It follows from that discussion that the function

$$z_{x,t}=\frac{1}{2\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^\pi \frac{\sin(2x\theta-2t\sin\theta)\,d\theta}{\sin\theta\left(\cos\theta-\cos\frac{\pi}{n}\right)}$$

involves a first part corresponding to θ nearly 0, which is $\pm\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)$, according as $x-t$, being large, is positive or negative; but is $=\pm\frac{1}{12}\left(1+\cos\frac{\pi}{n}\right)$, according as t, being large, is positive or negative, if $x=t$.

The same function or integral $z_{x,t}$ involves a 2nd part, corresponding to θ nearly $=\pi$; which 2nd part is $\pm\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)$, according as $x+t$, being large, is positive or negative; but is $=\mp\frac{1}{12}\left(1-\cos\frac{\pi}{n}\right)$, according as t, being large, is positive or negative, if $x=-t$.

The function $z_{x,t}$ involves finally a 3rd part, corresponding to θ nearly $=\frac{\pi}{n}$; which 3rd part vanishes when $x=t\cos\frac{\pi}{n}$, if t be large, positive or negative; but is $=\mp\frac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)$, according as $x-t\cos\frac{\pi}{n}$, being large, is positive or negative.

More precisely, by saying that $x-t\cos\frac{\pi}{n}$ is *large*, we mean that it is numerically equal to or numerically greater than $\frac{1}{2}s\sqrt{\pm t\sin\frac{\pi}{n}}$, s being a positive number so great that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\cos\left(\phi\pm\frac{\psi^2}{s^2}\right)\frac{\sin\psi}{\psi}\,d\psi$$

is sensibly equal to $\cos\phi$. Perhaps the number $s=1000$ may sufficiently satisfy this condition. At all events, whatever degree of accuracy may be required, this number s is independent of t & of x.

By saying that $x-t$ is *large*, we mean that it is numerically equal to or greater than $\frac{1}{2}s't^{\frac{1}{3}}$, s' being a positive number so great that $\frac{2}{\pi}\int_0^\infty \sin\left(\psi\pm\frac{\psi^3}{3s'^3}\right)\frac{d\psi}{\psi}$ is sensibly $=1$. Perhaps s' may be taken $=200$.

Finally, by saying that $x+t$ is large, we mean that it is numerically equal to, or numerically greater than, the same quantity $\frac{1}{2}s't^{\frac{1}{3}}$, s' being the same positive number.

Now let the number in be considerably greater than $\frac{s}{2}\sqrt{\pm t\sin\frac{\pi}{n}}$, & also than $\frac{s'}{2}\sqrt[3]{\pm t}$. (What are the relations between these 2 limits?)

And first, let i be infinite; that is, let us return to the *original problem of this manuscript*; in which it is supposed that, at the origin of time, all particles behind the origin are disturbed according to the law of the versed sine, & all beyond it are at rest. The number n of molecular intervals in *one* wave-length is not obliged to be large. We shall suppose that $1-\cos\frac{\pi}{n}$ is sensibly different from 0.

We have now, (compare p. 554,)

$$\eta_{x,t}=\tfrac{1}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)-z_{x,t}=\tfrac{1}{2}\operatorname{vers}\phi-z_{x,t}.$$

The law of $z_{x,t}$ is sufficiently stated on the preceding page; and we are to deduce the corresponding law of $\eta_{x,t}$.

I. *Let t be large and positive.**

(1.) If x be $=$ or $>t+\frac{1}{2}s't^{\frac{1}{3}}$, then

$$z_{x,t}=\frac{1}{4}\left(1+\cos\frac{\pi}{n}\right)+\frac{1}{4}\left(1-\cos\frac{\pi}{n}\right)-\frac{1}{2}\cos\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right)=\frac{1}{2}\operatorname{vers}\left(\frac{2x\pi}{n}-2t\sin\frac{\pi}{n}\right),$$

and $\eta_{x,t}=0$; the disturbance is insensible at a distance beyond the origin so great as $t+\frac{1}{2}s't^{\frac{1}{3}}$.

(2.) If x be $=t$, then $\eta_{x,t}=\frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$; there is then a certain constant and positive displacement, which travels with a velocity $=1=$ the square root of the attractive (accelerative) force exerted by any one particle on its next neighbour, and therefore with a velocity independent of the wave-length.

(3.) If x be $=$ or $<t-\frac{1}{2}s't^{\frac{1}{3}}$, but $>$ or $=t\cos\frac{\pi}{n}+\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$, then $\eta_{x,t}=\left(\cos\frac{\pi}{2n}\right)^2=$ a certain other constant & positive displacement, three times as great as that last mentioned. This new displacement is sensibly constant, within an extent $=t\operatorname{vers}\frac{\pi}{n}-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\frac{1}{2}s't^{\frac{1}{3}}$; whereas for a range $=s't^{\frac{1}{3}}$ next following, there is a variable displacement, which vanishes (sensibly) at the end of that range, & is, at the middle, reduced to one third of what it was at the beginning.

(4.) If x be $=t\cos\frac{\pi}{n}$, then $\eta_{x,t}=\frac{1}{2}\operatorname{vers}\phi+\frac{1}{2}\cos\frac{\pi}{n}=$ a periodical displacement, which travels with a constant velocity $\left(=\cos\frac{\pi}{n}=\text{a function of the wave-length}\right)$, & of which the amplitude is $=1=$ half the amplitude that corresponds to the most complete effect of the initial disturbance.

(5.) If x be $=$ or $<t\cos\frac{\pi}{n}-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$, but $>$ or $=-t+\frac{1}{2}s't^{\frac{1}{3}}$, then $\eta_{x,t}=\operatorname{vers}\phi-\left(\sin\frac{\pi}{2n}\right)^2=$ a periodical displacement, which has a constant amplitude $=2$, & of which the period, for any given value of x, is $\pi\operatorname{cosec}\frac{\pi}{n}$ [wave-length $=n$]. This part of space corresponds very well to the phenomena of light. Its extent is $t\left(1+\cos\frac{\pi}{n}\right)-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\frac{1}{2}s't^{\frac{1}{3}}$.

(6.) If x be $=-t$, then $\eta_{x,t}=\operatorname{vers}\phi-\frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$.

* [See Appendix, Note 12, p. 640.]

(7.) If x be $=$ or $< -t-\frac{1}{2}s't^{\frac{1}{3}}$, then $\eta_{x,t}=\text{vers}\,\phi$.

Thus the law of the versed sine may be considered as holding good, without sensible error, for all those particles which are at least as far behind the origin as the position $x=-t-\frac{1}{2}s't^{\frac{1}{3}}$; but during the range $=s't^{\frac{1}{3}}$ next following in the positive direction, that is, between the bounds $x=-t\mp\frac{1}{2}s't^{\frac{1}{3}}$, a negative term is added to vers ϕ, which negative term is sensibly $=-\frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$ at the middle of this interval, & $=-\left(\sin\frac{\pi}{2n}\right)^2$ at the end thereof. It then remains sensibly constant, as also does the law vers ϕ remain unchanged, during a large range

$$=2t\left(\cos\frac{\pi}{2n}\right)^2-\tfrac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\tfrac{1}{2}s't^{\frac{1}{3}},$$

that is, till $x=t\cos\frac{\pi}{n}-\frac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}$; but in the interval $=s\sqrt{t\sin\frac{\pi}{n}}$ next following, the displacement changes from $\left(\cos\frac{\pi}{2n}\right)^2-\cos\phi$ to $\left(\cos\frac{\pi}{2n}\right)^2$, the coefficient of $-\cos\phi$ decreasing from 1 to 0 & being $=\frac{1}{2}$ at the middle of this interval, that is, when $x=t\cos\frac{\pi}{n}$. In another range

$$=2t\left(\sin\frac{\pi}{2n}\right)^2-\tfrac{1}{2}s\sqrt{t\sin\frac{\pi}{n}}-\tfrac{1}{2}s't^{\frac{1}{3}},$$

that is, till $x=t-\frac{1}{2}s't^{\frac{1}{3}}$, the constant term $\left(\cos\frac{\pi}{2n}\right)^2$ remains sensibly the same; but in the interval $=s't^{\frac{1}{3}}$ next following, this term is reduced to an insensible quantity, being $=\frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$ at the middle of the interval, that is, for $x=t$. Finally, for $x>t+\frac{1}{2}s't^{\frac{1}{3}}$, the disturbance or displacement is insensible.

II. *Let t be large and negative.*

That is, let us trace back what *must have been* the state of the disturbances before the origin of t.

(1.)′ If $x=$ or $> -t-\frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t}=0$.

(2.)′ If $x=-t$; $\eta_{x,t}=\frac{1}{3}\left(\sin\frac{\pi}{2n}\right)^2$.

(3.)′ If $x=$ or $< -t+\frac{1}{2}s't^{\frac{1}{3}}$, but $>$ or $=t\cos\frac{\pi}{n}+\frac{1}{2}s\sqrt{-t\sin\frac{\pi}{n}}$; $\eta_{x,t}=\left(\sin\frac{\pi}{2n}\right)^2$.

(4.)′ If $x=t\cos\frac{\pi}{n}$; $\eta_{x,t}=\left(\sin\frac{\pi}{2n}\right)^2-\frac{1}{2}\cos\phi$.

(5.)′ If $x=$ or $<t\cos\frac{\pi}{n}-\frac{1}{2}s\sqrt{-t\sin\frac{\pi}{n}}$, but $>$ or $=t-\frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t}=\text{vers}\,\phi-\left(\cos\frac{\pi}{2n}\right)^2$.

(6.)′ If $x=t$; $\eta_{x,t}=\text{vers}\,\phi-\frac{1}{3}\left(\cos\frac{\pi}{2n}\right)^2$.

(7.)′ If $x=$ or $<t+\frac{1}{2}s't^{\frac{1}{3}}$; $\eta_{x,t}=\text{vers}\,\phi$.

[*Extension of original problem.*]

(June 14th, 1839.)

If we have the equation

$$\eta''_{x,t}=a_1^2(\eta_{x+1,t}-2\eta_{x,t}+\eta_{x-1,t})+a_2^2(\eta_{x+2,t}-2\eta_{x,t}+\eta_{x-2,t}),$$

we may satisfy it, generally & rigorously, by the expression

$$\eta_{x,t}=\frac{2}{\pi}\Sigma_{(l)-\infty}^{\infty}\eta_{l,0}\int_0^{\frac{\pi}{2}}d\theta\cos(2l\theta-2x\theta)\cos(2t\Theta_\theta)$$

$$+\frac{1}{\pi}\Sigma_{(l)-\infty}^{\infty}\eta'_{l,0}\int_0^{\frac{\pi}{2}}d\theta\cos(2l\theta-2x\theta)\sin(2t\Theta_\theta)\,\Theta_\theta^{-1},$$

in which $\Theta_\theta=\sqrt{a_1^2\sin\theta^2+a_2^2\sin 2\theta^2}$; x and l being integers.

Let the initial conditions be

$$\eta_{-l,0}=\eta\operatorname{vers}\left(\frac{2l\pi}{n}\right)\quad\text{or }=0,$$

and

$$\eta'_{-l,0}=2\eta\sin\left(\frac{2l\pi}{n}\right)\Theta_{\frac{\pi}{n}}\quad\text{or }=0,$$

according as $-l$ is or is not included between 0 and in; then

$$\eta_{x,t}=\frac{2\eta}{\pi}\Sigma_{(l)0}^{in}\left\{\operatorname{vers}\left(\frac{2l\pi}{n}\right)\int_0^{\frac{\pi}{2}}d\theta\cos(2l\theta+2x\theta)\cos(2t\Theta_\theta)\right.$$

$$\left.+\Theta_{\frac{\pi}{n}}\sin\frac{2l\pi}{n}\int_0^{\frac{\pi}{2}}d\theta\cos(2l\theta+2x\theta)\sin(2t\Theta_\theta)\,\Theta_\theta^{-1}\right\}.$$

We have

$$\Sigma_{(l)0}^{in}\operatorname{vers}\frac{2l\pi}{n}\cos(2l\theta+2x\theta)=\frac{\sin in\theta}{2\sin\theta}\cos(2x\theta+in\theta)\left(\sin\frac{\pi}{n}\right)^2\left\{\frac{1}{\cos\theta-\cos\frac{\pi}{n}}+\frac{1}{\cos\theta+\cos\frac{\pi}{n}}\right\},$$

$$\Sigma_{(l)0}^{in}\sin\frac{2l\pi}{n}\cos(2l\theta+2x\theta)=\frac{\sin in\theta}{2}\sin(2x\theta+in\theta)\sin\frac{\pi}{n}\left\{\frac{1}{\cos\theta-\cos\frac{\pi}{n}}-\frac{1}{\cos\theta+\cos\frac{\pi}{n}}\right\};$$

$$\therefore\quad\eta_{x,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi}d\theta\frac{\sin in\theta}{\sin\theta}\frac{\cos(2x\theta+in\theta)}{\cos\theta-\cos\frac{\pi}{n}}\cos(2t\Theta_\theta)$$

$$+\frac{\eta}{\pi}\Theta_{\frac{\pi}{n}}\sin\frac{\pi}{n}\int_0^{\pi}d\theta\frac{\sin in\theta}{\Theta_\theta}\frac{\sin(2x\theta+in\theta)}{\cos\theta-\cos\frac{\pi}{n}}\sin(2t\Theta_\theta).$$

In general

$$4\sin a\cos b\cos c=\sin(b+c+a)-\sin(b+c-a)+\sin(b-c+a)-\sin(b-c-a),$$

$$4\sin a\sin b\sin c=-\sin(b+c+a)+\sin(b+c-a)+\sin(b-c+a)-\sin(b-c-a);$$

$$\therefore\quad\eta_{x,t}=A-B+C-D,$$

in which

$$A=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}-\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta+2in\theta+2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}},$$

$$B=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}-\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta+2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}},$$

$$C=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}+\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta+2in\theta-2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}},$$

$$D=\frac{\eta}{4\pi}\sin\frac{\pi}{n}\int_0^\pi d\theta\left(\frac{\sin\frac{\pi}{n}}{\sin\theta}+\frac{\Theta_{\frac{\pi}{n}}}{\Theta_\theta}\right)\frac{\sin(2x\theta-2t\Theta_\theta)}{\cos\theta-\cos\frac{\pi}{n}}.$$

[*Two-dimensional waves.*]

(June 14th, 1839.)

If we consider the vibrations of an indefinite *square system* of points & suppose that each attracts only the 4 nearest to itself, the distance between any 2 nearest points being $=1$ in the state of equilibrium, & the attraction being sensibly $=a^2+b^2(r-1)$ for any distance r differing little from 1, but being sensibly $=0$ for any much greater distance such as even $\sqrt{2}$; we shall then have the 2 indefinite equations in mixed differences

$$0=\left\{-\left(\frac{d}{dt}\right)^2+\frac{a^2\Delta_y^2}{1+\Delta_y}+\frac{b^2\Delta_x^2}{1+\Delta_x}\right\}\xi_{x,y,t},$$

$$0=\left\{-\left(\frac{d}{dt}\right)^2+\frac{a^2\Delta_x^2}{1+\Delta_x}+\frac{b^2\Delta_y^2}{1+\Delta_y}\right\}\eta_{x,y,t};$$

of which the complete and general integrals may be thus written:

$$\xi_{x,y,t}=\frac{4}{\pi^2}\Sigma_{(l,m)-\infty,-\infty}^{\infty,\infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2x\theta-2l\theta)\cos(2y\phi-2m\phi)\left\{\xi_{l,m,0}\cos(2t\Theta)+\xi'_{l,m,0}\frac{\sin(2t\Theta)}{2\Theta}\right\},$$

$$\eta_{x,y,t}=\frac{4}{\pi^2}\Sigma_{(l,m)-\infty,-\infty}^{\infty,\infty}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2y\theta-2m\theta)\cos(2x\phi-2l\phi)\left\{\eta_{l,m,0}\cos(2t\Theta)+\eta'_{l,m,0}\frac{\sin(2t\Theta)}{2\Theta}\right\},$$

if

$$\Theta=\sqrt{a^2\sin\phi^2+b^2\sin\theta^2}.$$

Suppose now that for every integer value of l between 0 & $-in$, and for every integer value of m between $-m_{,}$ & $+m_{,}$, (the extremes being included,) we have

$$\eta_{l,m,0}=\eta\operatorname{vers}\frac{2l\pi}{n},\quad \eta'_{l,m,0}=-2a\eta\sin\frac{\pi}{n}\sin\frac{2l\pi}{n};$$

but for all other values of l or m we have $\eta_{l,m,0}=0$, $\eta'_{l,m,0}=0$, & for all values of l & m, $\xi_{l,m,0}=0$, $\xi'_{l,m,0}=0$. Then $\xi_{x,y,t}=0$ and

$$\eta_{x,y,t}=\frac{4\eta}{\pi^2}\Sigma^{m,}_{(m)-m,}\Sigma^{in}_{(l)0}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\phi\cos(2y\theta-2m\theta)\cos(2x\phi+2l\phi)$$
$$\times\left\{\operatorname{vers}\frac{2l\pi}{n}\cos(2t\Theta)+\frac{a\sin\frac{\pi}{n}}{\Theta}\sin\left(\frac{2l\pi}{n}\right)\sin(2t\Theta)\right\}.$$

But

$$\Sigma^{m,}_{(m)-m,}\cos(2y\theta-2m\theta)=\frac{\cos 2y\theta\sin(2m,\theta+\theta)}{\sin\theta};$$

$$\therefore\quad \eta_{x,y,t}=\frac{2\eta}{\pi^2}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\frac{\pi}{2}}d\theta\int_0^{\pi}d\phi\,\frac{\cos 2y\theta\sin(2m,\theta+\theta)}{\sin\theta}\,\frac{\sin in\phi}{\sin\phi}$$
$$\times\left\{\frac{\cos(2x\phi+in\phi)}{\cos\phi-\cos\frac{\pi}{n}}\cos(2t\Theta)+\frac{a\sin\phi}{\Theta}\frac{\sin(2x\phi+in\phi)}{\cos\phi-\cos\frac{\pi}{n}}\sin(2t\Theta)\right\}.$$

As $m,$ increases without limit, this expression tends to become independent of y,* namely

$$\eta_{x,y,t}=\frac{\eta}{\pi}\left(\sin\frac{\pi}{n}\right)^2\int_0^{\pi}d\phi\,\frac{\sin in\phi}{\sin\phi}\,\frac{\cos(2x\phi+in\phi-2at\sin\phi)}{\cos\phi-\cos\frac{\pi}{n}},$$

agreeing with the value found before. The same result is obtained if y be large, but $m,$ much larger.

If on the contrary y be much larger than $m,$, then the relation

$$2\cos 2y\theta\sin(2m,\theta+\theta)=\sin(2y\theta+2m,\theta+\theta)-\sin(2y\theta-2m,\theta-\theta)$$

causes $\eta_{x,y,t}$ to be sensibly $=0$; & we have thus an example of the possibility of representing in dynamical calculation the sensible *rectilinearity of the propagation of light*.

Diffraction corresponds to the case of y nearly $=m,$.

[*Three-dimensional waves in dispersive medium.*]

(June 18th, 1839.)

In general, the equations of infinitely small vibrations of a system of attracting or repelling particles,

$$\frac{d^2\delta x}{dt^2}=\Sigma\{\Delta\delta x\,\phi(r)+\Delta x\,\phi'(r)\,\delta r\},\ \&c.,$$

* [This means, of course, that in the integrand, with the exception of the factor $\sin(2m,\theta+\theta)/\sin\theta$, θ is made zero so that $\Theta=a\sin\phi$.]

may have their integrals expressed, as Cauchy* has shown, by equations of the form

$$(\delta x)_t=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{d_t\cos(ux+vy+wz)+g_t\sin(ux+vy+wz)\}\,du\,dv\,dw,$$

$$(\delta y)_t=\qquad\{e_t\qquad\qquad h_t\qquad\}\quad,$$

$$(\delta z)_t=\qquad\{f_t\qquad\qquad i_t\qquad\}\quad,$$

$d_t, \ldots i_t$ being 6 real functions of t and of u, v, w, depending on the nature of the system.

The initial values & rates of increase of these 6 functions may be deduced from those of δx, δy, δz by the 12 formulae

$$d_0=\left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\delta x)_0\cos(ux+vy+wz)\,dx\,dy\,dz,$$

......

$$i_0'=\left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\delta z)_0'\sin(ux+vy+wz)\,dx\,dy\,dz,$$

in which $(\delta x)_0$, $(\delta y)_0$, $(\delta z)_0$, $(\delta x)_0'$, $(\delta y)_0'$, $(\delta z)_0'$ are 6 known real functions of x, y, z.

The differential relations between d_t, e_t, f_t are of the forms

$$d_t''=-(Ld_t+Re_t+Qf_t),\quad e_t''=-(Me_t+Pf_t+Rd_t),\quad f_t''=-(Nf_t+Qd_t+Pe_t),$$

in which L, M, N, P, Q, R are 6 real and known functions of u, v, w, depending on the nature of the system; this system being supposed to have a certain symmetry of arrangement & to have the same arrangement throughout; & the same equations hold good when d, e, f are changed to g, h, i.

The expressions for these 6 functions $L, \ldots R$ are

$$L=\Sigma\,.\left(\frac{\Delta x^2}{r}\phi'(r)+\phi(r)\right)\operatorname{vers}(u\Delta x+v\Delta y+w\Delta z),\qquad M=\qquad,\quad N=\qquad,$$

$$P=\Sigma\,.\left(\frac{\Delta x\Delta y}{r}\phi'(r)\right)\operatorname{vers}(u\Delta x+v\Delta y+w\Delta z),\qquad Q=\qquad,\quad R=\qquad.$$

The integrals of the differential relations between d_t, e_t, f_t are of the forms

$$A_1d_t+B_1e_t+C_1f_t=(A_1d_0+B_1e_0+C_1f_0)\cos s_1t+(A_1d_0'+B_1e_0'+C_1f_0')\int_0^t\cos s_1t\,dt,$$

....

$$A_3d_t+B_3e_t+C_3f_t=(A_3d_0+B_3e_0+C_3f_0)\cos s_3t+(A_3d_0'+B_3e_0'+C_3f_0')\int_0^t\cos s_3t\,dt;$$

$A_1, \ldots C_3$ being 9 real cosines, namely of the inclinations of 3 rectangular lines to the 3 rectangular axes of coordinates, these 3 lines being the axes of a certain surface of the 2nd order, & depending only on u, v, w; but s_1, s_2, s_3 being (perhaps) not necessarily real.

The equations which determine them all are

$$s^2=L+\frac{BR+CQ}{A}=M+\frac{CP+AR}{B}=N+\frac{AQ+BP}{C}.$$

* [Cauchy, *Œuvres*, 1e Série, Tome IV, pp. 237–298. It is to be noted that $r\phi(r)$ is the law of force between two points.]

(June 19th.)

Let *only half the medium be originally agitated*; & in particular let the initial conditions be

$$(\delta x)_0=0,\quad (\delta y)_0=\text{vers}\, u\grave{}x,\quad (\delta z)_0=0,\quad (\delta x)'_0=0,\quad (\delta y)'_0=s\grave{}\sin(u\grave{}x),\quad (\delta z)'_0=0,$$

for all negative values of x, these 6 functions all vanishing for positive values of x.

Then

$$d_0=0,\quad f_0=0,\quad g_0=0,\quad i_0=0,\quad d'_0=0,\quad f'_0=0,\quad g'_0=0,\quad i'_0=0,$$

and

$$e_0=\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\,\text{vers}\,u\grave{}x\cos(ux+vy+wz),$$

$$h_0=\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\,\text{vers}\,u\grave{}x\sin(ux+vy+wz),$$

$$e'_0=s\grave{}\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\sin u\grave{}x\cos(ux+vy+wz),$$

$$h'_0=s\grave{}\frac{1}{(2\pi)^3}\int_{-\infty}^{0}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\sin u\grave{}x\sin(ux+vy+wz).$$

[Hamilton then calculates from integrals in which the integrands above have a factor $e^{-h^2(x^2+y^2+z^2)}$. Inserting these values in the equations

$$e_t=(B_1^2\cos s_1t+B_2^2\cos s_2t+B_3^2\cos s_3t)\,e_0+\left(B_1^2\frac{\sin s_1t}{s_1}+B_2^2\frac{\sin s_2t}{s_2}+B_3^2\frac{\sin s_3t}{s_3}\right)e'_0,\ \&c.$$

and making $h=0$ in the triple integral for $(\delta y)_t$, we get finally the following equation]*

$$\boxed{(\delta y)_t=\tfrac{1}{2}\text{vers}(u\grave{}x+s\grave{}t)+\frac{u\grave{}}{\pi}\int_0^\infty\frac{du}{u^2-u\grave{}^2}\left(\frac{u\grave{}}{u}\sin ux\cos s_1t+\frac{s\grave{}}{s_1}\cos ux\sin s_1t\right).}$$

This expression for $(\delta y)_t$ is rigorous in the present question; & accordingly it gives

$$(\delta y)_0=\text{vers}\,u\grave{}x\ \text{or}\ =0,\quad\text{and}\quad (\delta y)'_0=s\grave{}\sin u\grave{}x\ \text{or}\ =0,$$

according as x is <0 or >0; because, in these 2 respective cases, we have

$$2\int_0^\infty\frac{\sin ux\,.\,u\,du}{u^2-u\grave{}^2}=\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\sin ux}{u+u\grave{}}+\frac{\sin ux}{u-u\grave{}}\right)du=\mp\pi\cos u\grave{}x;$$

$$2u\grave{}\int_0^\infty\frac{\cos ux\,.\,du}{u^2-u\grave{}^2}=\pm\pi\sin u\grave{}x;\quad 2u\grave{}^2\int_0^\infty\frac{\sin ux\,.\,du}{u(u^2-u\grave{}^2)}=\pm\pi\,\text{vers}\,u\grave{}x.$$

Thus, rigorously,

$$\boxed{(\delta y)_t=\tfrac{1}{2}\text{vers}(u\grave{}x+s\grave{}t)+\frac{u\grave{}}{2\pi}\int_0^\infty\frac{du}{u^2-u\grave{}^2}\left\{\left(\frac{u\grave{}}{u}+\frac{s\grave{}}{s_1}\right)\sin(ux+s_1t)+\left(\frac{u\grave{}}{u}-\frac{s\grave{}}{s_1}\right)\sin(ux-s_1t)\right\};}$$

& its periodic part is 0 or $-\cos(u\grave{}x+s\grave{}t)$, according as $x+\frac{ds\grave{}}{du\grave{}}t$, being large, is $>$ or <0. *The velocity of propagation of vibration, or rather in this case the velocity of extinction, is expressed*

* [See Sommerfeld, Diss., Königsberg, 1891 and Carslaw, *Fourier's Series and Integrals*, p. 293.]

*not by the ratio of the coefficients of time and space, but by the ratio of their differentials; & I feel almost sure that this is the general law of the propagation of vibratory motion.**

$$\left[\textit{Velocity of propagation } \frac{ds'}{dk'}.\right]$$

(June 20th.)

Suppose that only half the medium is originally agitated, namely that half which is on the negative side (relatively to x) of the plane $u'x+v'y+w'z=0$, in such a way that, if

$$x < -\frac{v'y+w'z}{u'},$$

we have

$$(\delta x)_0 = A'\cos(u'x+v'y+w'z),\quad (\delta y)_0 = B'\cos(\quad),\quad (\delta z)_0 = C'\cos(\quad),$$

$$(\delta x)_0' = A's'\sin(\quad),\quad (\delta y)_0' = B's'\sin(\quad),\quad (\delta z)_0' = C's'\sin(\quad);$$

but $$(\delta x)_0 = (\delta y)_0 = (\delta z)_0 = (\delta x)_0' = (\delta y)_0' = (\delta z)_0' = 0,\ \text{if}\ x > -\frac{v'y+w'z}{u'}.$$

How are we now to effect the triple integrations for $d_0, \ldots i_0'$?

We may conceive 3 new rectangular coordinates x', y', z', of which x' is the perpendicular distance $\frac{u'x+v'y+w'z}{\sqrt{u'^2+v'^2+w'^2}}$ from the plane $u'x+v'y+w'z=0$; (u' may for simplicity be supposed >0;) $y' = \frac{w'y-v'z}{\sqrt{v'^2+w'^2}}$ = perpendicular distance from that plane which passes through [the axes of] x and x'; & the plane of $x'y'$ is of the form

$$x+\lambda'(v'y+w'z)=0,\quad \lambda' = -\frac{u'}{v'^2+w'^2},$$

$$\therefore\quad z' = \frac{(v'^2+w'^2)x - u'(v'y+w'z)}{\sqrt{v'^2+w'^2}\sqrt{u'^2+v'^2+w'^2}}.$$

Accordingly these expressions give $x'^2+y'^2+z'^2 = x^2+y^2+z^2$. They may be written thus: Making $u' = k'\cos\theta'$, $v' = k'\sin\theta'\cos\phi'$, $w' = k'\sin\theta'\sin\phi'$, we have

$$x' = x\cos\theta' + (y\cos\phi' + z\sin\phi')\sin\theta';\quad y' = y\sin\phi' - z\cos\phi';$$

$$z' = x\sin\theta' - (y\cos\phi' + z\sin\phi')\cos\theta';$$

and they give reciprocally

$$x = x'\cos\theta' + z'\sin\theta';\quad y = (x'\sin\theta' - z'\cos\theta')\cos\phi' + y'\sin\phi';$$

$$z = (x'\sin\theta' - z'\cos\theta')\sin\phi' - y'\cos\phi'.$$

Hence

$$ux+vy+wz = x'\{u\cos\theta' + (v\cos\phi' + w\sin\phi')\sin\theta'\}$$
$$+ y'(v\sin\phi' - w\cos\phi') + z'\{u\sin\theta' - (v\cos\phi' + w\sin\phi')\cos\theta'\};$$

* [This is obtained by considering the contribution of the values of u in the neighbourhood of $u=u'$ to the value of the integral. It is to be observed that s' is the same function of u' as s_1 is of u.]

and we are to multiply both the cosine and sine of this expression by the cosine & sine of $k^{\backprime} x^{\backprime}$, & integrate relatively to $x^{\backprime}$ from $-\infty$ to 0 but relatively to $y^{\backprime}$ and $z^{\backprime}$ from $-\infty$ to ∞. The results, multiplied by $\left(\frac{1}{2\pi}\right)^3$ & by $A^{\backprime}, B^{\backprime}, C^{\backprime}, A^{\backprime}s^{\backprime}, B^{\backprime}s^{\backprime}, C^{\backprime}s^{\backprime}$, will give $d_0, \ldots i_0'$.

We find, in this way,

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \cos k^{\backprime} x^{\backprime} \cos (ux+vy+wz)$$

$$= \int_{0}^{\infty} dx^{\backprime} \cos k^{\backprime} x^{\backprime} \cos [x^{\backprime} \{u \cos \theta^{\backprime} + (v \cos \phi^{\backprime} + w \sin \phi^{\backprime}) \sin \theta^{\backprime}\}]$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy^{\backprime} dz^{\backprime} \cos [y^{\backprime} (v \sin \phi^{\backprime} - w \cos \phi^{\backprime}) + z^{\backprime} \{u \sin \theta^{\backprime} - (v \cos \phi^{\backprime} + w \sin \phi^{\backprime}) \cos \theta^{\backprime}\}]$$

$$= \frac{\pi^{\frac{3}{2}}}{4h^3} \left\{ e^{-\left(\frac{k_{,}-k^{\backprime}}{2h}\right)^2} + e^{-\left(\frac{k_{,}+k^{\backprime}}{2h}\right)^2} \right\} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)},$$

if $k_{,} = u \cos \theta^{\backprime} + (v \cos \phi^{\backprime} + w \sin \phi^{\backprime}) \sin \theta^{\backprime}$, and $k = \sqrt{u^2+v^2+w^2}$, as $k^{\backprime} = \sqrt{u^{\backprime 2}+v^{\backprime 2}+w^{\backprime 2}}$. In like manner

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \sin k^{\backprime} x^{\backprime} \sin (ux+vy+wz) = \frac{\pi^{\frac{3}{2}}}{4h^3} \left\{ e^{-\left(\frac{k_{,}-k^{\backprime}}{2h}\right)^2} - e^{-\left(\frac{k_{,}+k^{\backprime}}{2h}\right)^2} \right\} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \sin k^{\backprime} x^{\backprime} \cos (ux+vy+wz) = \frac{\pi}{h^2} \frac{k^{\backprime}}{k_{,}^2 - k^{\backprime 2}} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

$$\int_{-\infty}^{0} dx^{\backprime} \int_{-\infty}^{\infty} dy^{\backprime} \int_{-\infty}^{\infty} dz^{\backprime} \cos k^{\backprime} x^{\backprime} \sin (ux+vy+wz) = -\frac{\pi}{h^2} \frac{k_{,}}{k_{,}^2 - k^{\backprime 2}} e^{-\left(\frac{k^2-k_{,}^2}{4h^2}\right)};$$

in which it is to be remembered that

$$k^2 - k_{,}^2 = (v \sin \phi^{\backprime} - w \cos \phi^{\backprime})^2 + \{u \sin \theta^{\backprime} - (v \cos \phi^{\backprime} + w \sin \phi^{\backprime}) \cos \theta^{\backprime}\}^2.$$

Let $\quad u_{,} = k_{,}, \quad v_{,} = v \sin \phi^{\backprime} - w \cos \phi^{\backprime}, \quad w_{,} = u \sin \theta^{\backprime} - (v \cos \phi^{\backprime} + w \sin \phi^{\backprime}) \cos \theta^{\backprime};$

then $\quad k^2 - k_{,}^2 = v_{,}^2 + w_{,}^2, \quad u = u_{,} \cos \theta^{\backprime} + w_{,} \sin \theta^{\backprime},$

$$v = (u_{,} \sin \theta^{\backprime} - w_{,} \cos \theta^{\backprime}) \cos \phi^{\backprime} + v_{,} \sin \phi^{\backprime}, \quad w = (u_{,} \sin \theta^{\backprime} - w_{,} \cos \theta^{\backprime}) \sin \phi^{\backprime} - v_{,} \cos \phi^{\backprime};$$

$$du\,dv\,dw = du_{,}\,dv_{,}\,dw_{,};$$

$$ux+vy+wz = u_{,} x^{\backprime} + v_{,} y^{\backprime} + w_{,} z^{\backprime}; \quad u^2+v^2+w^2 = u_{,}^2 + v_{,}^2 + w_{,}^2 = k^2.$$

Hence

$$\frac{d_0}{A^{\backprime}} = \frac{e_0}{B^{\backprime}} = \frac{f_0}{C^{\backprime}} = \frac{1}{4} \left(\frac{1}{2h\sqrt{\pi}} \right)^3 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \left\{ e^{-\left(\frac{u_{,}-k^{\backprime}}{2h}\right)^2} + e^{-\left(\frac{u_{,}+k^{\backprime}}{2h}\right)^2} \right\};$$

$$\frac{g_0'}{A^{\backprime}s^{\backprime}} = \frac{h_0'}{B^{\backprime}s^{\backprime}} = \frac{i_0'}{C^{\backprime}s^{\backprime}} = \frac{1}{4} \left(\frac{1}{2h\sqrt{\pi}} \right)^3 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \left\{ \qquad - \qquad \right\};$$

$$\frac{d_0'}{A^{\backprime}s^{\backprime}} = \frac{e_0'}{B^{\backprime}s^{\backprime}} = \frac{f_0'}{C^{\backprime}s^{\backprime}} = \frac{1}{2\pi} \left(\frac{1}{2h\sqrt{\pi}} \right)^2 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \frac{k^{\backprime}}{u_{,}^2 - k^{\backprime 2}};$$

$$\frac{g_0}{A^{\backprime}} = \frac{h_0}{B^{\backprime}} = \frac{i_0}{C^{\backprime}} = -\frac{1}{2\pi} \left(\frac{1}{2h\sqrt{\pi}} \right)^2 e^{-\frac{v_{,}^2+w_{,}^2}{4h^2}} \frac{u_{,}}{u_{,}^2 - k^{\backprime 2}};$$

these are to be multiplied by $\overset{\cos}{\underset{\sin}{.}}(u_, x^{\backprime} + v_, y^{\backprime} + w_, z^{\backprime})\, du_, dv_, dw_,$, & also by some known functions of $u_,$, $v_,$, $w_,$, & integrated relatively to $u_,$, $v_,$, $w_,$ from $-\infty$ to ∞.

This integration may be simplified, by 1st suppressing

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_, dw_, \left(\frac{1}{2h\sqrt{\pi}}\right)^2 e^{-\frac{v_,^2+w_,^2}{4h^2}},$$

& then changing $u_,$, $v_,$, $w_,$ to k, 0, 0; which latter change is equivalent to making

$$u = k\cos\theta^{\backprime}, \quad v = k\sin\theta^{\backprime}\cos\phi^{\backprime}, \quad w = k\sin\theta^{\backprime}\sin\phi^{\backprime}.$$

We are ∴ to multiply

$$\frac{1}{8h\sqrt{\pi}}\left(e^{-\left(\frac{k-k^{\backprime}}{2h}\right)^2} \pm e^{-\left(\frac{k+k^{\backprime}}{2h}\right)^2}\right), \quad -\frac{1}{4\pi}\left(\frac{1}{k-k^{\backprime}} \mp \frac{1}{k+k^{\backprime}}\right),$$

by some known functions of k & t, & to perform the operations $\int_{-\infty}^{\infty} \overset{\cos}{\underset{\sin}{.}} kx^{\backprime}\, dk$.

But $$\lim_{h=0}\int_{-\infty}^{\infty} dk\,(2h\sqrt{\pi})^{-1} e^{-\left(\frac{k\mp k^{\backprime}}{2h}\right)^2} f(k) = f(\pm k^{\backprime});$$

the sign of integration is ∴ affixed only (after all reductions) to terms of the form $\dfrac{\overset{\cos}{\underset{\sin}{.}} kx^{\backprime}}{k \mp k^{\backprime}} F(k,t)$. This latter function $F(k,t)$ involves $\overset{\cos}{\underset{\sin}{.}} st$; & here also the velocity of propagation is $\dfrac{ds^{\backprime}}{dk^{\backprime}}$.

Thus the parts of the 2 respective sets of integrals

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, d_t \cos(ux+vy+wz), \quad \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, g_t \sin(ux+vy+wz),$$
$$e_t \qquad , \qquad h_t \qquad ,$$
$$f_t \qquad , \qquad i_t \qquad ,$$

which are respectively even and odd functions of t, are

$$\tfrac{1}{2}A^{\backprime}\cos k^{\backprime}x^{\backprime}\cos s^{\backprime}t, \quad \tfrac{1}{2}A^{\backprime}\sin k^{\backprime}x^{\backprime}\sin s^{\backprime}t,$$
$$B^{\backprime} \qquad , \qquad B^{\backprime} \qquad ,$$
$$C^{\backprime} \qquad , \qquad C^{\backprime} \qquad ;$$

& ∴ by adding these we obtain

$$\tfrac{1}{2}A^{\backprime}\cos(k^{\backprime}x^{\backprime} - s^{\backprime}t), \quad \tfrac{1}{2}B^{\backprime}\cos(k^{\backprime}x^{\backprime} - s^{\backprime}t), \quad \tfrac{1}{2}C^{\backprime}\cos(k^{\backprime}x^{\backprime} - s^{\backprime}t),$$

as the parts of the final expressions for $(\delta x)_t$, $(\delta y)_t$, $(\delta z)_t$ which do not (after all reductions) involve the sign of integration.

The other parts will be less simple. The parts of $\iiint \ldots d_t\cos(ux+vy+wz)$, &c., which are odd functions of t, are the sums of the three values of

$$\frac{s^{\backprime}k^{\backprime}}{2\pi}\int_{-\infty}^{\infty}\frac{dk\cos kx^{\backprime}\sin st}{s(k^2-k^{\backprime 2})} A(AA^{\backprime}+BB^{\backprime}+CC^{\backprime}), \text{ \&c.,}$$

corresponding to the three systems of values of A, B, C and s; and the parts of

$$\iiint \ldots g_t \sin(ux+vy+wz), \text{ \&c.,}$$

which are even functions of t, are the sums of the 3 values of

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{k \sin kx` \cos st}{k^2 - k`^2} A\,(AA` + BB` + CC`), \text{ \&c.}$$

Thus the whole remaining parts of $(\delta x)_t$, $(\delta y)_t$, $(\delta z)_t$ are the sums of the 3 values of

$$\int_0^{\infty} dk \frac{s`k` \cos kx` \sin st - sk \sin kx` \cos st}{\pi s\,(k^2 - k`^2)} A\,(AA` + BB` + CC`), \text{ \&c.,}$$

which sums consequently are

$$= -\tfrac{1}{2}A` \cos(k`x` - s`t) + (\delta x)_t, \text{ \&c.}$$

And thus we express rigorously the effect at the time t, corresponding to an initial system of velocities and displacements extending only to half the medium. And we confirm the theorem that the velocity of propagation of vibration is $\frac{ds`}{dk`}$*; for we find that the vibration does not sensibly attain a place* $x`$ *till* $x` < t\frac{ds`}{dk`}$.