XIV.

CALCULUS OF PRINCIPAL RELATIONS

A NEW SERIES OF INVESTIGATIONS

[1836.]

[Note Book 42.]

[CONTENTS

[*Statement of the theory for a single differential equation of the first order.*]

(Feb. 18th, 1836.)

[1.] Let the following be any proposed or original ordinary differential equation of the first order:

$$(1) \qquad 0 = f(x, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'),$$

in which

$$(2) \qquad x_1' = \frac{dx_1}{dx}, \quad x_2' = \frac{dx_2}{dx}, \quad \ldots, \quad x_n' = \frac{dx_n}{dx}.$$

If $n = 1$, the equation is simply

$$(3) \qquad 0 = f(x, x_1, x_1'),$$

and then it has always an integral of the form

$$(4) \qquad 0 = F(a, a_1, x, x_1),$$

a_1 being the value of x_1 which corresponds to the value a of x. But if $n > 1$, the one original differential equation (1) is not in general sufficient to determine the forms of the two or more functions $x_1, \ldots, x_n$; however it will still conduct in general to what I call the *principal integral relation* between those functions and their initial values, of the form

$$(5) \qquad 0 = F(a, a_1, \ldots, a_n, x, x_1, \ldots, x_n),$$

provided that we combine it with the following $n-1$ *principal supplementary differential equations* of the second order:

$$(6) \qquad \frac{\dfrac{\delta f}{\delta x_1} - \dfrac{d}{dx}\dfrac{\delta f}{\delta x_1'}}{\dfrac{\delta f}{\delta x_1'}} = \ldots\ldots = \frac{\dfrac{\delta f}{\delta x_n} - \dfrac{d}{dx}\dfrac{\delta f}{\delta x_n'}}{\dfrac{\delta f}{\delta x_n'}},$$

assigned by the Calculus of Variations. Since these supplementary equations enable us to choose λ so that

$$(7)\qquad \lambda\frac{\delta f}{\delta x_1}=\frac{d}{dx}\left(\lambda\frac{\delta f}{\delta x_1'}\right),\quad \ldots,\quad \lambda\frac{\delta f}{\delta x_n}=\frac{d}{dx}\left(\lambda\frac{\delta f}{\delta x_n'}\right),$$

we see that if we multiply by λ, thus chosen, the variation of the original differential equation (1), namely

$$(8)\qquad 0=\frac{\delta f}{\delta x}\delta x+\frac{\delta f}{\delta x_1}\delta x_1+\ldots+\frac{\delta f}{\delta x_n}\delta x_n+\frac{\delta f}{\delta x_1'}\delta x_1'+\ldots+\frac{\delta f}{\delta x_n'}\delta x_n',$$

in which

$$(9)\qquad 0=\frac{\delta f}{\delta x}+\frac{\delta f}{\delta x_1}x_1'+\ldots+\frac{\delta f}{\delta x_n}x_n'+\frac{\delta f}{\delta x_1'}x_1''+\ldots+\frac{\delta f}{\delta x_n'}x_n'',$$

we shall get

$$(10)\qquad 0=\left(\lambda\frac{\delta f}{\delta x_1'}\right)'(\delta x_1-x_1'\delta x)+\ldots+\left(\lambda\frac{\delta f}{\delta x_n'}\right)'(\delta x_n-x_n'\delta x)$$
$$+\lambda\frac{\delta f}{\delta x_1'}(\delta x_1'-x_1''\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n'-x_n''\delta x),$$

in which $\delta x_i'=\delta\frac{dx_i}{dx}=\frac{d}{dx}\delta x_i-x_i'\frac{d}{dx}\delta x$, and therefore

$$(11)\qquad \delta x_i'-x_i''\delta x=\frac{d}{dx}(\delta x_i-x_i'\delta x)=(\delta x_i-x_i'\delta x)'.$$

Hence

$$(12)\qquad 0=\Delta\left\{\lambda\frac{\delta f}{\delta x_1'}(\delta x_1-x_1'\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n-x_n'\delta x)\right\};$$

that is,

$$(13)\qquad 0=\lambda_x\left\{\frac{\delta f_x}{\delta x_1'}(\delta x_1-x_1'\delta x)+\ldots+\frac{\delta f_x}{\delta x_n'}(\delta x_n-x_n'\delta x)\right\}$$
$$-\lambda_a\left\{\frac{\delta f_a}{\delta a_1'}(\delta a_1-a_1'\delta a)+\ldots+\frac{\delta f_a}{\delta a_n'}(\delta a_n-a_n'\delta a)\right\},$$

if we denote by λ_x and f_x the final, and by λ_a and f_a the initial values of λ and f.

Comparing the equation (13), which is the *integral of the variation* of the original differential equation (1), with the *variation of the principal integral relation* (5), namely

$$(14)\qquad 0=\frac{\delta F}{\delta a}\delta a+\frac{\delta F}{\delta a_1}\delta a_1+\ldots+\frac{\delta F}{\delta a_n}\delta a_n+\frac{\delta F}{\delta x}\delta x+\frac{\delta F}{\delta x_1}\delta x_1+\ldots+\frac{\delta F}{\delta x_n}\delta x_n,$$

and observing that the coefficients of the one must be proportional to those of the other, we find the n final equations

$$(15)\qquad \begin{cases} 0=\dfrac{\delta F}{\delta x_1}\left(x_1'\dfrac{\delta f_x}{\delta x_1'}+\ldots+x_n'\dfrac{\delta f_x}{\delta x_n'}\right)+\dfrac{\delta F}{\delta x}\dfrac{\delta f_x}{\delta x_1'}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0=\dfrac{\delta F}{\delta x_n}\left(x_1'\dfrac{\delta f_x}{\delta x_1'}+\ldots+x_n'\dfrac{\delta f_x}{\delta x_n'}\right)+\dfrac{\delta F}{\delta x}\dfrac{\delta f_x}{\delta x_n'}; \end{cases}$$

the n initial equations

$$(16)\qquad \begin{cases} 0=\dfrac{\delta F}{\delta a_1}\left(a_1'\dfrac{\delta f_a}{\delta a_1'}+\ldots+a_n'\dfrac{\delta f_a}{\delta a_n'}\right)+\dfrac{\delta F}{\delta a}\dfrac{\delta f_a}{\delta a_1'},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ 0=\dfrac{\delta F}{\delta a_n}\left(a_1'\dfrac{\delta f_a}{\delta a_1'}+\ldots+a_n'\dfrac{\delta f_a}{\delta a_n'}\right)+\dfrac{\delta F}{\delta a}\dfrac{\delta f_a}{\delta a_n'};\end{cases}$$

and this other equation

$$(17)\qquad 0=\lambda_a\left(a_1'\frac{\delta f_a}{\delta a_1'}+\ldots+a_n'\frac{\delta f_a}{\delta a_n'}\right)\frac{\delta F}{\delta x}+\lambda_x\left(x_1'\frac{\delta f_x}{\delta x_1'}+\ldots+x_n'\frac{\delta f_x}{\delta x_n'}\right)\frac{\delta F}{\delta a}.$$

The n differential coefficients or derived functions $x_1', \ldots, x_n'$ can in general be eliminated between the $n+1$ equations (1) and (15), and the result will be a partial differential equation of the first order and of the form

$$(18)\qquad 0=\Psi\left(\frac{\dfrac{\delta F}{\delta x_1}}{\dfrac{\delta F}{\delta x}}, \ldots, \frac{\dfrac{\delta F}{\delta x_n}}{\dfrac{\delta F}{\delta x}}, x, x_1, \ldots, x_n\right),$$

which the principal integral relation (5) must satisfy. In like manner, by eliminating $a_1', \ldots, a_n'$ between the n equations (16) and the initial form of (1), namely

$$(19)\qquad 0=f(a, a_1, \ldots, a_n, a_1', \ldots, a_n'),$$

we obtain this other partial differential equation to be satisfied also by (5),

$$(20)\qquad 0=\Psi\left(\frac{\dfrac{\delta F}{\delta a_1}}{\dfrac{\delta F}{\delta a}}, \ldots, \frac{\dfrac{\delta F}{\delta a_n}}{\dfrac{\delta F}{\delta a}}, a, a_1, \ldots, a_n\right),$$

Ψ being the same function as in (18).

When the form of F is known, as well as that of f, and when both these forms are substituted in the equations (16), those n equations along with the principal integral relation (5) reduce themselves to only n distinct equations, which however are in general sufficient to determine the forms of the n functions $x_1, \ldots, x_n$. These n functions involve the independent variable x and the $2n$ arbitrary constants $a_1, \ldots, a_n, a_1', \ldots, a_n'$, which are not however all independent, being connected by the relation (19) when a is given or assumed. Thus the system of $n+1$ equations (5) and (16) is equivalent to a system of only n distinct equations and is a form for the complete integral, with $2n-1$ arbitrary constants, of the system of n equations (1) and (6).

[*Extension to a number of differential equations of any order.*]

[2.] More generally, let there be a proposed differential equation:

$$(21)\qquad 0=f(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)});$$

and let it be combined with the following supplementary differential equations:

$$(22)\qquad \begin{cases} 0=\lambda\dfrac{\delta f}{\delta x_1}-\dfrac{d}{dx}\left(\lambda\dfrac{\delta f}{\delta x_1'}\right)+\ldots+(-1)^{\omega_1}\left(\dfrac{d}{dx}\right)^{\omega_1}\left(\lambda\dfrac{\delta f}{\delta x_1^{(\omega_1)}}\right),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ 0=\lambda\dfrac{\delta f}{\delta x_n}-\dfrac{d}{dx}\left(\lambda\dfrac{\delta f}{\delta x_n'}\right)+\ldots+(-1)^{\omega_n}\left(\dfrac{d}{dx}\right)^{\omega_n}\left(\lambda\dfrac{\delta f}{\delta x_n^{(\omega_n)}}\right).\end{cases}$$

We shall then have

$$(23)\quad 0=\lambda(\delta f-f'\delta x)=\lambda\left\{\frac{\delta f}{\delta x_1}(\delta x_1-x_1'\delta x)+\ldots+\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1^{(\omega_1)}-x_1^{(\omega_1+1)}\delta x)\right\}$$
$$+\ldots\ldots\ldots$$
$$+\lambda\left\{\frac{\delta f}{\delta x_n}(\delta x_n-x_n'\delta x)+\ldots+\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n^{(\omega_n)}-x_n^{(\omega_n+1)}\delta x)\right\}$$
$$=\lambda\frac{\delta f}{\delta x_1'}(\delta x_1'-x_1''\delta x)+\left(\lambda\frac{\delta f}{\delta x_1'}\right)'(\delta x_1-x_1'\delta x)+\ldots+\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1^{(\omega_1)}-x_1^{(\omega_1+1)}\delta x)$$
$$+(-1)^{\omega_1+1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)}(\delta x_1-x_1'\delta x)$$
$$+\ldots\ldots\ldots$$
$$+\lambda\frac{\delta f}{\delta x_n'}(\delta x_n'-x_n''\delta x)+\left(\lambda\frac{\delta f}{\delta x_n'}\right)'(\delta x_n-x_n'\delta x)+\ldots$$
$$+\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n^{(\omega_n)}-x_n^{(\omega_n+1)}\delta x)+(-1)^{\omega_n+1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)}(\delta x_n-x_n'\delta x);$$

in which

$$(24)\quad \delta x_i^{(\omega_i)}-x_i^{(\omega_i+1)}\delta x=(\delta x_i-x_i'\delta x)^{(\omega_i)}.$$

Also in general

$$(25)\quad yz^{(\omega_i)}+(-1)^{\omega_i+1}y^{(\omega_i)}z=\{yz^{(\omega_i-1)}-y'z^{(\omega_i-2)}+y''z^{(\omega_i-3)}-\ldots+(-1)^{\omega_i+1}y^{(\omega_i-1)}z\}'.$$

Therefore

$$(26)\quad 0=\lambda(\delta f-f'\delta x)$$
$$=\left[\left\{\lambda\frac{\delta f}{\delta x_1'}-\left(\lambda\frac{\delta f}{\delta x_1''}\right)'+\ldots+(-1)^{\omega_1-1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-1)}\right\}(\delta x_1-x_1'\delta x)\right]'$$
$$+\ldots\ldots$$
$$+\left[\left\{\lambda\frac{\delta f}{\delta x_n'}-\left(\lambda\frac{\delta f}{\delta x_n''}\right)'+\ldots+(-1)^{\omega_n-1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-1)}\right\}(\delta x_n-x_n'\delta x)\right]'$$
$$+\left[\left\{\lambda\frac{\delta f}{\delta x_1''}-\left(\lambda\frac{\delta f}{\delta x_1'''}\right)'+\ldots+(-1)^{\omega_1-2}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-2)}\right\}(\delta x_1-x_1'\delta x)'\right]'$$
$$+\ldots\ldots$$
$$+\left[\left\{\lambda\frac{\delta f}{\delta x_n''}-\left(\lambda\frac{\delta f}{\delta x_n'''}\right)'+\ldots+(-1)^{\omega_n-2}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-2)}\right\}(\delta x_n-x_n'\delta x)'\right]'$$
$$+\&c.$$
$$+\left[\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1-x_1'\delta x)^{(\omega_1-1)}\right]'+\ldots+\left[\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n-x_n'\delta x)^{(\omega_n-1)}\right]';$$

so that, by integration,

$$(27)\quad 0=\int_a^x \lambda(\delta f - f'\delta x)\,dx$$

$$=\Delta\Bigg[\left\{\lambda\frac{\delta f}{\delta x_1'}-\left(\lambda\frac{\delta f}{\delta x_1''}\right)'+\ldots+(-1)^{\omega_1-1}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-1)}\right\}(\delta x_1-x_1'\delta x)$$

$$+\left\{\lambda\frac{\delta f}{\delta x_1''}-\left(\lambda\frac{\delta f}{\delta x_1'''}\right)'+\ldots+(-1)^{\omega_1-2}\left(\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1-2)}\right\}(\delta x_1-x_1'\delta x)'$$

$$+\ldots+\lambda\frac{\delta f}{\delta x_1^{(\omega_1)}}(\delta x_1-x_1'\delta x)^{(\omega_1-1)}$$

$$+\&\text{c.}$$

$$+\left\{\lambda\frac{\delta f}{\delta x_n'}-\left(\lambda\frac{\delta f}{\delta x_n''}\right)'+\ldots+(-1)^{\omega_n-1}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-1)}\right\}(\delta x_n-x_n'\delta x)$$

$$+\left\{\lambda\frac{\delta f}{\delta x_n''}-\left(\lambda\frac{\delta f}{\delta x_n'''}\right)'+\ldots+(-1)^{\omega_n-2}\left(\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n-2)}\right\}(\delta x_n-x_n'\delta x)'$$

$$+\ldots+\lambda\frac{\delta f}{\delta x_n^{(\omega_n)}}(\delta x_n-x_n'\delta x)^{(\omega_n-1)}\Bigg].$$

The general term is

$$\Delta\left[(-1)^\alpha\left(\lambda\frac{\delta f}{\delta x_m^{(\beta+1)}}\right)^{(\alpha)}(\delta x_m-x_m'\delta x)^{(\beta-\alpha)}\right],$$

where α may have any value $\not> \beta$ and β may have any value $<\omega_m$, m having any value >0 but $\not> n$. Thus

$$(28)\qquad 0=\ \Sigma_{(\alpha)0}^{\beta}\,\Sigma_{(\beta)0}^{\omega_m-1}\,\Sigma_{(m)1}^{n}\left\{(-1)^\alpha\left(\lambda_x\frac{\delta f_x}{\delta x_m^{(\beta+1)}}\right)^{(\alpha)}(\delta x_m-x_m'\delta x)^{(\beta-\alpha)}\right\}$$
$$-\Sigma_{(\alpha)0}^{\beta}\,\Sigma_{(\beta)0}^{\omega_m-1}\,\Sigma_{(m)1}^{n}\left\{(-1)^\alpha\left(\lambda_a\frac{\delta f_a}{\delta a_m^{(\beta+1)}}\right)^{(\alpha)}(\delta a_m-a_m'\delta a)^{(\beta-\alpha)}\right\}.$$

More generally still, let

$$(29)\qquad\begin{cases}0=f_1\,(x,x_1,x_1',\ldots,x_1^{(\omega_{1,1})},x_2,x_2',\ldots,x_2^{(\omega_{1,2})},\ldots,x_n,x_n',\ldots,x_n^{(\omega_{1,n})}),\\ 0=f_2\,(x,x_1,x_1',\ldots,x_1^{(\omega_{2,1})},x_2,x_2',\ldots,x_2^{(\omega_{2,2})},\ldots,x_n,x_n',\ldots,x_n^{(\omega_{2,n})}),\\ \ldots\ldots\\ 0=f_m(x,x_1,x_1',\ldots,x_1^{(\omega_{m,1})},x_2,x_2',\ldots,x_2^{(\omega_{m,2})},\ldots,x_n,x_n',\ldots,x_n^{(\omega_{m,n})})\end{cases}$$

be any m original differential equations between any n functions $x_1, \ldots, x_n$ of an independent variable x; and let them be combined with the n following supplemental differential equations between the same functions and m multipliers $\lambda_1, \ldots, \lambda_m$:

$$(30)\qquad\begin{cases}0=\ \lambda_1\dfrac{\delta f_1}{\delta x_1}-\left(\lambda_1\dfrac{\delta f_1}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_{1,1}}\left(\lambda_1\dfrac{\delta f_1}{\delta x_1^{(\omega_{1,1})}}\right)^{(\omega_{1,1})}\\ \quad+\ldots\ldots\\ \quad+\lambda_m\dfrac{\delta f_m}{\delta x_1}-\left(\lambda_m\dfrac{\delta f_m}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_{m,1}}\left(\lambda_m\dfrac{\delta f_m}{\delta x_1^{(\omega_{m,1})}}\right)^{(\omega_{m,1})},\\ \ldots\ldots\ldots\ldots\\ 0=\ \lambda_1\dfrac{\delta f_1}{\delta x_n}-\left(\lambda_1\dfrac{\delta f_1}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_{1,n}}\left(\lambda_1\dfrac{\delta f_1}{\delta x_n^{(\omega_{1,n})}}\right)^{(\omega_{1,n})}\\ \quad+\ldots\ldots\\ \quad+\lambda_m\dfrac{\delta f_m}{\delta x_n}-\left(\lambda_m\dfrac{\delta f_m}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_{m,n}}\left(\lambda_m\dfrac{\delta f_m}{\delta x_n^{(\omega_{m,n})}}\right)^{(\omega_{m,n})}.\end{cases}$$

Then

$$(31)\qquad \lambda_1(\delta f_1 - f_1'\delta x) + \ldots + \lambda_m(\delta f_m - f_m'\delta x)$$

is an exact derived function, or total differential coefficient, namely the differential coefficient of the expression $\Sigma L_{i,k,l}$, in which

$$(32)\qquad L'_{i,k,l} = \lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}(\delta x_k^{(l+1)} - x_k^{(l+2)}\delta x) + (-1)^l \left(\lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}\right)^{(l+1)} (\delta x_k - x_k'\delta x),$$

and

$$(33)\qquad L_{i,k,l} = \lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}(\delta x_k^{(l)} - x_k^{(l+1)}\delta x) - \ldots + (-1)^l \left(\lambda_i \frac{\delta f_i}{\delta x_k^{(l+1)}}\right)^{(l)} (\delta x_k - x_k'\delta x);$$

and hence, by integration,*

$$(34)\qquad 0 = \Sigma_{(\alpha)0}^{\beta} \Sigma_{(\beta)0}^{\omega_{\mu,\nu}-1} \Sigma_{(\mu)1}^{m} \Sigma_{(\nu)1}^{n} \left\{ (-1)^\alpha \left(\lambda_{\mu,x} \frac{\delta f_{\mu,x}}{\delta x_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} - (-1)^\alpha \left(\lambda_{\mu,a} \frac{\delta f_{\mu,a}}{\delta a_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta a_\nu - a_\nu'\delta a)^{(\beta-\alpha)} \right\}.$$

(Feb. 19th.)

So far the Calculus of Variations conducts. But, proceeding to the Calculus of Principal Relations, it may be shown that the auxiliary or supplementary system of differential equations (30) conducts to a *principal integral* of the original system (29) of the form†

$$(35)\qquad 0 = F\begin{pmatrix} x, x_1, x_1', \ldots, x_1^{(\omega_1-1)}, x_2, x_2', \ldots, x_2^{(\omega_2-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n-1)} \\ a, a_1, a_1', \ldots, a_1^{(\omega_1-1)}, a_2, a_2', \ldots, a_2^{(\omega_2-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega_n-1)} \end{pmatrix};$$

ω_1 being the greatest exponent of the series $\omega_{1,1}, \omega_{2,1}, \ldots, \omega_{m,1}$, or more precisely an exponent of that series not less than any other exponent of the same series; ω_2 being an exponent similarly selected from the series $\omega_{1,2}, \omega_{2,2}, \ldots, \omega_{m,2}$, and so on. Hence, taking the variation of the principal integral (35), we find

$$(36)\qquad 0 = \frac{\delta F}{\delta x}\delta x + \frac{\delta F}{\delta x_1}\delta x_1 + \ldots + \frac{\delta F}{\delta x_1^{(\omega_1-1)}}\delta x_1^{(\omega_1-1)} + \ldots + \frac{\delta F}{\delta x_n}\delta x_n + \ldots + \frac{\delta F}{\delta x_n^{(\omega_n-1)}}\delta x_n^{(\omega_n-1)}$$
$$+ \frac{\delta F}{\delta a}\delta a + \frac{\delta F}{\delta a_1}\delta a_1 + \ldots + \frac{\delta F}{\delta a_1^{(\omega_1-1)}}\delta a_1^{(\omega_1-1)} + \ldots + \frac{\delta F}{\delta a_n}\delta a_n + \ldots + \frac{\delta F}{\delta a_n^{(\omega_n-1)}}\delta a_n^{(\omega_n-1)};$$

and the fundamental theorem of the Calculus of Principal Relations, for the integration of total differential equations, is that the coefficients of this last variation are proportional to the coefficients of the formula (34), or that

$$(37)\qquad \delta F = \lambda \Sigma_{(\alpha)0}^{\beta} \Sigma_{\beta(0)}^{\omega_{\mu,\nu}-1} \Sigma_{(\mu)1}^{m} \Sigma_{(\nu)1}^{n} (-1)^\alpha \left\{ \left(\lambda_{\mu,x} \frac{\delta f_{\mu,x}}{\delta x_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} - \left(\lambda_{\mu,a} \frac{\delta f_{\mu,a}}{\delta a_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta a_\nu - a_\nu'\delta a)^{(\beta-\alpha)} \right\}.$$

We have also, by the Calculus of Variations

$$(38)\qquad (\delta x_\nu - x_\nu'\delta x)^{(\beta-\alpha)} = \delta x_\nu^{(\beta-\alpha)} - x_\nu^{(\beta-\alpha+1)}\delta x,$$

* [$\lambda_{\mu,x}, f_{\mu,x}$ and $\lambda_{\mu,a}, f_{\mu,a}$ are the final and initial values of λ_μ, f_μ.]

† See subsequent investigations of these pages, respecting the general existence of this relation.

and

$$(39) \qquad (\delta a_\nu - a'_\nu \delta a)^{(\beta-\alpha)} = \delta a_\nu^{(\beta-\alpha)} - a_\nu^{(\beta-\alpha+1)} \delta a;$$

so that the fundamental theorem of the Calculus of Principal Relations may be thus written:

$$(40) \qquad \boxed{\begin{aligned}\delta F = \lambda \Sigma_{(\alpha)0}^{\beta} \Sigma_{\beta(0)}^{\omega_{\mu,\nu}-1} \Sigma_{(\mu)1}^{m} \Sigma_{(\nu)1}^{n} (-1)^\alpha \Bigg\{ &\left(\lambda_{\mu,x} \frac{\delta f_{\mu,x}}{\delta x_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta x_\nu^{(\beta-\alpha)} - x_\nu^{(\beta-\alpha+1)} \delta x) \\ &- \left(\lambda_{\mu,a} \frac{\delta f_{\mu,a}}{\delta a_\nu^{(\beta+1)}}\right)^{(\alpha)} (\delta a_\nu^{(\beta-\alpha)} - a_\nu^{(\beta-\alpha+1)} \delta a) \Bigg\}.\end{aligned}}$$

[*The case of one total differential equation of the first order resumed.*]

[3.] As an example of the application of this fundamental formula of the Calculus of Principal Relations, let $m=1$, $\omega_{1,1}=\omega_{1,2}=\ldots=\omega_{1,n}=1$. Then $\omega_{\mu,\nu}=1$, $\beta=0$, $\alpha=0$, $\mu=1$, and the formula (40) becomes

$$(41) \qquad \delta F = \lambda \Sigma_{(\nu)1}^{n} \left\{ \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_\nu} (\delta x_\nu - x'_\nu \delta x) - \lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_\nu} (\delta a_\nu - a'_\nu \delta a) \right\};$$

and resolves itself into the $2n+2$ separate equations following:

$$(42) \qquad \frac{\delta F}{\delta x} = -\lambda\lambda_{1,x} \Sigma_{(\nu)1}^{n} \left(x'_\nu \frac{\delta f_{1,x}}{\delta x'_\nu} \right), \quad \frac{\delta F}{\delta x_1} = \lambda\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_1}, \ \ldots, \ \frac{\delta F}{\delta x_n} = \lambda\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_n},$$

and

$$(43) \qquad \frac{\delta F}{\delta a} = \lambda\lambda_{1,a} \Sigma_{(\nu)1}^{n} \left(a'_\nu \frac{\delta f_{1,a}}{\delta a'_\nu} \right), \quad \frac{\delta F}{\delta a_1} = -\lambda\lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_1}, \ \ldots, \ \frac{\delta F}{\delta a_n} = -\lambda\lambda_{1,a} \frac{\delta f_{1,a}}{\delta a'_n}.$$

In this example there is only one original differential equation of the form

$$(44) \qquad 0 = f_1(x, x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n) = f_{1,x},$$

and the n supplementary differential equations are of the form

$$(45) \qquad 0 = \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x_1} - \left(\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_1} \right)', \ \ldots, \ 0 = \lambda_{1,x} \frac{\delta f_{1,x}}{\delta x_n} - \left(\lambda_{1,x} \frac{\delta f_{1,x}}{\delta x'_n} \right)';$$

they may also be thus written,

$$(46) \qquad \frac{\lambda'_{1,x}}{\lambda_{1,x}} = \frac{\dfrac{\delta f_{1,x}}{\delta x_1} - \left(\dfrac{\delta f_{1,x}}{\delta x'_1}\right)'}{\dfrac{\delta f_{1,x}}{\delta x'_1}} = \frac{\dfrac{\delta f_{1,x}}{\delta x_2} - \left(\dfrac{\delta f_{1,x}}{\delta x'_2}\right)'}{\dfrac{\delta f_{1,x}}{\delta x'_2}} = \ldots = \frac{\dfrac{\delta f_{1,x}}{\delta x_n} - \left(\dfrac{\delta f_{1,x}}{\delta x'_n}\right)'}{\dfrac{\delta f_{1,x}}{\delta x'_n}},$$

and they give in general $n-1$ supplementary differential equations of the second order between the n functions $x_1, \ldots, x_n$. If we join to these the equation

$$(47) \qquad 0 = f'_{1,x},$$

which is the differential coefficient of the original equation, we shall have n equations of the second order between the n functions $x_1, \ldots, x_n$, which can thus in general be found in terms of the final and initial values of the variable x and of the $2n$ initial data or constants $a_1, a'_1, a_2, a'_2, \ldots, a_n, a'_n$. But these initial constants are not all arbitrary and independent, being connected by the given integral relation

$$(48) \qquad 0 = f_{1,a}.$$

We may therefore in general conceive the n constants $a_1', a_2', \ldots, a_n'$ eliminated by this relation (48) between the n expressions of $x_1, x_2, \ldots, x_n$, and thus a relation obtained of the form

$$(49) \qquad 0 = F(x, x_1, x_2, \ldots, x_n, a, a_1, a_2, \ldots, a_n),$$

which is in this example the *principal integral relation* between the n functions $x_1, \ldots, x_n$, or the *principal integral of the original differential equation* (44).

The $n+1$ equations (42) give, by elimination of $\lambda\lambda_{1,x}$, the n equations following:

$$(50) \qquad 0 = \frac{\delta F}{\delta x}\frac{\delta f_{1,x}}{\delta x_1'} + \frac{\delta F}{\delta x_1}\Sigma_{(\nu)1}^n \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'}\right), \quad \ldots, \quad 0 = \frac{\delta F}{\delta x}\frac{\delta f_{1,x}}{\delta x_n'} + \frac{\delta F}{\delta x_n}\Sigma_{(\nu)1}^n \left(x_\nu' \frac{\delta f_{1,x}}{\delta x_\nu'}\right);$$

and if between the n equations (50) and the original differential equation (44) we eliminate $x_1', \ldots, x_n'$, we obtain a partial differential equation of the first order, of the form

$$(51) \qquad 0 = \Psi\left(\frac{\delta F}{\delta x}, \frac{\delta F}{\delta x_1}, \ldots, \frac{\delta F}{\delta x_n}, x, x_1, \ldots, x_n\right),$$

which the function F must satisfy. In like manner, the $n+1$ equations (43) give, by elimination of $\lambda\lambda_{1,a}$,

$$(52) \qquad 0 = \frac{\delta F}{\delta a}\frac{\delta f_{1,a}}{\delta a_1'} + \frac{\delta F}{\delta a_1}\Sigma_{(\nu)1}^n \left(a_\nu' \frac{\delta f_{1,a}}{\delta a_\nu'}\right), \quad \ldots, \quad 0 = \frac{\delta F}{\delta a}\frac{\delta f_{1,a}}{\delta a_n'} + \frac{\delta F}{\delta a_n}\Sigma_{(\nu)1}^n \left(a_\nu' \frac{\delta f_{1,a}}{\delta a_\nu'}\right);$$

and if between these and equation (48) we eliminate $a_1', \ldots a_n'$, we find this other partial differential equation of the first order

$$(53) \qquad 0 = \Psi\left(\frac{\delta F}{\delta a}, \frac{\delta F}{\delta a_1}, \ldots, \frac{\delta F}{\delta a_n}, a, a_1, \ldots, a_n\right),$$

which the function F must satisfy. In these two partial differential equations the form of Ψ is the same and is homogeneous of dimension 0 with respect to the $n+1$ partial differential coefficients, final or initial, of the function F: so that in the notation of derived functions

$$(54) \qquad 0 = \frac{\delta F}{\delta x}\Psi'\left(\frac{\delta F}{\delta x}\right) + \frac{\delta F}{\delta x_1}\Psi'\left(\frac{\delta F}{\delta x_1}\right) + \ldots + \frac{\delta F}{\delta x_n}\Psi'\left(\frac{\delta F}{\delta x_n}\right),$$

and

$$(55) \qquad 0 = \frac{\delta F}{\delta a}\Psi'\left(\frac{\delta F}{\delta a}\right) + \frac{\delta F}{\delta a_1}\Psi'\left(\frac{\delta F}{\delta a_1}\right) + \ldots + \frac{\delta F}{\delta a_n}\Psi'\left(\frac{\delta F}{\delta a_n}\right).$$

The n equations (52) and equation (49) are equivalent by (53) only to n distinct equations; but these are in general sufficient, when the form of F is known, to determine the forms of the n functions $x_1, \ldots, x_n$ in terms of the independent variable x, its assumed initial value a and any $2n-1$ of the $2n$ initial data $a_1, a_1', a_2, a_2', \ldots, a_n, a_n'$, which are themselves connected by the one equation (48). The system (49)–(52) is therefore a form for the complete integral of the system of the original differential equation (44) and the $n-1$ supplementary differential equations deduced by elimination from (45), and this integration of a system of several total differential equations, by means of one principal integral relation, is the chief use of such principal relations and the reason for giving them that name.

We have, by differentiating the principal integral relation,

$$(56)\qquad 0=F'=\frac{\delta F}{\delta x}+x_1'\frac{\delta F}{\delta x_1}+\ldots+x_n'\frac{\delta F}{\delta x_n};$$

so that if we put, for abbreviation,

$$(57)\qquad \frac{\delta F}{\delta x_1}=t_1\frac{\delta F}{\delta x},\quad \ldots,\quad \frac{\delta F}{\delta x_n}=t_n\frac{\delta F}{\delta x},$$

we shall have

$$(58)\qquad 0=1+x_1't_1+\ldots+x_n't_n.$$

At the same time, by (50),

$$(59)\qquad \frac{\delta f_{1,x}}{\delta x_1'}=-t_1\Sigma_{(\nu)1}^n\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right),\quad \ldots,\quad \frac{\delta f_{1,x}}{\delta x_n'}=-t_n\Sigma_{(\nu)1}^n\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right).$$

The variation of (58) gives

$$(60)\qquad x_1'\delta t_1+\ldots+x_n'\delta t_n=-t_1\delta x_1'-\ldots-t_n\delta x_n'.$$

Therefore by (59)

$$(61)\qquad (x_1'\delta t_1+\ldots+x_n'\delta t_n)\Sigma_{(\nu)1}^n\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right)=\frac{\delta f_{1,x}}{\delta x_1'}\delta x_1'+\ldots+\frac{\delta f_{1,x}}{\delta x_n'}\delta x_n'$$

$$=-\frac{\delta f_{1,x}}{\delta x}\delta x-\frac{\delta f_{1,x}}{\delta x_1}\delta x_1-\ldots-\frac{\delta f_{1,x}}{\delta x_n}\delta x_n;$$

but also

$$(62)\qquad x_1'\delta t_1+\ldots+x_n'\delta t_n=x_1'\delta\frac{\frac{\delta F}{\delta x_1}}{\frac{\delta F}{\delta x}}+\ldots+x_n'\delta\frac{\frac{\delta F}{\delta x_n}}{\frac{\delta F}{\delta x}}$$

$$=\left(\frac{\delta F}{\delta x}\right)^{-1}\left\{\delta\frac{\delta F}{\delta x}+x_1'\delta\frac{\delta F}{\delta x_1}+\ldots+x_n'\delta\frac{\delta F}{\delta x_n}\right\};$$

therefore

$$(63)\qquad 0=\left(\delta\frac{\delta F}{\delta x}+x_1'\delta\frac{\delta F}{\delta x_1}+\ldots+x_n'\delta\frac{\delta F}{\delta x_n}\right)\Sigma_{(\nu)1}^n\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right)$$

$$+\frac{\delta F}{\delta x}\left(\frac{\delta f_{1,x}}{\delta x}\delta x+\frac{\delta f_{1,x}}{\delta x_1}\delta x_1+\ldots+\frac{\delta f_{1,x}}{\delta x_n}\delta x_n\right).$$

Comparing this with the variation of the partial differential equation (51), that is, with the following formula:

$$(64)\qquad 0=\Psi'\left(\frac{\delta F}{\delta x}\right)\delta\frac{\delta F}{\delta x}+\Psi'\left(\frac{\delta F}{\delta x_1}\right)\delta\frac{\delta F}{\delta x_1}+\ldots+\Psi'\left(\frac{\delta F}{\delta x_n}\right)\delta\frac{\delta F}{\delta x_n}$$

$$+\Psi'(x)\delta x+\Psi'(x_1)\delta x_1+\ldots+\Psi'(x_n)\delta x_n,$$

we find

$$(65)\qquad \Psi'\left(\frac{\delta F}{\delta x_1}\right)=x_1'\Psi'\left(\frac{\delta F}{\delta x}\right),\quad \ldots,\quad \Psi'\left(\frac{\delta F}{\delta x_n}\right)=x_n'\Psi'\left(\frac{\delta F}{\delta x}\right),$$

$$(66)\qquad \Psi'(x_1)\frac{\delta f_{1,x}}{\delta x}=\Psi'(x)\frac{\delta f_{1,x}}{\delta x_1},\quad \ldots,\quad \Psi'(x_n)\frac{\delta f_{1,x}}{\delta x}=\Psi'(x)\frac{\delta f_{1,x}}{\delta x_n},$$

and

$$(67)\qquad \frac{\delta F}{\delta x}\Psi'\left(\frac{\delta F}{\delta x}\right)\frac{\delta f_{1,x}}{\delta x}=\Psi'(x)\Sigma_{(\nu)1}^n\left(x_\nu'\frac{\delta f_{1,x}}{\delta x_\nu'}\right).$$

The equations (65) agree evidently with (54) and (56).

Let $F+\Delta F$ be a function nearly equal to F, and let it be substituted instead of F in the partial differential equation (51) and the result developed by Taylor's theorem as far as the first powers of the coefficients of the small function ΔF. We get

$$(68)\qquad \Psi\left(\frac{\delta F}{\delta x}+\frac{\delta\Delta F}{\delta x}, \frac{\delta F}{\delta x_1}+\frac{\delta\Delta F}{\delta x_1}, \ldots, \frac{\delta F}{\delta x_n}+\frac{\delta\Delta F}{\delta x_n}, x, x_1, \ldots, x_n\right)=\Psi+\Delta\Psi=\Delta\Psi$$

$$=\Psi'\left(\frac{\delta F}{\delta x}\right)\frac{\delta\Delta F}{\delta x}+\Psi'\left(\frac{\delta F}{\delta x_1}\right)\frac{\delta\Delta F}{\delta x_1}+\ldots+\Psi'\left(\frac{\delta F}{\delta x_n}\right)\frac{\delta\Delta F}{\delta x_n},$$

that is, by (65),

$$(69)\qquad \Delta\Psi=\Psi'\left(\frac{\delta F}{\delta x}\right)\left(\frac{\delta\Delta F}{\delta x}+x_1'\frac{\delta\Delta F}{\delta x_1}+\ldots+x_n'\frac{\delta\Delta F}{\delta x_n}\right)=\Psi'\left(\frac{\delta F}{\delta x}\right)\frac{d\Delta F}{dx};$$

and therefore

$$(70)\qquad \Delta F=\int\left\{\Psi'\left(\frac{\delta F}{\delta x}\right)\right\}^{-1}\Delta\Psi\, dx.$$

And if we choose that F and ΔF shall vanish when $x=a$ (which we are always at liberty to do), we may then take a and x for the limits of the integration and write

$$(71)\qquad \boxed{\Delta F=\int_a^x\left\{\Psi'\left(\frac{\delta F}{\delta x}\right)\right\}^{-1}\Delta\Psi\, dx.}$$

This theorem is very important in the Calculus of Principal Relations for it enables us theoretically to express and often practically to calculate the small correction of an approximate form $F+\Delta F=0$ for the principal relation (40) by means of a definite integral.*

The important equations (65) might have been obtained more simply by observing that

$$(72)\qquad 0=d_x\delta_x F-\delta_x d_x F=d\left(\frac{\delta F}{\delta x}\delta x+\frac{\delta F}{\delta x_1}\delta x_1+\ldots+\frac{\delta F}{\delta x_n}\delta x_n\right)$$

$$-\delta\left(\frac{\delta F}{\delta x}dx+\frac{\delta F}{\delta x_1}dx_1+\ldots+\frac{\delta F}{\delta x_n}dx_n\right)$$

$$=d\frac{\delta F}{\delta x}.\delta x+d\frac{\delta F}{\delta x_1}.\delta x_1+\ldots+d\frac{\delta F}{\delta x_n}.\delta x_n-\delta\frac{\delta F}{\delta x}.dx-\delta\frac{\delta F}{\delta x_1}.dx_1-\ldots-\delta\frac{\delta F}{\delta x_n}.dx_n,$$

that is,

$$(73)\qquad 0=\delta\frac{\delta F}{\delta x}+x_1'\delta\frac{\delta F}{\delta x_1}+\ldots+x_n'\delta\frac{\delta F}{\delta x_n}-\left(\frac{\delta F}{\delta x}\right)'\delta x-\left(\frac{\delta F}{\delta x_1}\right)'\delta x_1-\ldots-\left(\frac{\delta F}{\delta x_n}\right)'\delta x_n.$$

Comparing this with (64) we find not only equations (65) but also

$$(74)\qquad \Psi'(x)=-\left(\frac{\delta F}{\delta x}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),\quad \Psi'(x_1)=-\left(\frac{\delta F}{\delta x_1}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),\quad \ldots,$$

$$\Psi'(x_n)=-\left(\frac{\delta F}{\delta x_n}\right)'\Psi'\left(\frac{\delta F}{\delta x}\right),$$

equations simpler than (66) and (67).

With respect to the logic of (72), we may observe that $x, x_1, \ldots, x_n$ are not all arbitrary when $a, a_1, \ldots, a_n$ are given but are then connected by the principal integral relation $F=0$, if the functions $x_1, \ldots, x_n$ are to satisfy the original differential equation and the principal supple-

* [This method of approximating to the principal function is exactly the same as Hamilton used in the Second Essay on Dynamics. Cf. equation (F), p. 170.]

mentary equations assigned above. But, consistently with these differential equations and data, we may vary $x, x_1, \ldots, x_n$ provided that we make the variations satisfy the equation

$$0=\delta_x F=\frac{\delta F}{\delta x}\delta x+\frac{\delta F}{\delta x_1}\delta x_1+\ldots+\frac{\delta F}{\delta x_n}\delta x_n, \tag{75}$$

omitting squares and products of the variations. At the same time $x_1', \ldots, x_n'$ will in general vary but must satisfy (among others) the following equation, deduced by differentiation from (75),

$$\begin{aligned}0&=\frac{d}{dx}\left\{\frac{\delta F}{\delta x_1}(\delta x_1-x_1'\delta x)+\ldots+\frac{\delta F}{\delta x_n}(\delta x_n-x_n'\delta x)\right\}\\&=\left(\frac{\delta F}{\delta x_1}\right)'(\delta x_1-x_1'\delta x)+\ldots+\left(\frac{\delta F}{\delta x_n}\right)'(\delta x_n-x_n'\delta x)\\&\qquad+\frac{\delta F}{\delta x_1}(\delta x_1'-x_1''\delta x)+\ldots+\frac{\delta F}{\delta x_n}(\delta x_n'-x_n''\delta x)\\&=\left(\frac{\delta F}{\delta x}\right)'\delta x+\left(\frac{\delta F}{\delta x_1}\right)'\delta x_1+\ldots+\left(\frac{\delta F}{\delta x_n}\right)'\delta x_n+\frac{\delta F}{\delta x_1}\delta x_1'+\ldots+\frac{\delta F}{\delta x_n}\delta x_n';\end{aligned} \tag{76}$$

and also the following, deduced by variation from (56),

$$0=\delta\frac{\delta F}{\delta x}+x_1'\delta\frac{\delta F}{\delta x_1}+\ldots+x_n'\delta\frac{\delta F}{\delta x_n}+\frac{\delta F}{\delta x_1}\delta x_1'+\ldots+\frac{\delta F}{\delta x_n}\delta x_n'. \tag{77}$$

Comparing therefore (76) and (77) we find that we can eliminate between these two equations all the n variations $\delta x_1', \ldots, \delta x_n'$, and that the equation (73) results.

[*Two or more differential equations of the first order.*]

[4.] As another example of the application of the formulae (29)–(40), let there be two original differential equations of the first order

$$\begin{cases}0=f_1(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n'),\\0=f_2(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n'),\end{cases} \tag{78}$$

so that now $m=2$, $\omega_{\mu,\nu}=1$, $\beta=0$, $\alpha=0$, and equation (40) becomes

$$\delta F=\lambda\Sigma_{(\mu)1}^2\Sigma_{(\nu)1}^n\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}(\delta x_\nu-x_\nu'\delta x)-\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}(\delta a_\nu-a_\nu'\delta a)\right\}. \tag{79}$$

It resolves itself into $2n+2$ separate equations

$$\frac{\delta F}{\delta x}=-\lambda\Sigma_{(\mu)1}^2\Sigma_{(\nu)1}^n\left(\lambda_{\mu,x}x_\nu'\frac{\delta f_{\mu,x}}{\delta x_\nu'}\right),\ \frac{\delta F}{\delta x_1}=\lambda\Sigma_{(\mu)1}^2\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right),\ \ldots,\ \frac{\delta F}{\delta x_n}=\lambda\Sigma_{(\mu)1}^2\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right), \tag{80}$$

and

$$\frac{\delta F}{\delta a}=\lambda\Sigma_{(\mu)1}^2\Sigma_{(\nu)1}^n\left(\lambda_{\mu,a}a_\nu'\frac{\delta f_{\mu,a}}{\delta a_\nu'}\right),\ \frac{\delta F}{\delta a_1}=-\lambda\Sigma_{(\mu)1}^2\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_1'}\right),\ \ldots,\ \frac{\delta F}{\delta a_n}=-\lambda\Sigma_{(\mu)1}^2\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_n'}\right). \tag{81}$$

(Feb. 20th.)

The supplementary differential equations are now

$$0=\Sigma_{(\mu)1}^2\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'\right\},\ \ldots,\ 0=\Sigma_{(\mu)1}^2\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'\right\}. \tag{82}$$

They are n in number but they involve 2 unknown multipliers $\lambda_{1,x}$, $\lambda_{2,x}$ and their first differential coefficients $\lambda'_{1,x}$, $\lambda'_{2,x}$. They may be differentiated any number i of times, and thus we may obtain in new equations of the form

$$(83)\quad \left\{\begin{array}{l} 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}\right)' - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x'_1}\right)''\right\}, \ldots, 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}\right)' - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x'_n}\right)''\right\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}\right)^{(i)} - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x'_1}\right)^{(i+1)}\right\}, \ldots, \\ \qquad\qquad 0=\Sigma^2_{(\mu)1}\left\{\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}\right)^{(i)} - \left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x'_n}\right)^{(i+1)}\right\}; \end{array}\right.$$

but these involve the $2i$ additional differential coefficients $\lambda''_{1,x}, \lambda''_{2,x}, \ldots, \lambda^{(i+1)}_{1,x}, \lambda^{(i+1)}_{2,x}$. If we take $i>0$, then $n+in$ will be greater than $3+2i$ (n being supposed greater than 2), and then by eliminating the $3+2i$ ratios of the $4+2i$ multipliers $\lambda_{1,x}, \lambda_{2,x}, \lambda'_{1,x}, \lambda'_{2,x}, \ldots, \lambda^{(i+1)}_{1,x}, \lambda^{(i+1)}_{2,x}$ between the $n+in$ equations (82) and (83) we shall obtain a number $i(n-2)+n-3$ of resulting differential equations of the order $i+2$ between the functions $x_1, \ldots, x_n$. Even if we take $i=0$, we shall thus have $n-3$ equations of the second order between these n functions by eliminating the 3 ratios of $\lambda_{1,x}, \lambda_{2,x}, \lambda'_{1,x}, \lambda'_{2,x}$ between the n equations (82); but we must join to these another equation, which will be of the third order, in order to determine the forms of the n functions $x_1, \ldots, x_n$; and then we shall have two equations of the first order, $n-3$ of the second and one of the third, giving for the sum of the exponents of the orders $2\times 1+(n-3)\times 2+1\times 3=2n-1$: so that the complete expressions of the n functions $x_1, \ldots, x_n$ will involve $2n-1$ independent constants and, by eliminating $n-1$ of them, we can in general obtain a relation of the form (49), which I have called the principal integral relation between the final and initial values of the n functions $x_1, \ldots, x_n$ and of the independent variable x, namely,

$$(49)\qquad 0=F(x, x_1, \ldots, x_n, a, a_1, \ldots, a_n).$$

Comparing its variation with (79), we find (80) and (81). Any three of equations (82) enable us to express $\frac{\lambda_{2,x}}{\lambda_{1,x}}$ in terms of $x, x_1, \ldots, x_n, x'_1, \ldots, x'_n, x''_1, \ldots, x''_n$, and therefore $\frac{\lambda_{2,a}}{\lambda_{1,a}}$ in terms of $a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$. Thus the n equations obtained by eliminating λ between (81) will be n equations between

$$\frac{\frac{\delta F}{\delta a_1}}{\frac{\delta F}{\delta a}}, \ldots, \frac{\frac{\delta F}{\delta a_n}}{\frac{\delta F}{\delta a}}, a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n,$$

when this value of $\frac{\lambda_{2,a}}{\lambda_{1,a}}$ is substituted. Also the $n-3$ supplementary differential equations of the second order, combined with the differentials of the two original differential equations (78) of the first order and with those two original equations themselves, will give, when referred to initial values, $n-1$ equations between $a, a_1, \ldots, a_n, a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$ and two equations between $a, a_1, \ldots, a_n, a'_1, \ldots a'_n$; so that we can in general eliminate the $2n$ quantities $a'_1, \ldots, a'_n, a''_1, \ldots, a''_n$ between the $n+1$ equations last mentioned and the n equations obtained from (81), and shall thus be conducted in general to a partial differential equation of the form (53), which will be subject to the conditions (55). The rigorous equations (65) will also hold and therefore the formula of approximation (71), as connected with a partial differential equation of the form (51) subject to the condition (54), which may be obtained by eliminating the $2n+4$ quantities

$$x_1', \ldots, x_n', x_1'', \ldots, x_n'', \lambda, \frac{\lambda_{2,x}}{\lambda_{1,x}}, \frac{\lambda_{1,x}'}{\lambda_{1,x}}, \frac{\lambda_{2,x}'}{\lambda_{1,x}}$$

between the $n+1$ equations (80), the n equations (82), the two equations (78) and the two differentials of these. Also the equations (81) or those deduced from them by elimination of λ, combined if necessary with the equation (49), are forms for the integrals of the system of the original and supplementary differential equations of the question.

As a more general example, let the original differential system be

$$(84)\quad 0=f_1(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n')=f_{1,x}, \ldots, 0=f_m(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n')=f_{m,x};$$

so that the supplementary equations are

$$(85)\quad 0=\Sigma_{(\mu)1}^m\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right)'\right\}, \ldots, 0=\Sigma_{(\mu)1}^m\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right)'\right\}.$$

If $n \nless 2m$, we can eliminate the $2m-1$ ratios of the $2m$ functions $\lambda_{\mu,x}$ and $\lambda_{\mu,x}'$ between these n supplementary equations; and so obtain $n-2m+1$ equations of the second order between the n functions $x_1, \ldots, x_n$, to be combined with the m original equations (84). There will then remain $m-1$ equations to be assigned between the same n functions, and these will be of the third order and will be had by elimination from any $2m-1$ of the n equations obtained by differentiating (85). And the sum of the exponents of the orders of the n equations between the n functions $x_1, \ldots, x_n$ will be $m\times 1+\overline{n-2m+1}\times 2+\overline{m-1}\times 3=2n-1$; so that we shall still have a principal integral relation of the form (49) and its variation will be of the form

$$(86)\quad \delta F=\lambda\Sigma_{(\mu)1}^m\Sigma_{(\nu)1}^n\left\{\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}(\delta x_\nu-x_\nu'\delta x)-\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}(\delta a_\nu-a_\nu'\delta a)\right\}.$$

Hence

$$(87)\quad \frac{\delta F}{\delta x}=-\lambda\Sigma_{(\mu)1}^m\Sigma_{(\nu)1}^n\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_\nu'}x_\nu'\right),\quad \frac{\delta F}{\delta x_1}=\lambda\Sigma_{(\mu)1}^m\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_1'}\right),\quad \ldots,$$
$$\frac{\delta F}{\delta x_n}=\lambda\Sigma_{(\mu)1}^m\left(\lambda_{\mu,x}\frac{\delta f_{\mu,x}}{\delta x_n'}\right),$$

and

$$(88)\quad \frac{\delta F}{\delta a}=\lambda\Sigma_{(\mu)1}^m\Sigma_{(\nu)1}^n\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_\nu'}a_\nu'\right),\quad \frac{\delta F}{\delta a_1}=-\lambda\Sigma_{(\mu)1}^m\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_1'}\right),\quad \ldots,$$
$$\frac{\delta F}{\delta a_n}=-\lambda\Sigma_{(\mu)1}^m\left(\lambda_{\mu,a}\frac{\delta f_{\mu,a}}{\delta a_n'}\right).$$

If we eliminate the $2m+2n$ quantities

$$\lambda, \frac{\lambda_{2,x}}{\lambda_{1,x}}, \ldots, \frac{\lambda_{m,x}}{\lambda_{1,x}}, \frac{\lambda_{1,x}'}{\lambda_{1,x}}, \frac{\lambda_{2,x}'}{\lambda_{1,x}}, \ldots, \frac{\lambda_{m,x}'}{\lambda_{1,x}}, x_1', \ldots, x_n', x_1'', \ldots, x_n''$$

between the $n+1$ equations (87), the $m+n$ equations (84), (85) and the m differentials of (84), we are in general conducted to a partial differential equation of the form (51); and an analogous elimination conducts to another partial differential equation (53)—together with conclusions similar to those drawn in pages 364–368. In particular, the equations (88), when the form of F

is known, will give the complete integrals of the system of the n differential equations between the n functions $x_1, \ldots, x_n$.

If $n < 2m$ but $2n \nless 3m$, we can then eliminate the $3m-1$ ratios of $\lambda_{\mu,x}, \lambda'_{\mu,x}, \lambda''_{\mu,x}$ between the n equations (85) and their n differentials, and so obtain $2n-3m+1$ equations of the third order between the functions $x_1, \ldots, x_n$, to be combined with the m original equations of the first order and with $2m-n-1$ equations of the fourth order (observing that $2m-n-1 < n-m$, because $2n+1 > 3m$ and therefore $2m+n-(2n+1) < 2m+n-3m$). Thus the sum of the exponents of the orders is $(m \times 1) + (\overline{2n-3m+1} \times 3) + (\overline{2m-n-1} \times 4) = 2n-1$. In this case, therefore, as in the last, we are conducted to a principal integral of the form (49) and to conclusions altogether similar.

In general the n equations (85) and their in differentials as far as the ith order contain $(i+2)m-1$ ratios of $\lambda_{\mu,x}, \lambda'_{\mu,x}, \ldots, \lambda^{(i+1)}_{\mu,x}$, and

$$n + in - (i+2)m + 1 = n(i+1) - m(i+2) + 1 = n - 2m + 1 + i(n-m).$$

If i be the least integer which makes this last remainder >0 and k the corresponding value of the remainder, so that

$$(89) \qquad k = i(n-m) + n - 2m + 1$$

is >0 but $\ngtr n-m$, we shall then have k supplementary equations of order $i+2$ and $n-m-k$ equations of order $(i+3)$, to be combined with the m original differential equations of the first order. The sum of the exponents of all these n differential equations between the n functions $x_1, \ldots, x_n$ is therefore

$$(m \times 1) + (k \times \overline{i+2}) + (\overline{n-m-k} \times \overline{i+3}) = n(i+3) - m(i+2) - k = 2n-1;$$

conclusions therefore follow in general of the same kind as those deduced for the particular cases above.

i is the least integer which makes $n(i+1) \nless m(i+2)$, that is, $\dfrac{n}{m} \nless \dfrac{i+2}{i+1}$, or

$$(90) \qquad i+1 \nless \frac{m}{n-m}.$$

Therefore i is the next less integer to

$$\frac{m}{n-m} = \frac{\text{number of original differential equations}}{\text{number of supplementary differential equations}},$$

the supplementary differential equations being between the n functions $x_1, x_2, \ldots, x_n$ only and not involving the multipliers $\lambda_{\mu,x}$ nor their differential coefficients, which are supposed to have been eliminated. And $k = (i+1)(n-m) - m + 1$.

Thus, let $n = 10$ and let $m = 1, 2, \ldots, 9$ successively. We shall have

m	i	k	$n-m-k$	Original equations	Supplementary equations	Sum of exponents
1	0	9	0	1 of 1st order	9 of 2nd order	1+9.2 =19
2	0	7	1	2 ,, ,, ,,	7 ,, ,, ,, , 1 of 3rd order	2+7.2+1.3=19
3	0	5	2	3 ,, ,, ,,	5 ,, ,, ,, , 2 ,, ,, ,,	3+5.2+2.3=19
4	0	3	3	4 ,, ,, ,,	3 ,, ,, ,, , 3 ,, ,, ,,	4+3.2+3.3=19
5	0	1	4	5 ,, ,, ,,	1 ,, ,, ,, , 4 ,, ,, ,,	5+1.2+4.3=19
6	1	3	1	6 ,, ,, ,,	3 ,, 3rd ,, , 1 ,, 4th ,,	6+3.3+1.4=19
7	2	3	0	7 ,, ,, ,,	3 ,, 4th ,,	7+3.4 =19
8	3	1	1	8 ,, ,, ,,	1 ,, 5th ,, , 1 ,, 6th ,,	8+1.5+1.6=19
9	8	1	0	9 ,, ,, ,,	1 ,, 10th ,,	9+1.10 =19

[*Differential equations of order higher than the first.*]

[5.] If the original differential system be of the second order,

(91) $0=f_{1,x}=f_1(x, x_1, x_1', x_1'', \ldots, x_n, x_n', x_n''), \ldots, 0=f_{m,x}=f_m(x, x_1, x_1', x_1'', \ldots, x_n, x_n', x_n'')$,

then the n supplementary equations between the n original functions $x_1, \ldots, x_n$ and the m supplementary functions $\lambda_{1,x}, \ldots, \lambda_{m,x}$ are

$$(92)\quad \begin{cases} 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1'}\right)'+\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1''}\right)''\right\}=\sigma_{1,x}, \\ \ldots\ldots\ldots \\ 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n'}\right)'+\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n''}\right)''\right\}=\sigma_{n,x}. \end{cases}$$

With these we may join their derived equations up to any order i,

$$(93)\qquad 0=\sigma_{1,x}'=\sigma_{2,x}'=\ldots=\sigma_{n,x}', \ldots, 0=\sigma_{1,x}^{(i)}=\sigma_{2,x}^{(i)}=\ldots=\sigma_{n,x}^{(i)};$$

and if we take i large enough, it will always be possible to eliminate between these $n+in$ equations (92), (93) the $m(i+3)-1$ ratios of the $m(i+3)$ functions $\lambda_{\mu,x}, \lambda_{\mu,x}', \ldots, \lambda_{\mu,x}^{(i+2)}$, and so to get a number k of equations of the order $(i+4)$ between the n functions $x_1, \ldots, x_n$, where

$$(94)\qquad k=n(i+1)-m(i+3)+1=(n-m)(i+3)-2n+1.$$

Let i be the least integer which makes k positive, so that i is the least integer which gives

$$(95)\qquad i+3 \not< \frac{2n}{n-m},$$

that is, $i+2$ is the next less integer to $\dfrac{2n}{n-m}$ or i is the next less integer to $\dfrac{2m}{n-m}$. Then

$$(96)\qquad k>0, \quad k \not> n-m,$$

and the supplementary $n-m$ equations between the functions $x_1, \ldots, x_n$ will be k of the order $i+4$ and $n-m-k$ of the order $i+5$. The total sum of exponents will be

$$(97)\qquad (m.2)+(k.\overline{i+4})+(\overline{n-m-k}.\overline{i+5})=n(i+5)-m(i+3)-k=4n-1.$$

This therefore will be the total number of independent and arbitrary constants (besides a) in the expressions of the n functions $x_1, \ldots, x_n$, and only the same constants will enter into the n expressions of $x_1', \ldots, x_n'$. Eliminating therefore $2n-1$ constants between the $2n$ expressions of $x_1, \ldots, x_n, x_1', \ldots, x_n'$, we shall in general obtain a *principal integral* relation of the form

$$(98)\qquad 0=F(x, x_1, x_1', x_2, x_2', \ldots, x_n, x_n', a, a_1, a_1', a_2, a_2', \ldots, a_n, a_n').$$

If the original differential system be of order ω,

$$(99)\quad \begin{cases} 0=f_{1,x}=f_1(x, x_1, x_1', \ldots, x_1^{(\omega)}, x_2, x_2', \ldots, x_2^{(\omega)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega)}), \\ \ldots\ldots \\ 0=f_{m,x}=f_m(x, x_1, x_1', \ldots, x_1^{(\omega)}, x_2, x_2', \ldots, x_2^{(\omega)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega)}), \end{cases}$$

then the n supplementary equations are

$$(100)\quad \begin{cases} 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1'}\right)'+\ldots+(-1)^{\omega}\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_1^{(\omega)}}\right)^{(\omega)}\right\}=\sigma_{1,x}, \\ \ldots\ldots \\ 0=\Sigma_{(\mu)1}^{m}\left\{\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n}-\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n'}\right)'+\ldots+(-1)^{\omega}\left(\lambda_{\mu,x}\dfrac{\delta f_{\mu,x}}{\delta x_n^{(\omega)}}\right)^{(\omega)}\right\}=\sigma_{n,x}, \end{cases}$$

to which we may join in equations of the form (93), making in all $n(i+1)$ supplementary equations between the $m(i+\omega+1)-1$ ratios of the $m(i+\omega+1)$ functions $\lambda_{\mu,x}, \lambda'_{\mu,x}, \ldots, \lambda^{(\omega)}_{\mu,x}$. Let i be the least integer which makes

$$(101) \qquad i+\omega+1 \nless \frac{n\omega}{n-m}$$

and therefore

$$(102) \qquad n(i+1) \nless m(i+\omega+1).$$

Putting

$$(103) \qquad k = n(i+1) - m(i+\omega+1) + 1,$$

we have

$$(104) \qquad k > 0, \quad k \ngtr n-m;$$

we have also m original equations of the ωth order, k supplementary of order $i+2\omega$ and $n-m-k$ of order $i+2\omega+1$ between $x_1, x_2, \ldots, x_n$. The sum of all the exponents of these n equations is

$$(105) \qquad (m\,.\,\omega) + (k\,.\,\overline{i+2\omega}) + (\overline{n-m-k}\,.\,\overline{i+2\omega+1})$$
$$= n(i+2\omega+1) - m(i+\omega+1) - k = 2\omega n - 1.$$

This therefore is the total number of independent constants in the expressions of $x_1, x_1', \ldots, x_1^{(\omega-1)}, x_2, x_2', \ldots, x_2^{(\omega-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega-1)}$, and by eliminating $\omega n - 1$ of the constants we can deduce finally a principal integral relation of the form

$$(106) \qquad 0 = F\begin{pmatrix} x, x_1, x_1', \ldots, x_1^{(\omega-1)}, x_2, x_2', \ldots, x_2^{(\omega-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega-1)} \\ a, a_1, a_1', \ldots, a_1^{(\omega-1)}, a_2, a_2', \ldots, a_2^{(\omega-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega-1)} \end{pmatrix}.$$

(Feb. 23rd.)

Let there be one original differential equation of the form

$$(107) \qquad 0 = f(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)}) = f_x.$$

The n supplementary equations are then

$$(108) \qquad \begin{cases} 0 = \lambda\dfrac{\delta f}{\delta x_1} - \left(\lambda\dfrac{\delta f}{\delta x_1'}\right)' + \ldots + (-1)^{\omega_1}\left(\lambda\dfrac{\delta f}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)} = \sigma_{1,x}, \\ \ldots\ldots \\ 0 = \lambda\dfrac{\delta f}{\delta x_n} - \left(\lambda\dfrac{\delta f}{\delta x_n'}\right)' + \ldots + (-1)^{\omega_n}\left(\lambda\dfrac{\delta f}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)} = \sigma_{n,x}. \end{cases}$$

We shall suppose $\omega_1 \nless 1, \omega_2 \nless \omega_1, \ldots, \omega_n \nless \omega_{n-1}$. The $n+1$ equations (107) and (108), which may be written for abridgment thus

$$(109) \qquad 0 = f,\ 0 = \sigma_1,\ \ldots,\ 0 = \sigma_n,$$

may be joined with the $(n+1)\,\omega_n - \Sigma\omega$ derived equations following:

$$(110) \qquad 0 = f', \ldots, 0 = f^{(\omega_n)}, 0 = \sigma_1', \ldots 0 = \sigma_1^{(\omega_n - \omega_1)}, \ldots, 0 = \sigma_{n-1}', \ldots, 0 = \sigma_{n-1}^{(\omega_n - \omega_{n-1})},$$

and with these other $(n+1)\,i$ derived equations

$$(111) \qquad 0 = f^{(\omega_n+1)}, \ldots, 0 = f^{(\omega_n+i)}, 0 = \sigma_1^{(\omega_n-\omega_1+1)}, \ldots, 0 = \sigma_1^{(\omega_n-\omega_1+i)}, \ldots,$$
$$0 = \sigma_{n-1}^{(\omega_n-\omega_{n-1}+1)}, \ldots, 0 = \sigma_{n-1}^{(\omega_n-\omega_{n-1}+i)}, 0 = \sigma_n', \ldots, \sigma_n^{(i)}.$$

Thus we shall have in all $(n+1)(1+\omega_n+i)-\Sigma\omega$ equations between the ω_n+i ratios of $\frac{\lambda'}{\lambda}, \ldots, \frac{\lambda^{(\omega_n+i)}}{\lambda}$, the independent variable x, the $1+\omega_n+\omega_1+i$ functions $x_1, x_1', \ldots, x_1^{(\omega_n+\omega_1+i)}$, the $1+\omega_n+\omega_2+i$ functions $x_2, x_2', \ldots, x_2^{(\omega_n+\omega_2+i)}$, ..., and finally the $1+2\omega_n+i$ functions $x_n, x_n', \ldots, x_n^{(2\omega_n+i)}$. The number of these latter functions, when we leave out $x_1, x_1', \ldots, x_1^{(\omega_n+\omega_1+i)}$, is $(n-1)(1+\omega_n+i)+\Sigma\omega-\omega_1$; and, when added to the number ω_n+i of ratios of $\frac{\lambda'}{\lambda}$ &c. it gives $n(1+\omega_n+i)+\Sigma\omega-\omega_1-1$, which will be less than the number

$$(n+1)(1+\omega_n+i)-\Sigma\omega$$

of equations provided that

$$i=2\Sigma\omega-\omega_1-\omega_n-2+k, \quad k>0. \tag{112}$$

When i is taken sufficiently large to satisfy this condition, we shall have, by elimination, k equations between $x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2+k)}$; we shall therefore have one equation between $x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-1)}$, of which differential equation the integral will contain $2\Sigma\omega-1$ arbitrary constants. These will be the only constants which will enter into the complete expressions of the n functions $x_1, \ldots, x_n$ and of all their differential coefficients. They can therefore in general be eliminated between the $2\Sigma\omega$ expressions

$$x_1, x_1', \ldots, x_1^{(\omega_1-1)}, x_2, x_2', \ldots, x_2^{(\omega_2-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n-1)},$$
$$a_1, a_1', \ldots, a_1^{(\omega_1-1)}, a_2, a_2', \ldots, a_2^{(\omega_2-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega_n-1)},$$

and thus a principal integral relation obtained of the form

$$0=F\begin{pmatrix} x, x_1, x_1', \ldots, x_1^{(\omega_1-1)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n-1)} \\ a, a_1, a_1', \ldots, a_1^{(\omega_1-1)}, \ldots, a_n, a_n', \ldots, a_n^{(\omega_n-1)} \end{pmatrix}. \tag{113}$$

Let there be an original differential system

$$\begin{cases} 0=f_1(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)})=f_{1,x}, \\ \ldots\ldots \\ 0=f_m(x, x_1, x_1', \ldots, x_1^{(\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_n)})=f_{m,x}, \end{cases} \tag{114}$$

which is to be combined with the supplementary system

$$\begin{cases} 0=\lambda_1\frac{\delta f_1}{\delta x_1}-\left(\lambda_1\frac{\delta f_1}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_1}\left(\lambda_1\frac{\delta f_1}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)} \\ \quad +\ldots\ldots \\ \quad +\lambda_m\frac{\delta f_m}{\delta x_1}-\left(\lambda_m\frac{\delta f_m}{\delta x_1'}\right)'+\ldots+(-1)^{\omega_1}\left(\lambda_m\frac{\delta f_m}{\delta x_1^{(\omega_1)}}\right)^{(\omega_1)}=\sigma_{1,x}, \\ \quad \ldots\ldots \\ 0=\lambda_1\frac{\delta f_1}{\delta x_n}-\left(\lambda_1\frac{\delta f_1}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_n}\left(\lambda_1\frac{\delta f_1}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)} \\ \quad +\ldots\ldots \\ \quad +\lambda_m\frac{\delta f_m}{\delta x_n}-\left(\lambda_m\frac{\delta f_m}{\delta x_n'}\right)'+\ldots+(-1)^{\omega_n}\left(\lambda_m\frac{\delta f_m}{\delta x_n^{(\omega_n)}}\right)^{(\omega_n)}=\sigma_{n,x}. \end{cases} \tag{115}$$

Let it still be supposed that

$$\omega_1 \not< 1, \ \omega_2 \not< \omega_1, \ \ldots, \ \omega_n \not< \omega_{n-1}. \tag{116}$$

The equation $\sigma_{1,x}=0$ contains

$$\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_1)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_1)}, x_1, x_1', \ldots, x_1^{(2\omega_1)}, x_2, x_2', \ldots, x_2^{(\omega_1+\omega_2)}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_1+\omega_n)};$$

the equation $\sigma_{2,x}=0$ contains

$$\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_2)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_2)}, x_1, x_1', \ldots, x_1^{(\omega_2+\omega_1)}, x_2, x_2', \ldots, x_2^{2\omega_2}, \ldots, x_n, x_n', \ldots, x_n^{(\omega_2+\omega_n)};$$

and so on for the remaining $n-2$ equations of (115). Hence the equations $0=\sigma_{1,x}^{(\omega_n-\omega_1)}$, $0=\sigma_{2,x}^{(\omega_n-\omega_2)}$, &c., contain each the same things as $\sigma_{n,x}=0$; and by taking i large enough we can in general eliminate the $m(1+\omega_n+i)-1$ ratios of $\lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_n+i)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_n+i)}$ and also the $(n-1)(1+\omega_n+i)-\Sigma\omega-\omega_1$ functions $x_2, \ldots, x_2^{(\omega_n+\omega_2+i)}, \ldots, x_n, \ldots, x_n^{(2\omega_n+i)}$, being in all $(n+m-1)(1+\omega_n+i)+\Sigma\omega-\omega_1-1$ things, between the $(n+m)(1+\omega_n+i)-\Sigma\omega$ equations

$$(117)\quad \begin{cases} 0=f_{1,x}=f_{1,x}'=\ldots=f_{1,x}^{(\omega_n+i)},\ 0=f_{2,x}=f_{2,x}'=\ldots=f_{2,x}^{(\omega_n+i)},\ \ldots,\ 0=f_{m,x}=f_{m,x}'=\ldots=f_{m,x}^{(\omega_n+i)}, \\ 0=\sigma_{1,x}=\sigma_{1,x}'=\ldots=\sigma_{1,x}^{(\omega_n-\omega_1+i)},\ 0=\sigma_{2,x}=\sigma_{2,x}'=\ldots=\sigma_{2,x}^{(\omega_n-\omega_2+i)},\ \ldots, \\ \qquad 0=\sigma_{n,x}=\sigma_{n,x}'=\ldots=\sigma_{n,x}^{(i)}, \end{cases}$$

and thus obtain $\omega_1+\omega_n+i+2-2\Sigma\omega$ equations between $x, x_1, x_1', \ldots, x_1^{(\omega_1+\omega_n+i)}$. If then we take

$$(118)\qquad i=2\Sigma\omega-\omega_1-\omega_n-1,$$

we shall have one differential equation of the order $2\Sigma\omega-1$ to determine the function x_1, and thus we are conducted as before to a principal integral relation of the form (113).*

Thus, from the system of the $m+n$ equations (114), (115), we can deduce by differentiation $(m+n)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ other equations and then by elimination transform the system of all these $(m+n)(2\Sigma\omega-\omega_1)-\Sigma\omega$ equations into a system of the following form:

$$(119)\qquad x_1^{(2\Sigma\omega-1)}=\phi(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)});$$

$$(120)\qquad x_2=\phi_{2,0}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)}), \ldots, x_n=\phi_{n,0}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)});$$

$$(121)\qquad \lambda_1'=\lambda_1\psi_{1,1}(x, x_1, x_1', \ldots, x_1^{(2\Sigma\omega-2)});$$

$$(122)\qquad \lambda_2=\lambda_1\psi_{2,0}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_m=\lambda_1\psi_{m,0}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)});$$

$$(123)\quad \begin{cases} x_2'=\phi_{2,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ x_2^{(2\Sigma\omega+\omega_2-\omega_1-1)}=\phi_{2,2\Sigma\omega+\omega_2-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}), \\ \ldots\ldots \\ x_n'=\phi_{n,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ x_n^{(2\Sigma\omega+\omega_n-\omega_1-1)}=\phi_{n,2\Sigma\omega+\omega_n-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}); \end{cases}$$

$$(124)\qquad \lambda_1''=\lambda_1\psi_{1,2}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_1^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{1,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)});$$

$$(125)\quad \begin{cases} \lambda_2'=\lambda_1\psi_{2,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_2^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{2,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}), \\ \ldots\ldots \\ \lambda_m'=\lambda_1\psi_{m,1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}),\ \ldots,\ \lambda_m^{(2\Sigma\omega-\omega_1-1)}=\lambda_1\psi_{m,2\Sigma\omega-\omega_1-1}(x, x_1, \ldots, x_1^{(2\Sigma\omega-2)}). \end{cases}$$

Now the equations (123) ought to be deducible by differentiation from (119) and (120); the equations (124) from (119) and (121); and the equations (125) from (119), (121) and (122).*

[*Particular cases treated in greater detail.*]

[**6.**] To illustrate this point let us consider some less general cases, and first let $m=1$, $\omega_n=1$ (and therefore $\Sigma\omega=n$), so that we have the case of equation (1), page 358. Thus for (114) and (115) we have now the $n+1$ equations (1) and (7),

$$(126)\qquad 0=f,\quad 0=\sigma_1=\sigma_2=\ldots=\sigma_n;$$

* Consult p. 384.

to which we are to join these other $2n^2-n-2$ derived equations

(127) $$0=f'=\ldots=f^{(2n-2)},\quad 0=\sigma_1'=\ldots=\sigma_1^{(2n-3)},\ \ldots,\quad 0=\sigma_n'=\ldots=\sigma_n^{(2n-3)}.$$

Then by elimination we are to deduce the $n+1$ equations

(128) $$x_1^{(2n-1)}=\phi(x,\,x_1,\,x_1',\,\ldots,\,x_1^{(2n-2)}),$$

(129) $$x_2=\phi_{2,0}(x,\,\ldots,\,x_1^{(2n-2)}),\ \ldots,\ x_n=\phi_{n,0}(x,\,\ldots,\,x_1^{(2n-2)}),$$

(130) $$\lambda'=\lambda\psi_1(x,\,\ldots,\,x_1^{(2n-2)}),$$

and these other $2n^2-n-2$ equations

(131) $$\begin{cases} x_2'=\phi_{2,1}(x,\,\ldots,\,x_1^{(2n-2)}),\ \ldots,\ x_2^{(2n-1)}=\phi_{2,\,2n-1}(x,\,\ldots,\,x_1^{(2n-2)}),\\ \ldots\ldots\\ x_n'=\phi_{n,1}(x,\,\ldots,\,x_1^{(2n-2)}),\ \ldots,\ x_n^{(2n-1)}=\phi_{n,\,2n-1}(x,\,\ldots,\,x_1^{(2n-2)}),\end{cases}$$

(132) $$\lambda''=\lambda\psi_2(x,\,\ldots,\,x_1^{(2n-2)}),\ \ldots,\ \lambda^{(2n-2)}=\lambda\psi_{2n-2}(x,\,\ldots,\,x_1^{(2n-2)}).$$

The equations (131) ought to be deducible by differentiation from the equations (128) and (129), and the equations (132) from (128) and (130).*

To particularise still further, let $n=2$. We have then the three equations

(133) $$0=f(x,\,x_1,\,x_1',\,x_2,\,x_2')=f,\quad 0=\lambda\frac{\delta f}{\delta x_1}-\left(\lambda\frac{\delta f}{\delta x_1'}\right)'=\sigma_1,\quad 0=\lambda\frac{\delta f}{\delta x_2}-\left(\lambda\frac{\delta f}{\delta x_2'}\right)'=\sigma_2,$$

and the four derived equations

(134) $$0=f'=f'',\quad 0=\sigma_1',\quad 0=\sigma_2',$$

in which

(135) $$f'=\frac{\delta f}{\delta x}+\frac{\delta f}{\delta x_1}x_1'+\frac{\delta f}{\delta x_1'}x_1''+\frac{\delta f}{\delta x_2}x_2'+\frac{\delta f}{\delta x_2'}x_2'',$$

and

(136) $$\begin{aligned} f''=&\frac{\delta^2 f}{\delta x^2}+2\frac{\delta^2 f}{\delta x\,\delta x_1}x_1'+2\frac{\delta^2 f}{\delta x\,\delta x_1'}x_1''+2\frac{\delta^2 f}{\delta x\,\delta x_2}x_2'+2\frac{\delta^2 f}{\delta x\,\delta x_2'}x_2''\\ &+\frac{\delta^2 f}{\delta x_1^2}x_1'^2+2\frac{\delta^2 f}{\delta x_1\,\delta x_1'}x_1'x_1''+2\frac{\delta^2 f}{\delta x_1\,\delta x_2}x_1'x_2'+\frac{\delta^2 f}{\delta x_1\,\delta x_2'}x_1'x_2''\\ &+\frac{\delta^2 f}{\delta x_1'^2}x_1''^2+2\frac{\delta^2 f}{\delta x_1'\,\delta x_2}x_1''x_2'+2\frac{\delta^2 f}{\delta x_1'\,\delta x_2'}x_1''x_2''\\ &+\frac{\delta^2 f}{\delta x_2^2}x_2'^2+2\frac{\delta^2 f}{\delta x_2\,\delta x_2'}x_2'x_2''+\frac{\delta^2 f}{\delta x_2'^2}x_2''^2\\ &+\frac{\delta f}{\delta x_1}x_1''+\frac{\delta f}{\delta x_1'}x_1'''+\frac{\delta f}{\delta x_2}x_2''+\frac{\delta f}{\delta x_2'}x_2'''.\end{aligned}$$

Also

(137) $$\sigma_1=\lambda\left\{\frac{\delta f}{\delta x_1}-\left(\frac{\delta f}{\delta x_1'}\right)'\right\}-\lambda'\frac{\delta f}{\delta x_1'},\quad \sigma_2=\lambda\left\{\frac{\delta f}{\delta x_2}-\left(\frac{\delta f}{\delta x_2'}\right)'\right\}-\lambda'\frac{\delta f}{\delta x_2'},$$

and*

(138 $$\begin{cases}\sigma_1'=\lambda\left\{\left(\frac{\delta f}{\delta x_1}\right)'-\left(\frac{\delta f}{\delta x_1'}\right)''\right\}+\lambda'\left\{\frac{\delta f}{\delta x_1}-2\left(\frac{\delta f}{\delta x_1'}\right)'\right\}-\lambda''\frac{\delta f}{\delta x_1'},\\ \sigma_2'=\lambda\left\{\qquad\qquad\right\}+\lambda'\left\{\qquad\qquad\right\}-\lambda''\frac{\delta f}{\delta x_2'}.\end{cases}$$

* See p. 383.

(Feb. 24th, 1836.)

We may simplify these equations by supposing that the original relation between x, x_1, x_1', x_2, x_2' has been resolved with respect to x_2', and thus put under the form

$$(139)\qquad 0=\chi(x,\, x_1,\, x_1',\, x_2)-x_2'=f.$$

We shall then have

$$(140)\quad \frac{\delta f}{\delta x_2'}=-1,\quad \frac{\delta^2 f}{\delta x\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_1\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_1'\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_2\,\delta x_2'}=0,\quad \frac{\delta^2 f}{\delta x_2'^2}=0,\quad \left(\frac{\delta f}{\delta x_2'}\right)'=0;$$

$$(141)\qquad x_2'=\chi(x,\, x_1,\, x_1',\, x_2)=f+x_2';$$

$$(142)\qquad x_2''=\frac{\delta\chi}{\delta x}+\frac{\delta\chi}{\delta x_1}x_1'+\frac{\delta\chi}{\delta x_1'}x_1''+\frac{\delta\chi}{\delta x_2}x_2'=f'+x_2'';$$

$$(143)\qquad x_2'''=\frac{\delta^2\chi}{\delta x^2}+2\frac{\delta^2\chi}{\delta x\,\delta x_1}x_1'+2\frac{\delta^2\chi}{\delta x\,\delta x_1'}x_1''+2\frac{\delta^2\chi}{\delta x\,\delta x_2}x_2'+\frac{\delta^2\chi}{\delta x_1^2}x_1'^2+2\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'x_1''$$

$$+2\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1'x_2'+\frac{\delta^2\chi}{\delta x_1'^2}x_1''^2+2\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1''x_2'+\frac{\delta^2\chi}{\delta x_2^2}x_2'^2+\frac{\delta\chi}{\delta x_1}x_1''+\frac{\delta\chi}{\delta x_1'}x_1'''+\frac{\delta\chi}{\delta x_2}x_2'';$$

$$(144)\qquad 0=\sigma_1=\lambda\left\{\frac{\delta\chi}{\delta x_1}-\left(\frac{\delta\chi}{\delta x_1'}\right)'\right\}-\lambda'\frac{\delta\chi}{\delta x_1'};$$

$$(145)\qquad 0=\sigma_2=\lambda\frac{\delta\chi}{\delta x_2}+\lambda';$$

therefore

$$(146)\qquad 0=\frac{1}{\lambda}\left(\sigma_1+\sigma_2\frac{\delta\chi}{\delta x_1'}\right)=\frac{\delta\chi}{\delta x_1}-\left(\frac{\delta\chi}{\delta x_1'}\right)'+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'},$$

that is,

$$(147)\qquad 0=\frac{\delta\chi}{\delta x_1}-\frac{\delta^2\chi}{\delta x\,\delta x_1'}+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'}-\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'-\frac{\delta^2\chi}{\delta x_1'^2}x_1''-\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\chi;$$

$$(148)\qquad 0=\sigma_1'=\lambda\left\{\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''\right\}+\lambda'\left\{\frac{\delta\chi}{\delta x_1}-2\left(\frac{\delta\chi}{\delta x_1'}\right)'\right\}-\lambda''\frac{\delta\chi}{\delta x_1'};$$

$$(149)\qquad 0=\sigma_2'=\lambda\left(\frac{\delta\chi}{\delta x_2}\right)'+\lambda'\frac{\delta\chi}{\delta x_2}+\lambda'';$$

therefore

$$(150)\quad 0=\sigma_1'+\sigma_2'\frac{\delta\chi}{\delta x_1'}=\lambda\left\{\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta\chi}{\delta x_2}\right)'\right\}+\lambda'\left\{\frac{\delta\chi}{\delta x_1}-2\left(\frac{\delta\chi}{\delta x_1'}\right)'+\frac{\delta\chi}{\delta x_1'}\frac{\delta\chi}{\delta x_2}\right\};$$

and

$$(151)\quad 0=\frac{1}{\lambda}\left(\sigma_1'+\sigma_2'\frac{\delta\chi}{\delta x_1'}\right)-\frac{\lambda'}{\lambda^2}\left(\sigma_1+\sigma_2\frac{\delta\chi}{\delta x_1'}\right)+\frac{\sigma_2}{\lambda}\left(\frac{\delta\chi}{\delta x_1'}\right)'=\left(\frac{\delta\chi}{\delta x_1}\right)'-\left(\frac{\delta\chi}{\delta x_1'}\right)''$$

$$+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta\chi}{\delta x_2}\right)'+\frac{\delta\chi}{\delta x_2}\left(\frac{\delta\chi}{\delta x_1'}\right)'.$$

We can eliminate x_2' and x_2'' between the four equations (141), (142), (146) and (151), and thus obtain two equations between $x, x_1, x_1', x_1'', x_1''', x_2$; namely, the equation (147) between x, x_1, x_1',

x_1'', x_2 and another equation deduced from (151) between $x, x_1, x_1', x_1'', x_1''', x_2$. To develope this last equation, we have

$$(152)\qquad \left(\frac{\delta\chi}{\delta x_1}\right)' = \frac{\delta^2\chi}{\delta x\,\delta x_1} + \frac{\delta^2\chi}{\delta x_1^2}x_1' + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_2',$$

$$(153)\qquad \left(\frac{\delta\chi}{\delta x_2}\right)' = \frac{\delta^2\chi}{\delta x\,\delta x_2} + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1'' + \frac{\delta^2\chi}{\delta x_2^2}x_2',$$

$$(154)\qquad \left(\frac{\delta\chi}{\delta x_1'}\right)' = \frac{\delta^2\chi}{\delta x\,\delta x_1'} + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1' + \frac{\delta^2\chi}{\delta x_1'^2}x_1'' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_2',$$

$$\begin{aligned}(155)\qquad \left(\frac{\delta\chi}{\delta x_1'}\right)'' = {} & \frac{\delta^3\chi}{\delta x^2\,\delta x_1'} + 2\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1'' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}x_2' + \frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2 \\ & + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1'' + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'x_2' + \frac{\delta^3\chi}{\delta x_1'^3}x_1''^2 + 2\frac{\delta^3\chi}{\delta x_1'\,\delta x_2}x_1''x_2' + \frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}x_2'^2 \\ & + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1'^2}x_1''' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_2'';\end{aligned}$$

therefore

$$\begin{aligned}(156)\qquad 0 = {} & \frac{\delta^2\chi}{\delta x\,\delta x_1} + \frac{\delta^2\chi}{\delta x_1^2}x_1' + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}\chi + \frac{\delta\chi}{\delta x_1'}\left(\frac{\delta^2\chi}{\delta x\,\delta x_2} + \frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1' + \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x_1'' + \frac{\delta^2\chi}{\delta x_2^2}\chi\right) \\ & + \frac{\delta\chi}{\delta x_2}\left(\frac{\delta^2\chi}{\delta x\,\delta x_1'} + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1' + \frac{\delta^2\chi}{\delta x_1'^2}x_1''\right) - \frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\left(\frac{\delta\chi}{\delta x} + \frac{\delta\chi}{\delta x_1}x_1' + \frac{\delta\chi}{\delta x_1'}x_1''\right) \\ & - \left(\frac{\delta^3\chi}{\delta x^2\,\delta x_1'} + 2\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1'' + 2\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}\chi + \frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2 + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1''\right. \\ & \left. + 2\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'\chi + \frac{\delta^3\chi}{\delta x_1'^3}x_1''^2 + 2\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''\chi + \frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}\chi^2 + \frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'' + \frac{\delta^2\chi}{\delta x_1'^2}x_1'''\right).\end{aligned}$$

The system of the three equations, (145), (147) and (156), conducts by elimination to expressions of x_1''', x_2 and $\frac{\lambda'}{\lambda}$ as functions of x, x_1, x_1', x_1'', which may be thus denoted:

$$(157)\qquad x_1''' = \phi(x, x_1, x_1', x_1''),$$

$$(158)\qquad x_2 = \phi_{2,0}(x, x_1, x_1', x_1''),$$

$$(159)\qquad \frac{\lambda'}{\lambda} = \psi_1(x, x_1, x_1', x_1'').$$

But the system of six equations (141), (142), (144), (145), (148) and (149) gives also three other expressions:

$$(160)\qquad x_2' = \phi_{2,1}(x, x_1, x_1', x_1''),$$

$$(161)\qquad x_2'' = \phi_{2,2}(x, x_1, x_1', x_1''),$$

$$(162)\qquad \frac{\lambda''}{\lambda} = \psi_2(x, x_1, x_1', x_1'');$$

and we wish to show that these last three equations can be deduced by differentiation and elimination from the equations (157)–(159), or, which will come to the same thing, that the two equations (141), (144) can be deduced from the three (145), (147) and (156). To prove this, we shall show first that (141) can be deduced by differentiation and elimination from (147) and (156). Differentiating (147), we find

$$(163)\qquad 0=\frac{\delta^2\chi}{\delta x\,\delta x_1}+\frac{\delta^2\chi}{\delta x_1^2}x_1'+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1''+\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_2'+\frac{\delta\chi}{\delta x_1'}\left(\frac{\delta^2\chi}{\delta x\,\delta x_2}+\frac{\delta^2\chi}{\delta x_1\,\delta x_2}x_1'+\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}x''+\frac{\delta^2\chi}{\delta x_2^2}x_2'\right)$$

$$+\frac{\delta\chi}{\delta x_2}\left(\frac{\delta^2\chi}{\delta x\,\delta x_1'}+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'+\frac{\delta^2\chi}{\delta x_1'^2}x_1''\right)-\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\left(\frac{\delta\chi}{\delta x}+\frac{\delta\chi}{\delta x_1}x_1'+\frac{\delta\chi}{\delta x_1'}x_1''\right)$$

$$-\left(\frac{\delta^3\chi}{\delta x^2\,\delta x_1'}+2\,\frac{\delta^3\chi}{\delta x\,\delta x_1\,\delta x_1'}x_1'+2\,\frac{\delta^3\chi}{\delta x\,\delta x_1'^2}x_1''+\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1^2\,\delta x_1'}x_1'^2\right.$$

$$+2\,\frac{\delta^3\chi}{\delta x_1\,\delta x_1'^2}x_1'x_1''+\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1'^3}x_1''^2+\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''(x_2'+\chi)+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}x_2'\chi$$

$$\left.+\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1''+\frac{\delta^2\chi}{\delta x_1'^2}x_1'''\right),$$

and subtracting (156) from this to eliminate x_1''' we have

$$(164)\qquad 0=L\,(x_2'-\chi),$$

in which the coefficient L is

$$(165)\qquad L=\frac{\delta^2\chi}{\delta x_1\,\delta x_2}+\frac{\delta\chi}{\delta x_1'}\frac{\delta^2\chi}{\delta x_2^2}-\left(\frac{\delta^3\chi}{\delta x\,\delta x_1'\,\delta x_2}+\frac{\delta^3\chi}{\delta x_1\,\delta x_1'\,\delta x_2}x_1'+\frac{\delta^3\chi}{\delta x_1'^2\,\delta x_2}x_1''+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}\chi\right)$$

$$=\frac{\delta^2\chi}{\delta x_1\,\delta x_2}-\left(\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\right)'+\frac{\delta\chi}{\delta x_1'}\frac{\delta^2\chi}{\delta x_2^2}+\frac{\delta^3\chi}{\delta x_1'\,\delta x_2^2}(x_2'-\chi).$$

Now this process of differentiating (147) and then eliminating x_1''' by (156) is evidently the process to be employed in order to get an equation between $x, x_1, x_1', x_1'', x_2, x_2'$, which when combined with (147) shall give x_2' as a function of the form (160); and we now see that this process gives $x_2'=\chi$, that is (141), because the coefficient L does not in general vanish. Having thus concluded (141), we can conclude also (146) from (147) and (156), and then (145) will enable us to conclude (144) also. We have then deduced, as we proposed, the equations (141) and (144), and can therefore deduce (142), (148) and (149) from (145), (147) and (156) or from the equivalent system (157), (158) and (159); so that this latter system conducts ultimately by differentiation and elimination to the system (160)–(162), which was the thing to be proved. (It might be useful to consider separately the cases in which the coefficient L, (165), vanishes.)

The argument may also be thus stated (without $\lambda, \lambda', \lambda''$). The original equation (141), being combined with the supplementary equation (146), which may be written

$$(166)\qquad x_2'=\psi(x, x_1, x_1', x_1'', x_2)=\left(\frac{\delta^2\chi}{\delta x_1'\,\delta x_2}\right)^{-1}\left\{\frac{\delta\chi}{\delta x_1}+\frac{\delta\chi}{\delta x_2}\frac{\delta\chi}{\delta x_1'}-\frac{\delta^2\chi}{\delta x\,\delta x_1'}-\frac{\delta^2\chi}{\delta x_1\,\delta x_1'}x_1'-\frac{\delta^2\chi}{\delta x_1'^2}x_1''\right\},$$

and with its own differential (142) and the differential of (166), namely

$$(167)\qquad x_2''=\frac{\delta\psi}{\delta x}+\frac{\delta\psi}{\delta x_1}x_1'+\frac{\delta\psi}{\delta x_1'}x_1''+\frac{\delta\psi}{\delta x_1''}x_1'''+\frac{\delta\psi}{\delta x_2}x_2',$$

gives the two following equations:

$$\chi(x, x_1, x_1', x_2) = \psi(x, x_1, x_1', x_1'', x_2), \tag{168}$$

$$\frac{\delta\chi}{\delta x} + \frac{\delta\chi}{\delta x_1} x_1' + \frac{\delta\chi}{\delta x_1'} x_1'' + \frac{\delta\chi}{\delta x_2}\chi = \frac{\delta\psi}{\delta x} + \frac{\delta\psi}{\delta x_1} x_1' + \frac{\delta\psi}{\delta x_1'} x_1'' + \frac{\delta\psi}{\delta x_1''} x_1''' + \frac{\delta\psi}{\delta x_2}\chi. \tag{169}$$

These two equations combined give, by elimination, two others of the forms (157) and (158), and reciprocally we may consider these two last equations as conducting to (168) and (169). Now the differential of (168), when (169) is subtracted from it, gives

$$\left(\frac{\delta\chi}{\delta x_2} - \frac{\delta\psi}{\delta x_2}\right)(x_2' - \chi) = 0, \tag{170}$$

and, since in general

$$\frac{\delta\chi}{\delta x_2} \mp \frac{\delta\psi}{\delta x_2}, \tag{171}$$

we conclude that the equations (168) and (169) give (141) and therefore also (166), (142) and (167). The equations therefore of the forms (160) and (161), which result along with (157) and (158) by elimination from the four equations (141), (142), (166) and (167), are consequences of equations (157) and (158) and can be deduced from them by differentiation and elimination.

When the original equation is left under the form

$$0 = f(x, x_1, x_1', x_2, x_2') = f, \tag{133}$$

and the supplementary equation (freed from λ) is of the form

$$0 = \frac{\sigma_1}{\lambda\dfrac{\delta f}{\delta x_1'}} - \frac{\sigma_2}{\lambda\dfrac{\delta f}{\delta x_2'}} = \frac{\dfrac{\delta f}{\delta x_1} - \left(\dfrac{\delta f}{\delta x_1'}\right)'}{\dfrac{\delta f}{\delta x_1'}} - \frac{\dfrac{\delta f}{\delta x_2} - \left(\dfrac{\delta f}{\delta x_2'}\right)'}{\dfrac{\delta f}{\delta x_2'}} = \psi(x, x_1, x_1', x_1'', x_2, x_2', x_2''), \tag{172}$$

the function ψ being linear with respect to x_1'' and x_2'', then these two equations, combined with the three derived equations

$$0 = f', \quad 0 = f'', \quad 0 = \psi', \tag{173}$$

will in general enable us to deduce by elimination x_1''', x_2, x_2', x_2'', x_2''' as functions of x, x_1, x_1', x_1'' of the forms (157), (158), (160), (161) and

$$x_2''' = \phi_{2,3}(x, x_1, x_1', x_1''). \tag{174}$$

We may propose to show that equations (160), (161) and (174) can be deduced from (157) and (158). We may begin by eliminating x_2'' and x_2''' between the four equations (172), (173), and so obtain two equations between $x, x_1, x_1', x_1'', x_1''', x_2, x_2'$ to be combined with (133). Then it must be shown that if x_1^{iv} be eliminated between the differentials of the three last mentioned equations the two resulting equations will both be satisfied by and will give the equation (172). Eliminating x_2''' between the equations $0 = f''$, $0 = \psi'$, we find

$$0 = \frac{\delta\psi}{\delta x_2''} f'' - \frac{\delta f}{\delta x_2'}\psi' = \text{funct}(x, x_1, x_1', x_1'', x_1''', x_2', x_2''), \tag{175}$$

quadratic with respect to x_2''; also eliminating x_2'' between the equations $0 = f'$, $0 = \psi$, we find

$$0 = \frac{\delta\psi}{\delta x_2''} f' - \frac{\delta f}{\delta x_2'}\psi = \text{funct}(x, x_1, x_1', x_1'', x_2, x_2'); \tag{176}$$

and another equation of the form

$$0 = \text{funct}\,(x,\ x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2') \tag{177}$$

will be obtained by eliminating x_2'' from (175) with the help of either $0=f'$ or $0=\psi$. The equations (133), (176) and (177) are the three above alluded to, which give x_1''', x_2, x_2' as functions of x, x_1, x_1', x_1''. Differentiating them all we should get three new expressions for x_1^{iv}, x_2', x_2'' as functions of the same four quantities x, x_1, x_1', x_1''. We must show that the two expressions for x_2' agree with each other and that the expressions for x_2, x_2', x_2'' agree with the equation $0=\psi$. We must show therefore that, on eliminating x_1''', x_1^{iv} between (133), (176), (177) and their three differentials, the four resulting equations between x, x_1, x_1', x_1'', x_2, x_2', x_2'' are equivalent to $0=f$, $0=f'$, $0=\psi$. This comes to showing that the two equations $0=f$, $0=f'$, combined with the three equations (175), (176) and the differential of (176), give by the elimination of x_1''' only the three equations $0=f$, $0=f'$, $0=\psi$. And this is easily seen to be true.

(Feb. 25th.)

We may also present the argument as follows. The system of five equations

$$0=f,\quad 0=f',\quad 0=f'',\quad 0=\psi,\quad 0=\psi' \tag{178}$$

may be conceived to determine x_1''', x_2, x_2', x_2'' and x_2''' as functions of x, x_1, x_1', x_1'', expressed by five equations of the forms (157), (158), (160), (161), (174). If we differentiate these five functions, making

$$dx_1 = x_1'\,dx,\quad dx_1' = x_1''\,dx,\quad dx_1'' = x_1'''\,dx, \tag{179}$$

we shall get expressions for the five first differential coefficients

$$\frac{dx_1'''}{dx},\quad \frac{dx_2}{dx},\quad \frac{dx_2'}{dx},\quad \frac{dx_2''}{dx},\quad \frac{dx_2'''}{dx}.$$

If we differentiate in the same way the five equations (178), without that previous elimination which would conduct to the five separate expressions (157), (158), (160), (161), (174), we shall still obtain five linear equations to determine the same five differential coefficients, namely

$$0=df,\quad 0=df',\quad 0=df'',\quad 0=d\psi,\quad 0=d\psi'; \tag{180}$$

and a subsequent elimination between these five linear equations will give these five coefficients. In order then to show that the two equations (157), (158) conduct in general to the equations (160), (161) and (174), it is necessary and sufficient to show that the equations (178), when combined with the three equations

$$0=\frac{df}{dx},\quad 0=\frac{df'}{dx},\quad 0=\frac{d\psi}{dx} \tag{181}$$

$\left(\text{which result from (180) by the elimination of } \frac{dx_1'''}{dx},\ \frac{dx_2'''}{dx}\right)$ and with (179), conduct to the following relations:

$$\frac{dx_2}{dx}=x_2',\quad \frac{dx_2'}{dx}=x_2'',\quad \frac{dx_2''}{dx}=x_2'''. \tag{182}$$

Accordingly equations (178), (179) and (181) give

$$(183)\quad \begin{cases} 0=\dfrac{df}{dx}-f'=\dfrac{\delta f}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta f}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right), \\ 0=\dfrac{df'}{dx}-f''=\dfrac{\delta f'}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta f'}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right)+\dfrac{\delta f'}{\delta x_2''}\left(\dfrac{dx_2''}{dx}-x_2'''\right), \quad \left[\dfrac{\delta f'}{\delta x_2''}=\dfrac{\delta f}{\delta x_2'}\right], \\ 0=\dfrac{d\psi}{dx}-\psi'=\dfrac{\delta\psi}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta\psi}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right)+\dfrac{\delta\psi}{\delta x_2''}\left(\dfrac{dx_2''}{dx}-x_2'''\right), \end{cases}$$

and these three linear equations conduct in general to the three relations (182).

Had we retained the λ's we might have argued thus. The equations (133), namely $0=f$, $0=\sigma_1$, $0=\sigma_2$, with their four derived equations

$$(134)\qquad 0=f',\quad 0=f'',\quad 0=\sigma_1',\quad 0=\sigma_2',$$

may be conceived to conduct, by elimination, to seven separate expressions of the forms (157), (158), (160), (161), (174), (159) and (162) for x_1''', x_2, x_2', x_2'', x_2''', $\frac{\lambda'}{\lambda}$, $\frac{\lambda''}{\lambda}$ as functions ϕ, $\phi_{2,0}$, $\phi_{2,1}$, $\phi_{2,2}$, $\phi_{2,3}$, ψ_1, ψ_2 of x, x_1, x_1', x_1''. If these seven functions had been actually found by performing this conceived elimination, we should then be able to deduce by differentiation and substitution expressions for the following differential coefficients:

$$(184)\qquad \frac{dx_1'''}{dx},\quad \frac{dx_2}{dx},\quad \frac{dx_2'}{dx},\quad \frac{dx_2''}{dx},\quad \frac{dx_2'''}{dx},$$

and also for the following ratios:

$$(185)\qquad \frac{1}{\lambda}\frac{d\lambda'}{dx},\quad \frac{1}{\lambda}\frac{d\lambda''}{dx},$$

provided that we substitute for dx_1, dx_1', dx_1'' their values (179) and also make

$$(186)\qquad d\lambda=\lambda' dx.$$

Or, instead of thus previously eliminating between the seven equations (133) and (134), we may at once differentiate these equations changing dx, dx_1', dx_1'', $d\lambda$ to their values (179), (186) and then determine the seven ratios (184), (185) by elimination between seven linear equations. And of these seven linear equations we need use only the four following:

$$(187)\qquad 0=\frac{df}{dx},\quad 0=\frac{df'}{dx},\quad 0=\frac{d\sigma_1}{dx},\quad 0=\frac{d\sigma_2}{dx},$$

(in combination with (133) and (134)) if we only want to deduce the four ratios

$$(188)\qquad \frac{dx_2}{dx},\quad \frac{dx_2'}{dx},\quad \frac{dx_2''}{dx},\quad \frac{1}{\lambda}\frac{d\lambda'}{dx},$$

and to examine whether they satisfy the equations (182) and also the following:

$$(189)\qquad \frac{1}{\lambda}\frac{d\lambda'}{dx}=\frac{\lambda''}{\lambda};$$

which four equations are the conditions requisite in order that the equations (160), (161), (174), (162) may result from the equations (157), (158), (159). Now the two first equations (187) combined with the two first equations (134) and with (182) give the two first equations (183), and the two last equations (187) combined with the two last equations (134) and with (182), (189) give in like manner the two following equations:

$$(190)\quad \begin{cases} 0=\dfrac{d\sigma_1}{dx}-\sigma_1'=\dfrac{\delta\sigma_1}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dfrac{\delta\sigma_1}{\delta x_2'}\left(\dfrac{dx_2'}{dx}-x_2''\right)+\dfrac{\delta\sigma_1}{\delta x_2''}\left(\dfrac{dx_2''}{dx}-x_2'''\right)+\lambda\dfrac{\delta\sigma_1}{\delta\lambda'}\left(\dfrac{1}{\lambda}\dfrac{d\lambda'}{dx}-\dfrac{\lambda''}{\lambda}\right), \\ 0=\dfrac{d\sigma_2}{dx}-\sigma_2'=\dfrac{\delta\sigma_2}{\delta x_2}\left(\dfrac{dx_2}{dx}-x_2'\right)+\dots \end{cases}$$

And accordingly these equations (190) combined with the two first equations (183) conduct in general, by the ordinary process of elimination between four equations of the first degree, to the equations (182) and (189).

In the case of the system of $2n^2-1$ equations (126), (127), pages 375, 376, we may seek the $(n-1)(2n-1)$ differential coefficients

$$(191)\quad \frac{dx_2}{dx},\ \frac{dx_2'}{dx},\ \dots,\ \frac{dx_2^{(2n-2)}}{dx},\ \dots,\ \frac{dx_n}{dx},\ \frac{dx_n'}{dx},\ \dots,\ \frac{dx_n^{(2n-2)}}{dx}$$

and the $2n-3$ ratios

$$(192)\quad \frac{1}{\lambda}\frac{d\lambda'}{dx},\ \frac{1}{\lambda}\frac{d\lambda''}{dx},\ \dots,\ \frac{1}{\lambda}\frac{d\lambda^{(2n-3)}}{dx}$$

by elimination between the $2n^2-n-2$ equations

$$(193)\quad 0=\frac{df}{dx},\ 0=\frac{df'}{dx},\ \dots,\ 0=\frac{df^{(2n-3)}}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_1}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_1'}{dx},\ \dots,\ 0=\frac{1}{\lambda}\frac{d\sigma_1^{(2n-4)}}{dx},\ \dots,$$
$$0=\frac{1}{\lambda}\frac{d\sigma_n}{dx},\ 0=\frac{1}{\lambda}\frac{d\sigma_n'}{dx},\ \dots,\ 0=\frac{1}{\lambda}\frac{d\sigma_n^{(2n-4)}}{dx},$$

in which we are to substitute the values

$$(194)\quad \frac{dx_1}{dx}=x_1',\ \dots,\ \frac{dx_1^{(2n-2)}}{dx}=x_1^{(2n-1)}$$

and the value

$$\frac{1}{\lambda}\frac{d\lambda}{dx}=\frac{\lambda'}{\lambda},$$

and with which we may combine the equations (126) and (127) themselves. In this manner we obtain $2n^2-n-2$ equations of the first degree of the forms

$$(195)\quad 0=\frac{df}{dx}-f',\ \dots,\ 0=\frac{df^{(2n-3)}}{dx}-f^{(2n-2)},\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_1}{dx}-\sigma_1'\right),\ \dots,$$
$$0=\frac{1}{\lambda}\left(\frac{d\sigma_1^{(2n-4)}}{dx}-\sigma_1^{(2n-3)}\right),\ \dots,\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_n}{dx}-\sigma_n'\right),\ \dots,\ 0=\frac{1}{\lambda}\left(\frac{d\sigma_n^{(2n-4)}}{dx}-\sigma_n^{(2n-3)}\right),$$

which give

$$(196)\quad \frac{dx_2}{dx}=x_2',\ \dots,\ \frac{dx_2^{(2n-2)}}{dx}=x_2^{(2n-1)},\ \dots,\ \frac{dx_n}{dx}=x_n',\ \dots,\ \frac{dx_n^{(2n-2)}}{dx}=x_n^{(2n-1)},$$
$$\frac{1}{\lambda}\frac{d\lambda'}{dx}=\frac{\lambda''}{\lambda},\ \dots,\ \frac{1}{\lambda}\frac{d\lambda^{(2n-3)}}{dx}=\frac{\lambda^{(2n-2)}}{\lambda}.$$

This shows that the $2n^2-n-2$ equations (131), (132) result by differentiation and elimination from the $n+1$ equations (128)–(130).

In the more general case of the system of $(n+m)(2\Sigma\omega-\omega_1)-\Sigma\omega$ equations (117), which may be written (on account of the value (118) of i)

$$(197)\quad \begin{cases} 0=f_1,\ 0=f_1',\ \ldots,\ 0=f_1^{(2\Sigma\omega-\omega_1-1)},\ \ldots,\ 0=f_m,\ 0=f_m',\ \ldots,\ 0=f_m^{(2\Sigma\omega-\omega_1-1)}, \\ 0=\sigma_1,\ 0=\sigma_1',\ \ldots,\ 0=\sigma_1^{(2\Sigma\omega-2\omega_1-1)},\ \ldots,\ 0=\sigma_n,\ 0=\sigma_n',\ \ldots,\ 0=\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-1)}, \end{cases}$$

if we establish the relations

$$(198)\quad \frac{dx_1}{dx}=x_1',\ \frac{dx_1'}{dx}=x_1'',\ \ldots,\ \frac{dx_1^{(2\Sigma\omega-2)}}{dx}=x_1^{(2\Sigma\omega-1)},\ \frac{1}{\lambda_1}\frac{d\lambda_1}{dx}=\frac{\lambda_1'}{\lambda_1},$$

then the following $(n+m)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ linear equations between $\frac{dx_2}{dx}$ &c., namely

$$(199)\quad \begin{cases} 0=\frac{df_1}{dx}-f_1',\ \ldots,\ 0=\frac{df_1^{(2\Sigma\omega-\omega_1-2)}}{dx}-f_1^{(2\Sigma\omega-\omega_1-1)},\ \ldots,\ 0=\frac{df_m}{dx}-f_m',\ \ldots, \\ \qquad 0=\frac{df_m^{(2\Sigma\omega-\omega_1-2)}}{dx}-f_m^{(2\Sigma\omega-\omega_1-1)}, \\ 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_1}{dx}-\sigma_1'\right),\ \ldots,\ 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_1^{(2\Sigma\omega-2\omega_1-2)}}{dx}-\sigma_1^{(2\Sigma\omega-2\omega_1-1)}\right),\ \ldots, \\ \qquad 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_n}{dx}-\sigma_n'\right),\ \ldots,\ 0=\frac{1}{\lambda_1}\left(\frac{d\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-2)}}{dx}-\sigma_n^{(2\Sigma\omega-\omega_1-\omega_n-1)}\right), \end{cases}$$

will give by elimination

$$(200)\quad \begin{cases} \frac{dx_2}{dx}=x_2',\ \ldots,\ \frac{dx_2^{(2\Sigma\omega+\omega_2-\omega_1-2)}}{dx}=x_2^{(2\Sigma\omega+\omega_2-\omega_1-1)},\ \ldots,\ \frac{dx_n}{dx}=x_n',\ \ldots, \\ \qquad \frac{dx_n^{(2\Sigma\omega+\omega_n-\omega_1-2)}}{dx}=x_n^{(2\Sigma\omega+\omega_n-\omega_1-1)}, \\ \frac{1}{\lambda_1}\frac{d\lambda_1'}{dx}=\frac{\lambda_1''}{\lambda_1},\ \ldots,\ \frac{1}{\lambda_1}\frac{d\lambda_1^{(2\Sigma\omega-\omega_1-2)}}{dx}=\frac{\lambda_1^{(2\Sigma\omega-\omega_1-1)}}{\lambda_1},\ \ldots,\ \frac{1}{\lambda_1}\frac{d\lambda_m}{dx}=\frac{\lambda_m'}{\lambda_1},\ \ldots, \\ \qquad \frac{1}{\lambda_1}\frac{d\lambda_m^{(2\Sigma\omega-\omega_1-2)}}{dx}=\frac{\lambda_m^{(2\Sigma\omega-\omega_1-1)}}{\lambda_1}. \end{cases}$$

Therefore the $(n+m)(2\Sigma\omega-\omega_1-1)-\Sigma\omega$ equations (123)–(125) result by differentiation and elimination from the $n+m$ equations (119)–(122).

Thus the system of these $n+m$ equations (119)–(122) is not only a necessary consequence of the system of the $n+m$ equations (114), (115) but also, reciprocally, the latter equations follow from the former. And the proof of this reciprocity was necessary, in strictness, in order to show that the complete expressions of the functions $x_1, \ldots, x_n$ in the equations (114), (115) involve *so many* as $2\Sigma\omega-1$ arbitrary constants. For the mere deduction of the system (119), (120) (with or without the equations (121), (122)) as a necessary consequence of (114), (115) only entitled us to infer that the complete expressions of $x_1, \ldots, x_n$, deduced from this last mentioned system, *could not contain more* than $2\Sigma\omega-1$ arbitrary constants; and left it doubtful whether some other combination of the equations of that system might not conduct to some new differential equation between x_1 and x *of an order lower than* $2\Sigma\omega-1$, in which case the number of arbitrary constants would be less than $2\Sigma\omega-1$, being always equal to the exponent of the lowest order in any given set of total differential equations between a function x_1 and a variable x which are supposed to be mutually compatible.

If we include the λ's, then the complete integral of the system of the $m+n$ equations (114), (115) will involve in all $2\Sigma\omega$ arbitrary constants, an additional constant being introduced by the integration of the equation (121) or by the circumstance that all the m multipliers λ may be together multiplied by any common constant without altering the relations expressed by the differential equations.

What if we had two original equations of the forms

$$(201) \quad 0=f_1(x, x_1, x_1', x_2, x_2', x_3, x_3')=f_1, \quad 0=f_2(x, x_1, x_1', x_2, x_2', x_3, x_3', x_3'')=f_2?$$

Could we then apply the same reasoning as in the case of a system of the form (114), in which all the equations were of some common order (ω_1 for x_1, ω_2 for x_2, &c.) for each of the n functions though this order might change in passing from one function to another, whereas here the first equation is only of the first order with respect to x_3 while the second is of the second order with respect to the same function?

We have now the case included in the general form (29), in which $m=2$, $n=3$, $\omega_{1,1}=1$, $\omega_{1,2}=1$, $\omega_{1,3}=1$, $\omega_{2,1}=1$, $\omega_{2,2}=1$, $\omega_{2,3}=2$. And the supplementary equations (30) become now

$$(202) \quad \begin{cases} 0=\lambda_1\dfrac{\delta f_1}{\delta x_1}-\left(\lambda_1\dfrac{\delta f_1}{\delta x_1'}\right)'+\lambda_2\dfrac{\delta f_2}{\delta x_1}-\left(\lambda_2\dfrac{\delta f_2}{\delta x_1'}\right)'=\sigma_1, \\ 0=\lambda_1\dfrac{\delta f_1}{\delta x_2}-\left(\lambda_1\dfrac{\delta f_1}{\delta x_2'}\right)'+\lambda_2\dfrac{\delta f_2}{\delta x_2}-\left(\lambda_2\dfrac{\delta f_2}{\delta x_2'}\right)'=\sigma_2, \\ 0=\lambda_1\dfrac{\delta f_1}{\delta x_3}-\left(\lambda_1\dfrac{\delta f_1}{\delta x_3'}\right)'-\lambda_2\dfrac{\delta f_2}{\delta x_3}-\left(\lambda_2\dfrac{\delta f_2}{\delta x_3'}\right)'+\left(\lambda_2\dfrac{\delta f_2}{\delta x_3''}\right)''=\sigma_3. \end{cases}$$

If the circumstance of f_1 not containing x_3'' be not sufficient to interrupt the application of our recent process, we are to consider $\omega_1=\omega_2=1$, $\omega_3=2$, that is, $\Sigma\omega=4$; and we are to form the system of 31 equations:

$$(203) \quad \begin{cases} 0=f_1,\ 0=f_1',\ 0=f_1'',\ 0=f_1''',\ 0=f_1^{\text{iv}},\ 0=f_1^{\text{v}},\ 0=f_1^{\text{vi}}, \\ 0=f_2,\ 0=f_2', \qquad\qquad\qquad\qquad\qquad , 0=f_2^{\text{vi}}, \\ 0=\sigma_1,\ 0=\sigma_1',\ 0=\sigma_1'',\ 0=\sigma_1''',\ 0=\sigma_1^{\text{iv}},\ 0=\sigma_1^{\text{v}}, \\ 0=\sigma_2,\ 0=\sigma_2', \qquad\qquad\qquad\qquad , 0=\sigma_2^{\text{v}}, \\ 0=\sigma_3,\ 0=\sigma_3', \qquad\qquad\qquad , 0=\sigma_3^{\text{iv}}. \end{cases}$$

Then, if possible, we are to eliminate the 26 quantities

$$(204) \quad x_2', x_2'', \ldots, x_2^{\text{vii}}, x_3', \ldots, x_3^{\text{viii}}, \frac{\lambda_1''}{\lambda_1}, \frac{\lambda_1'''}{\lambda_1}, \ldots, \frac{\lambda_1^{\text{vi}}}{\lambda_1}, \frac{\lambda_2'}{\lambda_1}, \frac{\lambda_2''}{\lambda_1}, \ldots, \frac{\lambda_2^{\text{vi}}}{\lambda_1}$$

between the 31 equations (203) and so to obtain a new system of five equations between the following 13 quantities

$$(205) \quad x, x_1, x_1', x_1'', \ldots, x_1^{\text{vii}}, x_2, x_3, \frac{\lambda_1'}{\lambda_1}, \frac{\lambda_2}{\lambda_1},$$

enabling us to express

$$(206) \quad x_1^{\text{vii}}, x_2, x_3, \frac{\lambda_1'}{\lambda_1}, \frac{\lambda_2}{\lambda_1}$$

as functions of $x, x_1, x_1', x_1'', x_1''', x_1^{\text{iv}}, x_1^{\text{v}}$ and x_1^{vi}. We must now endeavour to show that the deduction of equations (205) from equations (203), or ultimately from equations (201), (202), is possible; and that, reciprocally, these five equations (201), (202) can be deduced by differentiation and elimination from (205).

The equation $0=f_1^{(i)}$ involves

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+1)},\ x_2,\ x_2',\ \ldots,\ x_2^{(i+1)},\ x_3,\ x_3',\ \ldots,\ x_3^{(i+1)},$$

and the equation $0=f_2^{(i)}$ involves

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+1)},\ x_2,\ x_2',\ \ldots,\ x_2^{(i+1)},\ x_3,\ x_3',\ \ldots,\ x_3^{(i+2)}.$$

The equation $0=\sigma_1$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_2,\ x_2',\ x_2'',\ x_3,\ x_3',\ x_3'',\ x_3''',\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},$$

the equation $0=\sigma_2$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_2,\ x_2',\ x_2'',\ x_3,\ x_3',\ x_3'',\ x_3''',\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1}$$

and the equation $0=\sigma_3$ involves

$$x,\ x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2',\ x_2'',\ x_2''',\ x_3,\ x_3',\ x_3'',\ x_3''',\ x_3^{\text{iv}},\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},\ \frac{\lambda_2''}{\lambda_1}.$$

Hence the equations $0=\sigma_1^{\text{v}}$, $0=\sigma_2^{\text{v}}$, $0=\sigma_3^{\text{iv}}$ involve each the quantities

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{\text{vii}},\ x_2,\ x_2',\ \ldots,\ x_2^{\text{vii}},\ x_3,\ x_3',\ \ldots,\ x_3^{\text{viii}},\ \frac{\lambda_1'}{\lambda_1},\ \frac{\lambda_1''}{\lambda_1},\ \ldots,\ \frac{\lambda_1^{\text{vi}}}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \frac{\lambda_2'}{\lambda_1},\ \ldots,\ \frac{\lambda_2^{\text{vi}}}{\lambda_1},$$

except that $0=\sigma_3^{\text{iv}}$ does not involve $\dfrac{\lambda_1^{\text{vi}}}{\lambda_1}$. Generally the three equations

$$0=\sigma_1^{(i+1)},\quad 0=\sigma_2^{(i+1)},\quad 0=\sigma_3^{(i)}$$

contain all the same quantities, namely

$$x,\ x_1,\ x_1',\ \ldots,\ x_1^{(i+3)},\ x_2,\ \ldots,\ x_2^{(i+3)},\ x_3,\ \ldots,\ x_3^{(i+4)},\ \frac{\lambda_1'}{\lambda_1},\ \ldots,\ \frac{\lambda_1^{(i+2)}}{\lambda_1},\ \frac{\lambda_2}{\lambda_1},\ \ldots,\ \frac{\lambda_2^{(i+2)}}{\lambda_1},$$

except that $0=\sigma_3^{(i)}$ does not contain $\dfrac{\lambda_1^{(i+3)}}{\lambda_1}$. This last mentioned circumstance, however, does not hinder us to eliminate $\dfrac{\lambda_1^{\text{vi}}}{\lambda_1}, \dfrac{\lambda_2^{\text{vi}}}{\lambda_1}, x_2^{\text{vii}}, x_3^{\text{viii}}$ between the five equations $0=f_1^{\text{vi}}=f_2^{\text{vi}}=\sigma_1^{\text{v}}=\sigma_2^{\text{v}}=\sigma_3^{\text{vi}}$, nor does the circumstance that $0=f_1^{\text{vi}}$ does not involve x_3^{viii}. And generally the circumstance that some of the 26 things (204) to be eliminated do not appear in some of the 31 equations (203) does not hinder us to effect the elimination supposed above, but rather tends to facilitate it. Since then the equations (203) contain altogether only the 26 things (204) and the 13 things (205) and are in general 31 distinct and independent relations between these 39 things *so far as elimination is concerned* (although, *with respect to differentiation*, only 5 of these relations are independent and the 26 others are deduced from them), we can deduce by elimination *at least five* equations between the 13 things (205).

Indeed, a doubt might be felt whether we could not in general deduce *more than five* equations between these 13 things; or, in other words, whether we could in general return by differentiation and elimination from the five equations thus deduced to the system of the 31 equations (203). But if we suppose (as, setting aside singular exceptions, we may and ought to do) that *for purposes of elimination* the 31 relations (203) are distinct and independent and therefore suffice to determine the 31 functions (204), (205), we can conclude that the expressions of the 26 functions

(204) result by differentiation and elimination from those of the five functions (206); namely, by showing that the following 26 linear equations between $\frac{dx_2}{dx}$, &c.,

$$(207)\quad \begin{cases} 0=\frac{df_1}{dx}-f_1', \ldots, 0=\frac{df_1^{\mathrm{v}}}{dx}-f_1^{\mathrm{vi}}, 0=\frac{df_2}{dx}-f_2', \ldots, 0=\frac{df_2^{\mathrm{v}}}{dx}-f_2^{\mathrm{vi}}, \\ 0=\frac{d\sigma_1}{dx}-\sigma_1', \ldots, 0=\frac{d\sigma_1^{\mathrm{iv}}}{dx}-\sigma_1^{\mathrm{v}}, 0=\frac{d\sigma_2}{dx}-\sigma_2', \ldots, 0=\frac{d\sigma_2^{\mathrm{iv}}}{dx}-\sigma_2^{\mathrm{v}}, \\ \qquad 0=\frac{d\sigma_3}{dx}-\sigma_3', \ldots, 0=\frac{d\sigma_3'''}{dx}-\sigma_3^{\mathrm{iv}}, \end{cases}$$

conduct by elimination and by supposing

$$(208)\quad \frac{dx_1}{dx}=x_1', \ldots \frac{dx_1^{\mathrm{vi}}}{dx}=x_1^{\mathrm{vii}}, \frac{1}{\lambda_1}\frac{d\lambda_1}{dx}=\frac{\lambda_1'}{\lambda_1}$$

to the following 26 expressions

$$(209)\quad \begin{cases} \frac{dx_2}{dx}=x_2', \ldots, \frac{dx_2^{\mathrm{vi}}}{dx}=x_2^{\mathrm{vii}}, \frac{dx_3}{dx}=x_3', \ldots, \frac{dx_3^{\mathrm{vii}}}{dx}=x_3^{\mathrm{viii}}, \\ \frac{d\lambda_1'}{dx}=\lambda_1'', \ldots, \frac{d\lambda_1^{\mathrm{v}}}{dx}=\lambda_1^{\mathrm{vi}}, \frac{d\lambda_2}{dx}=\lambda_2', \ldots, \frac{d\lambda_2^{\mathrm{v}}}{dx}=\lambda_2^{\mathrm{vi}}. \end{cases}$$

(Feb. 26th.)

The five equations (201), (202), $0=f_1=f_2=\sigma_1=\sigma_2=\sigma_3$, might be proposed to be integrated by five series of the forms

$$(210)\quad \begin{cases} x_1+\Delta x_1=x_1+x_1'\Delta x+\tfrac{1}{2}x_1''\Delta x^2+\ldots, \quad \Delta x_2=x_2'\Delta x+\tfrac{1}{2}x_2''\Delta x^2+\ldots, \\ \qquad \Delta x_3=x_3'\Delta x+\tfrac{1}{2}x_3''\Delta x^2+\ldots, \\ \lambda_1+\Delta\lambda_1=\lambda_1+\lambda_1'\Delta x+\tfrac{1}{2}\lambda_1''\Delta x^2+\ldots, \quad \Delta\lambda_2=\lambda_2'\Delta x+\tfrac{1}{2}\lambda_2''\Delta x^2+\ldots, \end{cases}$$

by substitution of which series we are to satisfy, independently of Δx, the five equations

$$(211)\quad 0=f_1+\Delta f_1, \quad 0=f_2+\Delta f_2, \quad 0=\sigma_1+\Delta\sigma_1, \quad 0=\sigma_2+\Delta\sigma_2, \quad 0=\sigma_3+\Delta\sigma_3,$$

that is,

$$(212)\quad \begin{cases} 0=f_1+f_1'\Delta x+\tfrac{1}{2}f_1''\Delta x^2+\ldots, \quad 0=f_2+f_2'\Delta x+\tfrac{1}{2}f_2''\Delta x^2+\ldots, \\ 0=\sigma_1+\sigma_1'\Delta x+\tfrac{1}{2}\sigma_1''\Delta x^2+\ldots, \quad 0=\sigma_2+\sigma_2'\Delta x+\tfrac{1}{2}\sigma_2''\Delta x^2+\ldots, \\ \qquad 0=\sigma_3+\sigma_3'\Delta x+\tfrac{1}{2}\sigma_3''\Delta x^2+\ldots. \end{cases}$$

It is therefore necessary and, if Δx be small enough to allow the convergence of the three developments, it is in general sufficient to satisfy the five indefinite series of conditions:

$$(213)\quad \begin{cases} 0=f_1, 0=f_1', 0=f_1'', \ldots, 0=f_2, 0=f_2', 0=f_2'', \ldots, \\ 0=\sigma_1, 0=\sigma_1', \ldots, 0=\sigma_2, 0=\sigma_2', \ldots, 0=\sigma_3, 0=\sigma_3', \ldots, \end{cases}$$

for *any one particular value of* x in order to satisfy the equations (211) by the developments (210) *for any near value of* x, and so to obtain the five integrals of (211), at least in series. The question, "how many arbitrary constants does the system of these five integrals contain?," is therefore reducible to the question, "how many functions of the five following series

$$(214)\quad x_1, x_1', x_1'', \ldots, x_2, x_2', x_2'', \ldots, x_3, x_3', x_3'', \ldots, \lambda_1, \lambda_1', \lambda_1'', \ldots, \lambda_2, \lambda_2', \lambda_2'', \ldots$$

49-2

may have arbitrary values assigned to them, for an assumed particular value of x, so as to satisfy all the conditions of the five series (213)?"

The five conditions

(215) $$0=f_1,\ 0=f_2,\ 0=\sigma_1,\ 0=\sigma_2,\ 0=\sigma_3$$

involve only the 18 functions

(216) $$x_1,\ x_1',\ x_1'',\ x_1''',\ x_2,\ x_2',\ x_2'',\ x_2''',\ x_3,\ x_3',\ x_3'',\ x_3''',\ x_3^{\text{iv}},\ \lambda_1,\ \lambda_1',\ \lambda_2,\ \lambda_2',\ \lambda_2''$$

besides the variable x. The six additional conditions

(217) $$0=f_1',\ 0=f_1'',\ 0=f_2',\ 0=f_2'',\ 0=\sigma_1',\ 0=\sigma_2'$$

involve only one additional function, namely

(218) $$\lambda_1''.$$

Thus the 11 conditions (215), (217) involve only the variable x and the 19 functions (216), (218) and leave only 8, at most, of these functions arbitrary in value when the value of x is assumed. We might have eliminated λ_1'' between the two last of the six conditions (217) and then we should have five relations between the 18 functions (216) to combine with the five conditions (215), leaving thus only 8, at most, of those functions arbitrary in value for any one assumed value of x. Reciprocally, no new relation, deducible from the conditions (213), restricts the number of these arbitrary values so as to reduce it to be less than 8. For every new condition or set of conditions, taken out of the series (213), introduces a new function or functions which cannot in general be eliminated by means of those new conditions. Thus the condition $0=f_1'''$ introduces the two new functions $x_1^{\text{iv}}, x_2^{\text{iv}}$, the condition $0=f_2'''$ introduces $x_1^{\text{iv}}, x_2^{\text{iv}}, x_3^{\text{v}}$, the condition $0=\sigma_1''$ introduces $x_1^{\text{iv}}, x_2^{\text{iv}}, x_3^{\text{v}}, \lambda_1''', \lambda_2'''$, the condition $0=\sigma_2''$ introduces the same five functions as $0=\sigma_1''$, and the condition $0=\sigma_3'$ introduces $x_1^{\text{iv}}, x_2^{\text{iv}}, x_3^{\text{v}}, \lambda_2'''$. These five conditions are indeed sufficient to *determine*, but not in general sufficient to *eliminate*, the five new functions $x_1^{\text{iv}}, x_2^{\text{iv}}, x_3^{\text{v}}, \lambda_1''', \lambda_2'''$ when the 19 functions (216) and (218) are known. So that the five new conditions

(219) $$0=f_1''',\ 0=f_2''',\ 0=\sigma_1'',\ 0=\sigma_2'',\ 0=\sigma_3'$$

can in general all be satisfied, but only in one way, by choosing suitable values for these five new functions

(220) $$x_1^{\text{iv}},\ x_2^{\text{iv}},\ x_3^{\text{v}},\ \lambda_1''',\ \lambda_2''',$$

after 8 of the 19 functions (216), (218) have had their values arbitrarily assumed and after the remaining 11 values have been determined so as to satisfy the 11 conditions (215), (217). In like manner, the five new conditions

(221) $$0=f_1^{\text{iv}},\ 0=f_2^{\text{iv}},\ 0=\sigma_1''',\ 0=\sigma_2''',\ 0=\sigma_3''$$

can be satisfied by one but by only one set of values of the five new functions

(222) $$x_1^{\text{v}},\ x_2^{\text{v}},\ x_3^{\text{vi}},\ \lambda_1^{\text{iv}},\ \lambda_2^{\text{iv}},$$

and so on indefinitely. We may therefore conclude, by this mode of reasoning, that the five indefinite series of conditions (213) can all be satisfied together by one set of values of the functions (214), of which values eight and only eight are arbitrary after the value of x has been assumed. But of these eight arbitrary values, one is introduced by the consideration of the two series

(223) $$\lambda_1,\ \lambda_1',\ \lambda_1'',\ \ldots,\ \lambda_2,\ \lambda_2',\ \lambda_2'',\ \ldots$$

and only seven belong to the three series

(224) $$x_1, x_1', x_1'', \ldots, x_2, x_2', x_2'', \ldots, x_3, x_3', x_3'', \ldots.$$

For if the 11 conditions (215), (217) be distributed into these two new groups

(225) $$0=f_1,\ 0=f_1',\ 0=f_1'',\ 0=f_2,\ 0=f_2',\ 0=f_2''$$

and

(226) $$0=\sigma_1,\ 0=\sigma_1',\ 0=\sigma_2,\ 0=\sigma_2',\ 0=\sigma_3,$$

the six equations of the group (225) contain only the 13 functions

(227) $$x_1, x_1', x_1'', x_1''', x_2, x_2', x_2'', x_2''', x_3, x_3', x_3'', x_3''', x_3^{\text{iv}},$$

besides the variable x which is always understood. They leave therefore only 7 of these 13 values arbitrary and the eighth arbitrary value is that of any one of the six functions

(228) $$\lambda_1, \lambda_1', \lambda_1'', \lambda_2, \lambda_2', \lambda_2'',$$

of which the five equations of the group (226) determine only the five ratios.

It is important however to observe that we may not assume *any* seven of the values of the functions (227), although we have only the six equations (225) between them. Thus the first equation of this group, namely $0=f_1$, prevents us from assuming arbitrary values for all the functions $x_1, x_1', x_2, x_2', x_3, x_3'$, and therefore if the seven arbitrary values contain five of these they cannot contain the sixth. With this exception, however, we seem to be at liberty to select any seven of the functions (227) because the equation $0=f_2$ contains the same six functions as $0=f_1$ along with the new function x_3'' and because the differentials of these equations contain each more than seven different functions.

When the arbitrary values of some seven of the functions (227) have been assumed (not more than five belonging to the group $x_1, x_1', x_2, x_2', x_3, x_3'$ for the reason just now assigned) and when the other six of these functions have been determined by (225), we can then in general deduce the values of all the other functions of the series (224), without yet assuming that eighth arbitrary constant which is connected with the passage from the ratios to the absolute values of the functions (223). For the ratios only, and not the absolute values, of those multiplier-functions enter into the conditions (213).

We see, therefore, in this new way, that the complete expressions for the three functions x_1, x_2, x_3 (deduced from the differential equations (201) and (202) by differential elimination of λ_1, λ_2 and by integration) contain in general seven arbitrary constants and no more. We can therefore deduce a principal integral relation of the following form between those functions and x, x_3' and their initial values a_1, a_2, a_3, a, a_3':

(229) $$0=F(x, x_1, x_2, x_3, x_3', a, a_1, a_2, a_3, a_3').$$

Can we generalise the foregoing investigation so as to show, by a similar process, that a principal integral relation of the form (35) results from the two systems (29) and (30)?

In the first place we may observe that when equations of the forms (114) and (115) are given they are in number $m+n$, and they contain, taken altogether, the $(n+m)(\omega_n+1)+\Sigma\omega$ functions following (besides the independent variable x):

(230) $$\begin{cases} x_1, x_1', \ldots, x_1^{(\omega_1+\omega_n)}, x_2, x_2', \ldots, x_2^{(\omega_2+\omega_n)}, \ldots, x_n, x_n', \ldots, x_n^{(2\omega_n)}, \\ \lambda_1, \lambda_1', \ldots, \lambda_1^{(\omega_n)}, \lambda_2, \lambda_2', \ldots, \lambda_2^{(\omega_n)}, \ldots, \lambda_m, \lambda_m', \ldots, \lambda_m^{(\omega_n)}, \end{cases}$$

which functions, however, and no others will (at most) be contained in the following additional equations,

$$(231)\quad \begin{cases} 0=f_1', \ 0=f_1'', \ \ldots, \ 0=f_1^{(\omega_n)}, \ \ldots, \ 0=f_m', \ 0=f_m'', \ \ldots, \ 0=f_m^{(\omega_n)}, \\ 0=\sigma_1', \ 0=\sigma_1'', \ \ldots, \ 0=\sigma_1^{(\omega_n-\omega_1)}, \ \ldots, \ 0=\sigma_{n-1}', \ 0=\sigma_{n-1}'', \ \ldots, \ 0=\sigma_{n-1}^{(\omega_n-\omega_{n-1})}, \end{cases}$$

being in number $(n+m)\,\omega_n - \Sigma\omega$. Thus we have $(n+m)(\omega_n+1)-\Sigma\omega$ equations between $(n+m)(\omega_n+1)+\Sigma\omega$ functions and therefore, so far, $2\Sigma\omega$ of these functions remain arbitrary in value. Nor will any new derived equations of the forms

$$(232)\quad 0=f_1^{(\omega_n+i)}, \ \ldots, \ 0=f_m^{(\omega_n+i)}, \ 0=\sigma_1^{(\omega_n-\omega_1+i)}, \ 0=\sigma_2^{(\omega_n-\omega_2+i)}, \ \ldots, \ 0=\sigma_n^{(i)}$$

give any new relations between these arbitrary values. They remain therefore ultimately arbitrary and the complete integrals of the $m+n$ differential equations (114), (115) contain in general $2\Sigma\omega$ arbitrary constants. But of these only $2\Sigma\omega-1$ enter into the expressions of the functions $x_1, x_2, \ldots, x_n$, the other being introduced by the circumstance that the m multiplier-functions $\lambda_1, \ldots, \lambda_m$ may be multiplied by any arbitrary constant without disturbing the relations of the question.