# Part II

# CALCULUS OF PRINCIPAL RELATIONS

# XII.

## CALCULUS OF PRINCIPAL RELATIONS*

[1836.]

### Introduction.

**1.** It is a well-known theorem of the Integral Calculus that if $y_x$ be a function of $x$, respecting which we are only told that it satisfies a given differential equation of the first order,

$$0=f(x, y_x, y'_x), \tag{1}$$

$\left(y'_x \text{ denoting the differential coefficient or derived function } \dfrac{dy_x}{dx}\right)$, we are at liberty to assume an additional condition of the form

$$y_a=b, \tag{2}$$

$a$ and $b$ denoting any two assumed corresponding values of the two connected variables $x$ and $y_x$; but that after the assumption of this new condition, the functional relation between these two variables $x$ and $y_x$ is in general theoretically determined, whatever practical difficulties may remain in the actual discovery of its form. So that the differential equation (1) may in general be conceived as conducting, by an easy or difficult but always possible integration, to a relation between any one pair of corresponding values $a, y_a$ of the two connected variables of the question and any other pair of corresponding values $x, y_x$ of the same two connected variables; or to an equation in finite differences of the first order between $a$, $y_a$, $\Delta a$ and $\Delta y_a$, which may be thus denoted,

$$0=F(a, y_a, x, y_x)=F(a, y_a, a+\Delta a, y_a+\Delta y_a), \tag{3}$$

$\Delta a$ and $\Delta y_a$ being here equivalent to $x-a$ and $y_x-y_a$ respectively.

**2.** Again, if $y_x$ and $z_x$ denote two functions of $x$, respecting which we only know that their first derived functions $y'_x$ and $z'_x$ are connected with them and with $x$ by a given relation of the form

$$0=X+Yy'_x+Zz'_x, \tag{4}$$

in which $X$, $Y$, $Z$ denote given functions of $x$, $y_x$ and $z_x$; then, if we denote by

$$\frac{\delta X}{\delta x}, \frac{\delta X}{\delta y_x}, \frac{\delta X}{\delta z_x}, \quad \frac{\delta Y}{\delta x}, \frac{\delta Y}{\delta y_x}, \frac{\delta Y}{\delta z_x}, \quad \frac{\delta Z}{\delta x}, \frac{\delta Z}{\delta y_x}, \frac{\delta Z}{\delta z_x},$$

the nine partial differential coefficients of the first order of these three functions, taken with respect to the three variables $x$, $y_x$ and $z_x$, as if they were independent of each other, and if we suppose that these partial differential coefficients satisfy the condition

$$0=X\left(\frac{\delta Y}{\delta z_x}-\frac{\delta Z}{\delta y_x}\right)+Y\left(\frac{\delta Z}{\delta x}-\frac{\delta X}{\delta z_x}\right)+Z\left(\frac{\delta X}{\delta y_x}-\frac{\delta Y}{\delta x}\right), \tag{5}$$

* [It was Hamilton's intention to write a book on the Calculus of Principal Relations (see Graves, *Life of Hamilton*, II, p. 177). This is all that remains of the manuscript of the proposed treatise. Papers XIII and XIV, pp. 332–390, contain material which he obviously intended to incorporate in the book. A sketch of the calculus is to be found on page 408.]

it is known that the differential equation (4) conducts by integration to a relation of the form

$$f_{,}(x, y_x, z_x) = \text{const.} = f_{,}(a, y_a, z_a), \tag{6}$$

which may also be thus written, as an equation in finite differences of the first order,

$$0 = F(a, y_a, z_a, a+\Delta a, y_a+\Delta y_a, z_a+\Delta z_a). \tag{7}$$

**3**. But in general, if the derived functions $y'_x$ and $z'_x$ be only connected with $x, y_x$ and $z_x$ by a given relation, which may be thus denoted,

$$0 = f(x, y_x, z_x, y'_x, z'_x), \tag{8}$$

then (setting aside some particular cases of exception such as that which has been last considered) we may attribute any arbitrary values to the three initial and three final values, $a, y_a$, $z_a, a+\Delta a, y_a+\Delta y_a$ and $z_a+\Delta z_a$, without obliging these initial and final values to satisfy any determinate relation, such as (7), as a necessary consequence of the given differential equation (8). Or, to express the same thing otherwise, we may in general assume any arbitrary values $y_a$ and $z_a$ of the functions $y_x$ and $z_x$, as corresponding to an assumed value $a$ of the variable $x$, without thereby establishing any relation between the values of the same functions $y_{a+\Delta a}$ and $z_{a+\Delta a}$, which correspond to any other assumed value $a+\Delta a$ of the same variable $x$. According to the unlimited variety of forms which we may assume for the function $y_x$, consistently with previously assumed values $y_a$ and $y_{a+\Delta a}$, we shall have by the differential equation (8) an unlimited variety of corresponding forms of the function $z_x$, even when the initial value $z_a$ is given or previously assumed; and in passing from one such form of $z_x$ to another the final value $z_{a+\Delta a}$ will itself in general vary; so that it will be only in particular cases that these various forms of the function $z_x$ will fail to give an unlimited variety of final values $z_{a+\Delta a}$.

**4**. To illustrate this theorem by an example and to prove that it is true in an extensive class of cases before considering the general proof of it, let us assume the particular relation (4), supposing now that the condition (5) is *not* satisfied; and let us write, for abridgement,

$$-\frac{X}{Z} = p, \quad -\frac{Y}{Z} = q, \tag{9}$$

so that the condition (5) shall take this simplified form

$$\frac{\delta p}{\delta y_x} + q\frac{\delta p}{\delta z_x} = \frac{\delta q}{\delta x} + p\frac{\delta q}{\delta z_x}. \tag{10}$$

We have thus the differential equation

$$z'_x = p + qy'_x, \tag{11}$$

in which $p$ and $q$ are any two given functions of $x, y_x$ and $z_x$, *not* satisfying the condition (10). Let any function $y_x$ of $x$ be supposed to be assumed at pleasure, subject only to the initial and final conditions

$$y_a = b, \quad y_{a+\Delta a} = b + \Delta b, \tag{12}$$

in which the values of $b$ and $\Delta b$ are given or previously assumed; and then let the connected function $z_x$ be conceived to be determined (as we may always at least theoretically conceive it to be, by principles already referred to, whatever practical difficulties the actual determination may present) so as to satisfy the differential equation (11) and also the initial condition

$$z_a = c, \tag{13}$$

in which $c$ is given or assumed; and finally, from the form of the function $z_x$ thus determined, let the final value

$$z_{a+\Delta a}=c+\Delta c \tag{14}$$

be conceived to be calculated. We propose to show that this final value $c+\Delta c$ will in general vary with a variation of the assumed form of the function $y_x$, even when $a$, $b$, $c$, $\Delta a$ and $\Delta b$ remain unvaried. To show this, let $\epsilon$ be any small and arbitrary constant and let $y_x+\epsilon\eta_x$ and $z_x+\epsilon\zeta_x$ denote two new connected functions of $x$, which differ little from the two old functions $y_x$, $z_x$ and which satisfy, like them, the differential equation (11), or rather the new equation into which that transforms itself when the new functions are substituted for the old ones; let it be supposed also that these new functions $y_x+\epsilon\eta_x$, $z_x+\epsilon\zeta_x$ satisfy these three new conditions

$$y_a+\epsilon\eta_a=b, \quad y_{a+\Delta a}+\epsilon\eta_{a+\Delta a}=b+\Delta b, \quad z_a+\epsilon\zeta_a=c, \tag{15}$$

which are analogous to the three old conditions (12) and (13) and give, when combined with them,

$$\eta_a=0, \quad \eta_{a+\Delta a}=0, \quad \zeta_a=0. \tag{16}$$

The new differential equation, analogous to (11), may be thus written

$$z'_x+\epsilon\zeta'_x=p+\epsilon\eta_x\frac{\delta p}{\delta y_x}+\epsilon\zeta_x\frac{\delta p}{\delta z_x}+\left(q+\epsilon\eta_x\frac{\delta q}{\delta y_x}+\epsilon\zeta_x\frac{\delta q}{\delta z_x}\right)y'_x+\epsilon q\eta'_x+\epsilon^2E_x, \tag{17}$$

$E_x$ denoting, for abridgement, a factor the development of which is easily obtained but is not necessary for our present purpose; and if we subtract the old differential equation (11) from the new one (17) and divide by $\epsilon$, we find, more simply,

$$\zeta'_x=\left(\frac{\delta p}{\delta z_x}+y'_x\frac{\delta q}{\delta z_x}\right)\zeta_x+\left(\frac{\delta p}{\delta y_x}+y'_x\frac{\delta q}{\delta y_x}\right)\eta_x+q\eta'_x+\epsilon E_x: \tag{18}$$

from which equation we propose to deduce by integration an expression for the final value $\zeta_{a+\Delta a}$, which shall show that this final value does not in general vanish and therefore that the final value $z_{a+\Delta a}+\epsilon\zeta_{a+\Delta a}$ of the new function $z_x+\epsilon\zeta_x$ is in general different from the final value $z_{a+\Delta a}$ of the old function $z_x$.

With this view, we may multiply the differential equation (18) by a function $\lambda_x$, so chosen as to make the product

$$\lambda_x\left\{\zeta'_x-\left(\frac{\delta p}{\delta z_x}+y'_x\frac{\delta q}{\delta z_x}\right)\zeta_x\right\}$$

equal to the first differential coefficient or derived function $(\lambda_x\zeta_x)'$ of the product $\lambda_x\zeta_x$; a condition which gives

$$\lambda'_x=-\lambda_x\left(\frac{\delta p}{\delta z_x}+y'_x\frac{\delta q}{\delta z_x}\right), \tag{19}$$

and therefore, in the notation of definite integrals,

$$\lambda_x=\lambda_a e^{-\int_a^x\left(\frac{\delta p}{\delta z_x}+y'_x\frac{\delta q}{\delta z_x}\right)dx}. \tag{20}$$

When $\lambda_x$ is thus chosen, it is easy to see that the first differential coefficient or derived function of the product $\lambda_x q$ is

$$\begin{aligned}(\lambda_x q)' = \frac{d}{dx}(\lambda_x q) = \lambda_x \frac{dq}{dx} + q\lambda'_x \\ = \lambda_x \left(\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + z'_x \frac{\delta q}{\delta z_x} + q\frac{\lambda'_x}{\lambda_x}\right) \\ = \lambda_x \left\{\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + (p + qy'_x)\frac{\delta q}{\delta z_x} - q\left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right)\right\} \\ = \lambda_x \left(\frac{\delta q}{\delta x} + y'_x \frac{\delta q}{\delta y_x} + p\frac{\delta q}{\delta z_x} - q\frac{\delta p}{\delta z_x}\right),\end{aligned} \tag{21}$$

and that consequently

$$\lambda_x \left(\frac{\delta p}{\delta y_x} + y'_x \frac{\delta q}{\delta y_x}\right) = (\lambda_x q)' + \lambda_x \left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right); \tag{22}$$

changing therefore $(\lambda_x q)'\eta_x + \lambda_x q \eta'_x$ to the equivalent expression $(q\lambda_x\eta_x)'$, we find from the equation (18) the following

$$(\lambda_x \zeta_x)' = (q\lambda_x\eta_x)' + \lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right) + \epsilon\lambda_x E_x, \tag{23}$$

which gives, in the notation of definite integrals, by the conditions (16),

$$\lambda_{a+\Delta a}\zeta_{a+\Delta a} = \int_a^{a+\Delta a} \lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right)dx + \epsilon\int_a^{a+\Delta a}\lambda_x E_x dx, \tag{24}$$

that is, finally,

$$\zeta_{a+\Delta a} = \frac{1}{\lambda_{a+\Delta a}}\int_a^{a+\Delta a}\left\{\lambda_x\eta_x\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}\right) + \epsilon\lambda_x E_x\right\}dx. \tag{25}$$

From this expression it is easy to prove what was asserted, that when the condition (10) is *not* satisfied the final value $\zeta_{a+\Delta a}$ does *not* in general vanish independently of the form of the function $\eta_x$, even when the initial value $\zeta_a$ and the initial and final values $\eta_a$ and $\eta_{a+\Delta a}$ are each $=0$; for we now see that $\zeta_{a+\Delta a}$ may in general be developed according to ascending powers of the small multiplier $\epsilon$ in a series of the form

$$\zeta_{a+\Delta a} = \zeta^{(0)}_{a+\Delta a} + \epsilon\,\zeta^{(1)}_{a+\Delta a} + \text{etc.}, \tag{26}$$

and that the first term of this series does not in general vanish independently of $\eta_x$ since it may be expressed as follows

$$\zeta^{(0)}_{a+\Delta a} = e^{-\int_a^{a+\Delta a} L_x dx} \cdot \int_a^{a+\Delta a} e^{+\int_a^x L_x dx} M_x\eta_x dx, \tag{27}$$

if we put, for abridgement,

$$\left.\begin{aligned} L_x &= -\left(\frac{\delta p}{\delta z_x} + y'_x\frac{\delta q}{\delta z_x}\right), \\ M_x &= \frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} + q\frac{\delta p}{\delta z_x} - p\frac{\delta q}{\delta z_x}. \end{aligned}\right\} \tag{28}$$

We see, then, that except in the case expressed by the condition (10) (in which this last coefficient $M_x$ vanishes) the differential equation (11) is not sufficient to conduct, by necessary inference, to any one determinate relation between the three initial values $a$, $b$, $c$ and the three final values $a+\Delta a$, $b+\Delta b$, $c+\Delta c$ of the three connected variables $x$, $y_x$, $z_x$; and therefore that we cannot in general deduce, as a necessary consequence of the differential equation (11), any equation in finite differences of the form (7): since otherwise it would be necessary that $\zeta_{a+\Delta a}$ should in general vanish along with $\eta_a$, $\zeta_a$ and $\eta_{a+\Delta a}$ independently of the form of $\eta_x$ and of the value of $\epsilon$.

**5.** To make this reasoning more perfectly clear by a still more particular example, let us assume the following particular forms for the coefficients $p$ and $q$,

$$p=-y_x, \quad q=x; \tag{29}$$

so that the differential equation (11) becomes now

$$z_x'=xy_x'-y_x. \tag{30}$$

We have now

$$\frac{\delta p}{\delta x}=0, \quad \frac{\delta p}{\delta y_x}=-1, \quad \frac{\delta p}{\delta z_x}=0, \quad \frac{\delta q}{\delta x}=1, \quad \frac{\delta q}{\delta y_x}=0, \quad \frac{\delta q}{\delta z_x}=0; \tag{31}$$

so that the condition (10) is not satisfied. If we suppose that $y_x$ and $z_x$ are two functions of $x$, which satisfy the differential equation (30) and also the conditions (12) and (13); and that $y_x+\epsilon\eta_x$ and $z_x+\epsilon\zeta_x$ are two other functions of $x$, which satisfy the analogous differential equation

$$z_x'+\epsilon\zeta_x'=x(y_x'+\epsilon\eta_x')-(y_x+\epsilon\eta_x) \tag{32}$$

and the analogous conditions (15) or (16); we obtain this new differential equation of the form (18)

$$\zeta_x'=x\eta_x'-\eta_x, \tag{33}$$

and the general expression (20) for the multiplier $\lambda_x$ becomes here

$$\lambda_x=\lambda_a=\text{const.}, \tag{34}$$

so that we may suppress this multiplier or suppose it equal to 1; and we find this new equation of the form (23)

$$\zeta_x'=(x\eta_x)'-2\eta_x, \tag{35}$$

which gives, by integration and by the conditions (16), a result of the form (25), namely,

$$\zeta_{a+\Delta a}=\int_a^{a+\Delta a}(-2\eta_x)\,dx. \tag{36}$$

And it is clear that this final value of $\zeta_x$ does not vanish independently of the form of the function $\eta_x$, even when the conditions (16) are satisfied; and therefore that, in this example, the final values of the three connected variables $x$, $y_x$, $z_x$ are not connected with each other by any necessary relation when only the initial values and the differential equation (30) are given.

**6.** Resuming now the more general differential equation (8) and reasoning on it in a similar manner with a view to establish the general theorem of the third article, we find, instead of the equation (18), the following:

$$0=\frac{\delta f}{\delta z_x'}\zeta_x'+\frac{\delta f}{\delta z_x}\zeta_x+\frac{\delta f}{\delta y_x'}\eta_x'+\frac{\delta f}{\delta y_x}\eta_x+\epsilon E_x; \tag{37}$$

which is now to be multiplied by a function $\lambda_x$, chosen so as to make the product

$$\lambda_x\left(\frac{\delta f}{\delta z'_x}\zeta'_x+\frac{\delta f}{\delta z_x}\zeta_x\right)$$

equal to the first derived function or differential coefficient of the product $\lambda_x\frac{\delta f}{\delta z'_x}\zeta_x$, that is, so as to satisfy the following condition

$$\lambda'_x\frac{\delta f}{\delta z'_x}=\lambda_x\left\{\frac{\delta f}{\delta z_x}-\left(\frac{\delta f}{\delta z'_x}\right)'\right\}, \tag{38}$$

analogous to the condition (19). In this manner we find, for the multiplier $\lambda_x$, the following expression, more general than the expression (20),

$$\lambda_x=\lambda_a e^{\int_a^x L_x dx}, \tag{39}$$

in which we have made, for abridgement,

$$L_x=\frac{\dfrac{\delta f}{\delta z_x}-\left(\dfrac{\delta f}{\delta z'_x}\right)'}{\dfrac{\delta f}{\delta z'_x}}; \tag{40}$$

and since we have thus the relation

$$\lambda'_x=L_x\lambda_x, \tag{41}$$

we may put the product

$$\lambda_x\left(\frac{\delta f}{\delta y'_x}\eta'_x+\frac{\delta f}{\delta y_x}\eta_x\right)$$

under the form

$$\left(\lambda_x\frac{\delta f}{\delta y'_x}\eta_x\right)'+\lambda_x\eta_x\left\{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}\right\},$$

and consequently may present the differential equation (37) under the form

$$0=\left(\lambda_x\frac{\delta f}{\delta z'_x}\zeta_x\right)'+\left(\lambda_x\frac{\delta f}{\delta y'_x}\eta_x\right)'+\lambda_x\eta_x\left\{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}\right\}+\epsilon\lambda_x E_x, \tag{42}$$

which is analogous to the equation (23) and may be similarly integrated. We are thus conducted, when we suppose that the conditions (16) are satisfied, so that $\eta_a$, $\eta_{a+\Delta a}$ and $\zeta_a$ vanish, to a development of $\zeta_{a+\Delta a}$ of the form (26), in which the first and principal term $\zeta^{(0)}_{a+\Delta a}$ will not vanish independently of the form of the function $\eta_x$ unless the functions $y_x$, $z_x$ satisfy the following (additional) condition:

$$\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'-L_x\frac{\delta f}{\delta y'_x}=0; \tag{43}$$

that is, by (40),

$$\frac{\dfrac{\delta f}{\delta y_x}-\left(\dfrac{\delta f}{\delta y'_x}\right)'}{\dfrac{\delta f}{\delta y'_x}}=\frac{\dfrac{\delta f}{\delta z_x}-\left(\dfrac{\delta f}{\delta z'_x}\right)'}{\dfrac{\delta f}{\delta z'_x}}. \tag{44}$$

And since this condition (44) is not in general satisfied, either identically by all possible forms of the functions $y_x$, $z_x$ or even by all forms of those functions which satisfy the given differential equation (8), the theorem of the third article is true.

**7.** We see, however, that when the forms of the functions $y_x$, $z_x$ are such as to satisfy this new condition (44), along with the differential equation

$$0 = f(x, y_x, z_x, y'_x, z'_x) \tag{8}$$

and the initial and final conditions (12) and (13), we have then the final condition

$$\zeta^{(0)}_{a+\Delta a} = 0, \tag{45}$$

as a consequence of the differential equation (37) and of the initial and final conditions (16), whatever the form of the function $\eta_x$ may be for intermediate values of $x$; supposing always that no one of the functions here considered becomes infinite or indeterminate within the extent of the question. And because the condition (45), or the differential equation (44) which expresses it, conducts in general (in combination with the other conditions above mentioned) to a final value $z_{a+\Delta a}$ of the function $z_x$, which has its variation $\epsilon\zeta_{a+\Delta a}$ of the form $\epsilon^2\zeta^{(1)}_{a+\Delta a}+$ etc. and therefore ultimately proportional to the square of the small multiplier $\epsilon$, (to the first power of which multiplier the variation $\epsilon\eta_x$ of the function $y_x$ has been supposed proportional); whereas without the condition (45) the variation $\epsilon\zeta_{a+\Delta a}$ would be of the form $\epsilon\zeta^{(0)}_{a+\Delta a}+$ etc. and therefore ultimately proportional to the first power of the same small multiplier $\epsilon$; we see that this condition (45) conducts in general to forms of the functional relations between the variables $x$, $y_x$, $z_x$ which are remarkably distinguished from all other forms of those relations by a peculiar law of variation of the final value of $z_x$. On account of this important property and of others connected with it, we shall call the equation (44), which expresses the condition (45) and which is in general a differential equation of the second order, the *principal supplementary differential equation* for determining the forms of the functions $y_x$, $z_x$ in combination with the given differential equation of the first order

$$0 = f(x, y_x, z_x, y'_x, z'_x), \tag{8}$$

and with the initial and final conditions (12) and (13); and when these equations and conditions conduct to a determinate final value (or class of final values) $z_{a+\Delta a}$ of the function $z_x$, we shall call this value (or these values) $z_{a+\Delta a}$ the *principal final value* (or values) of the function $z_x$, corresponding to the given initial and final values $a$, $y_a$, $z_a$, $a+\Delta a$, $y_{a+\Delta a}$ and to the differential equation (8).

The values which we have thus called *principal final values* of a function $z_x$, which is connected with another function $y_x$ by a differential equation such as (8), are called more commonly the *maxima* or *minima* of the function $z_x$, though they are not always greater or less than all neighbouring values of that function; and it was the desire of discovering these particular or principal values which led to the invention of the Calculus of Variations. But although the rules of that Calculus conduct to what we have called the principal supplementary differential equations, such as (44), they offer no general method for investigating or even expressing the integrals of those differential equations. A general method for accomplishing this important object is supplied, however, by that new Calculus of Principal Relations, to the consideration of which we shall soon proceed.

**8.** For the particular case (4) of this general differential equation (8), that is, for the particular form

$$f = X + Yy'_x + Zz'_x \tag{46}$$

of the general function $f$, ($X$, $Y$ and $Z$ denoting here any given functions of $x$, $y_x$ and $z_x$,) we have the particular expressions

$$\left.\begin{aligned} \frac{\delta f}{\delta y_x} &= \frac{\delta X}{\delta y_x} + y'_x \frac{\delta Y}{\delta y_x} + z'_x \frac{\delta Z}{\delta y_x}, \quad \frac{\delta f}{\delta y'_x} = Y, \\ \frac{\delta f}{\delta z_x} &= \frac{\delta X}{\delta z_x} + y'_x \frac{\delta Y}{\delta z_x} + z'_x \frac{\delta Z}{\delta z_x}, \quad \frac{\delta f}{\delta z'_x} = Z, \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} \left(\frac{\delta f}{\delta y'_x}\right)' &= \frac{\delta Y}{\delta x} + y'_x \frac{\delta Y}{\delta y_x} + z'_x \frac{\delta Y}{\delta z_x}, \\ \left(\frac{\delta f}{\delta z'_x}\right)' &= \frac{\delta Z}{\delta x} + y'_x \frac{\delta Z}{\delta y_x} + z'_x \frac{\delta Z}{\delta z_x}, \end{aligned}\right\} \tag{48}$$

and therefore

$$\left.\begin{aligned} \frac{\delta f}{\delta y_x} - \left(\frac{\delta f}{\delta y'_x}\right)' &= \frac{\delta X}{\delta y_x} - \frac{\delta Y}{\delta x} + z'_x \left(\frac{\delta Z}{\delta y_x} - \frac{\delta Y}{\delta z_x}\right), \\ \frac{\delta f}{\delta z_x} - \left(\frac{\delta f}{\delta z'_x}\right)' &= \frac{\delta X}{\delta z_x} - \frac{\delta Z}{\delta x} + y'_x \left(\frac{\delta Y}{\delta z_x} - \frac{\delta Z}{\delta y_x}\right); \end{aligned}\right\} \tag{49}$$

so that when we attend to the original differential equation (8), which here reduces itself to the particular form (4), we find, after reductions, that in the present case the general supplementary equation (44) transforms itself to the particular condition

$$0 = X\left(\frac{\delta Y}{\delta z_x} - \frac{\delta Z}{\delta y_x}\right) + Y\left(\frac{\delta Z}{\delta x} - \frac{\delta X}{\delta z_x}\right) + Z\left(\frac{\delta X}{\delta y_x} - \frac{\delta Y}{\delta x}\right). \tag{5}$$

In like manner, if we assume the particular form

$$f = p + qy'_x - z'_x, \tag{50}$$

so as to reduce the general differential equation (8) to the particular differential equation (11), we find

$$\left.\begin{aligned} \frac{\dfrac{\delta f}{\delta y_x} - \left(\dfrac{\delta f}{\delta y'_x}\right)'}{\dfrac{\delta f}{\delta y'_x}} &= \frac{1}{q}\left(\frac{\delta p}{\delta y_x} - \frac{\delta q}{\delta x} - p\frac{\delta q}{\delta z_x} - qy'_x \frac{\delta q}{\delta z_x}\right), \\ \frac{\dfrac{\delta f}{\delta z_x} - \left(\dfrac{\delta f}{\delta z'_x}\right)'}{\dfrac{\delta f}{\delta z'_x}} &= -\left(\frac{\delta p}{\delta z_x} + y'_x \frac{\delta q}{\delta z_x}\right); \end{aligned}\right\} \tag{51}$$

so that in this particular case the general supplementary equation (44) reduces itself to the particular condition

$$\frac{\delta p}{\delta y_x} + q\frac{\delta p}{\delta z_x} = \frac{\delta q}{\delta x} + p\frac{\delta q}{\delta z_x}. \tag{10}$$

Thus for the two particular forms (46) and (50) of the function $f$ (of which two forms, indeed, the one includes the other) the condition (44) either cannot be satisfied at all (as in the example of the fifth article where $f$ was equal or proportional to $-y_x + xy'_x - z'_x$); is identically satisfied (as in the case considered in the second article); or else can only be satisfied by establishing a particular relation between the variable $x$ and the functions $y_x$, $z_x$ which does not involve the differential coefficients of those functions, and therefore does not permit us to assume as we have hitherto done the arbitrary initial conditions $y_a = b$, $z_a = c$. Peculiar considerations would also be required for the case when one or other of the differential coefficients $y'_x$, $z'_x$ disappears from the expression of $f$. But, in general, the equation (44) may evidently be put under the form

$$0 = P + Qy''_x + Rz''_x, \tag{52}$$

in which $P$, $Q$ and $R$ are known functions of $x$, $y_x$, $z_x$, $y'_x$ and $z'_x$, namely,

$$\left.\begin{aligned} P = {} & \frac{\delta f}{\delta z'_x}\left(\frac{\delta f}{\delta y_x} - \frac{\delta^2 f}{\delta x\, \delta y'_x} - y'_x \frac{\delta^2 f}{\delta y_x \delta y'_x} - z'_x \frac{\delta^2 f}{\delta z_x \delta y'_x}\right) \\ & - \frac{\delta f}{\delta y'_x}\left(\frac{\delta f}{\delta z_x} - \frac{\delta^2 f}{\delta x\, \delta z'_x} - y'_x \frac{\delta^2 f}{\delta y_x \delta z'_x} - z'_x \frac{\delta^2 f}{\delta z_x \delta z'_x}\right), \\ Q = {} & -\frac{\delta f}{\delta z'_x}\frac{\delta^2 f}{\delta y'^2_x} + \frac{\delta f}{\delta y'_x}\frac{\delta^2 f}{\delta y'_x \delta z'_x}, \\ R = {} & -\frac{\delta f}{\delta z'_x}\frac{\delta^2 f}{\delta y'_x \delta z'_x} + \frac{\delta f}{\delta y'_x}\frac{\delta^2 f}{\delta z'^2_x}, \end{aligned}\right\} \tag{53}$$

or functions proportional to these; and since the functions $Q$ and $R$ do not in general both vanish, this supplementary equation (44) or (52) is evidently in general a differential equation of the second order, as was remarked before.

**9.** Another differential equation of the same order, which is in general distinct from this *principal supplementary differential equation* (44) or (52), may be obtained by differentiating the *original differential equation* of the first order (8) and may be thus denoted,

$$0 = f' = P_{,} + Q_{,} y''_x + R_{,} z''_x; \tag{54}$$

in which the coefficients $P_{,}$, $Q_{,}$, $R_{,}$ are the following functions of $x$, $y_x$, $z_x$, $y'_x$, $z'_x$,

$$P_{,} = \frac{\delta f}{\delta x} + y'_x \frac{\delta f}{\delta y_x} + z'_x \frac{\delta f}{\delta z_x}, \quad Q_{,} = \frac{\delta f}{\delta y'_x}, \quad R_{,} = \frac{\delta f}{\delta z'_x}, \tag{55}$$

or functions proportional to these. If we had only the system of these two differential equations of the second order, (52) and (54), we could only in general deduce from them, by an easy or difficult but theoretically possible integration, two relations which might be thus denoted,

$$\left.\begin{aligned} 0 &= \chi(a, y_a, z_a, y'_a, z'_a, x, y_x, z_x), \\ 0 &= \chi_{,}(a, y_a, z_a, y'_a, z'_a, x, y_x, z_x), \end{aligned}\right\} \tag{56}$$

between the three connected variables $x$, $y_x$, $z_x$, their three initial values $a$, $y_a$, $z_a$, and the two initial differential coefficients $y'_a$, $z'_a$; or relations equivalent to these. But since the original differential equation of the first order (8) gives this other initial condition

$$0 = f_a = f(a, y_a, z_a, y'_a, z'_a), \tag{57}$$

we can in general conceive the two initial differential coefficients $y'_a$ and $z'_a$ to be eliminated between these three last equations and thus a relation obtained of the form

$$0 = F(a, y_a, z_a, x, y_x, z_x), \tag{58}$$

or

$$0 = F(a, y_a, z_a, a + \Delta a, y_{a+\Delta a}, z_{a+\Delta a}), \tag{59}$$

or, if we choose, of the form (7), or finally of this simpler form

$$0 = F(a, b, c, a + \Delta a, b + \Delta b, c + \Delta c) \tag{60}$$

between the three initial and three final values of the variables $x$, $y_x$, $z_x$, these variables being obliged to satisfy both the *original differential equation* (8) and also the *principal supplementary differential equation*, (44) or (52). And because the relation (58) or (59) or (60) thus obtained gives in general the *principal final value* $z_{a+\Delta a}$ or $c+\Delta c$ of the function $z_x$ when the five other initial and final values $a, y_a, z_a, a+\Delta a, y_{a+\Delta a}$, or $a, b, c, a+\Delta a, b+\Delta b$, are given, we shall call this relation the *principal integral* of the original differential equation (8), or the *principal integral relation* between the final and initial values of the three connected variables $x, y_x, z_x$ resulting from that differential equation.

**10**. To give an example of the application of this method of investigating a principal integral relation, let us take the following particular form of the function $f$,

$$f = gy_x + \tfrac{1}{2}y_x'^2 - z_x', \tag{61}$$

$g$ being any given constant; so that the proposed differential equation, of which we are to seek the principal integral, may now be put under the form

$$z_x' = gy_x + \tfrac{1}{2}y_x'^2. \tag{62}$$

We have now the following particular expressions for the partial differential coefficients of the function $f$,

$$\frac{\delta f}{\delta x} = 0, \quad \frac{\delta f}{\delta y_x} = g, \quad \frac{\delta f}{\delta z_x} = 0, \quad \frac{\delta f}{\delta y_x'} = y_x', \quad \frac{\delta f}{\delta z_x'} = -1; \tag{63}$$

so that, by (55), the general coefficients $P_{,}$, $Q_{,}$, $R_{,}$ of the differential equation (54) become here

$$P_{,} = gy_x', \quad Q_{,} = y_x', \quad R_{,} = -1: \tag{64}$$

we have also for the partial differential coefficients of the second order of the same function $f$ the expressions

$$\left.\begin{array}{l}
\dfrac{\delta^2 f}{\delta x^2} = 0, \quad \dfrac{\delta^2 f}{\delta x\,\delta y_x} = 0, \quad \dfrac{\delta^2 f}{\delta x\,\delta z_x} = 0, \quad \dfrac{\delta^2 f}{\delta x\,\delta y_x'} = 0, \quad \dfrac{\delta^2 f}{\delta x\,\delta z_x'} = 0, \\
\dfrac{\delta^2 f}{\delta y_x^2} = 0, \quad \dfrac{\delta^2 f}{\delta y_x\,\delta z_x} = 0, \quad \dfrac{\delta^2 f}{\delta y_x\,\delta y_x'} = 0, \quad \dfrac{\delta^2 f}{\delta y_x\,\delta z_x'} = 0, \\
\dfrac{\delta^2 f}{\delta z_x^2} = 0, \quad \dfrac{\delta^2 f}{\delta z_x\,\delta y_x'} = 0, \quad \dfrac{\delta^2 f}{\delta z_x\,\delta z_x'} = 0, \\
\dfrac{\delta^2 f}{\delta y_x'^2} = 1, \quad \dfrac{\delta^2 f}{\delta y_x'\,\delta z_x'} = 0, \\
\dfrac{\delta^2 f}{\delta z_x'^2} = 0;
\end{array}\right\} \tag{65}$$

and therefore, by (53), the general coefficients $P$, $Q$, $R$ of the differential equation (52) become, in the present example,

$$P = -g, \quad Q = 1, \quad R = 0. \tag{66}$$

The general differential equations of the second order

$$0 = P + Qy_x'' + Rz_x'' \tag{52}$$

and

$$0 = P_{,} + Q_{,}y_x'' + R_{,}z_x'' \tag{54}$$

become therefore now

$$0 = -g + y_x'' \tag{67}$$

and

$$z_x'' = gy_x' + y_x'y_x''; \tag{68}$$

the former of these two, namely the equation (67), being the principal supplementary differential equation and the latter, namely (68), being obtained by differentiation from the original differential equation (62). If we had only the system of these two differential equations of the second order we could only deduce from them by integration the system of the two following equations of the forms (56),

$$0=y_x-y_a-y'_a(x-a)-\tfrac{1}{2}g(x-a)^2 \tag{69}$$

and

$$0=z_x-z_a-z'_a(x-a)-gy'_a(x-a)^2-\tfrac{1}{3}g^2(x-a)^3; \tag{70}$$

or relations equivalent to these. But the original differential equation (62) gives also this initial condition of the form (57),

$$0=f_a=gy_a+\tfrac{1}{2}y'^2_a-z'_a; \tag{71}$$

and, by eliminating $y'_a$ and $z'_a$ between the three last equations (69), (70) and (71), we find this other equation of the form (58),

$$0=F=z_x-z_a-\tfrac{1}{2}g(y_x+y_a)(x-a)-\frac{(y_x-y_a)^2}{2(x-a)}+\tfrac{1}{24}g^2(x-a)^3, \tag{72}$$

which is the *principal integral* sought of the proposed differential equation (62). This integral may also be thus written, under the form (7), as an equation in finite differences,

$$0=\Delta z_a-g(y_a+\tfrac{1}{2}\Delta y_a)\Delta a-\frac{\Delta y_a^2}{2\Delta a}+\tfrac{1}{24}g^2\Delta a^3; \tag{73}$$

or more concisely thus, under the form (60),

$$0=\Delta c-g(b+\tfrac{1}{2}\Delta b)\Delta a-\frac{\Delta b^2}{2\Delta a}+\tfrac{1}{24}g^2\Delta a^3. \tag{74}$$

**11.** In a very extensive class of cases the proposed differential equation, though it contains the differential coefficient $z'_x$, does not contain the undifferentiated function $z_x$; and then, as in the last example, it may be put under the form

$$z'_x=\phi(x,\, y_x,\, y'_x), \tag{75}$$

so that the function $f$ may be regarded as being of the form

$$f=\phi(x,\, y_x,\, y'_x)-z'_x. \tag{76}$$

In all cases of this class the process of the 8th article for determining the principal integral admits of being simplified; for the principal supplementary differential equation (44) or (52) becomes simply

$$\frac{\delta\phi}{\delta y_x}=\left(\frac{\delta\phi}{\delta y'_x}\right)', \tag{77}$$

or

$$0=-\frac{\delta\phi}{\delta y_x}+\frac{\delta^2\phi}{\delta x\,\delta y'_x}+\frac{\delta^2\phi}{\delta y_x\,\delta y'_x}y'_x+\frac{\delta^2\phi}{\delta y'^2_x}y''_x, \tag{78}$$

and does not involve $z_x$, $z'_x$ nor $z''_x$, so that its integral, like the equation (69), is of the following form, more simple than the form of either of the two general equations (56),

$$0=\chi(a,\, y_a,\, y'_a,\, x,\, y_x). \tag{79}$$

And because this integral (79) of the principal supplementary differential equation (77) or (78) enables us to express the function $y_x$, and therefore also $y'_x$, in terms of the constants $a$, $y_a$, $y'_a$, and of the variable $x$, it enables us to express the difference $z_x - z_a$ in terms of the same constants and variable, by means of the original differential equation (75), as the value of a definite integral, namely,

$$z_x - z_a = \int_a^x \phi(x, y_x, y'_x)\, dx: \tag{80}$$

so that instead of eliminating the *two* initial differential coefficients $y'_a$ and $z'_a$ between the three equations assigned for that purpose in the 8th article, it is sufficient to eliminate the *one* initial differential coefficient $y'_a$ between the two equations (79) and (80) in order to obtain the principal integral relation connecting the initial and final values of $x$, $y_x$ and $z_x$; which relation will evidently be of the following form,

$$0 = F = z_x - z_a - \Phi(a, y_a, x, y_x) = \Delta c - \Phi(a, b, a + \Delta a, b + \Delta b). \tag{81}$$

Thus, in the last example, in which the function $\phi(x, y_x, y'_x) = gy_x + \frac{1}{2}y_x'^2$, so that

$$\frac{\delta\phi}{\delta y_x} = g, \quad \frac{\delta\phi}{\delta y'_x} = y'_x, \quad \frac{\delta^2\phi}{\delta x \delta y'_x} = 0, \quad \frac{\delta^2\phi}{\delta y_x \delta y'_x} = 0, \quad \frac{\delta^2\phi}{\delta y_x'^2} = 1, \tag{82}$$

the form (78) for the principal supplementary differential relation becomes

$$0 = -g + y''_x, \tag{67}$$

as was otherwise found before; and its integral is of the form (79), namely,

$$0 = y_x - y_a - y'_a(x - a) - \tfrac{1}{2}g(x - a)^2, \tag{69}$$

which gives for the functions $y_x$, $y'_x$ the expressions

$$\left.\begin{aligned} y_x &= y_a + y'_a(x - a) + \tfrac{1}{2}g(x - a)^2, \\ y'_x &= y'_a + g(x - a). \end{aligned}\right\} \tag{83}$$

Substituting these expressions in that of the function $\phi$, we find

$$\phi(x, y_x, y'_x) = gy_x + \tfrac{1}{2}y_x'^2 = gy_a + \tfrac{1}{2}y_a'^2 + 2gy'_a(x - a) + g^2(x - a)^2, \tag{84}$$

and therefore

$$\begin{aligned} z_x - z_a &= \int_a^x \{gy_a + \tfrac{1}{2}y_a'^2 + 2gy'_a(x - a) + g^2(x - a)^2\}\, dx \\ &= (gy_a + \tfrac{1}{2}y_a'^2)(x - a) + gy'_a(x - a)^2 + \tfrac{1}{3}g^2(x - a)^3; \end{aligned} \tag{85}$$

so that finally, by eliminating $y'_a$ between the two equations (69) and (85), we obtain the same principal integral relation

$$0 = z_x - z_a - \tfrac{1}{2}g(y_x + y_a)(x - a) - \frac{(y_x - y_a)^2}{2(x - a)} + \tfrac{1}{24}g^2(x - a)^3, \tag{72}$$

which was found before by a less simple process.

12. The differential equation

$$z'_x = \phi(x, y_x, y'_x) \tag{75}$$

conducts in general to the expression

$$z_x - z_a = \int_a^x \phi(x, y_x, y'_x)\, dx, \tag{80}$$

whether we do or do not employ the supplementary differential equation (77) or (78); but among all the values of the definite integral (80), corresponding to all possible forms of the function $y_x$ and to the initial and final conditions (12), the particular value (or values) determined by this principal supplementary differential equation may be called the *principal value* (or values) of that integral for reasons already explained. We shall therefore call the value (or values) of the definite integral (80), determined in this manner, the *principal definite integral* (or integrals) of the differential expression $\phi(x, y_x, y'_x)\,dx$, taken between the limits $a$ and $x$, or $a$ and $a+\Delta a$; and we shall distinguish in writing a *principal definite integral* of this kind by drawing a stroke under the sign of integration; so that the principal integral relation (81), deduced from a proposed differential equation of the form (75), may be denoted as follows:

$$z_x - z_a = \underline{\int_a^x} \phi(x, y_x, y'_x)\,dx = \underline{\int_a^{a+\Delta a}} \phi(x, y_x, y'_x)\,dx. \tag{86}$$

For example, by what has been already shown, the *principal definite integral* of the differential expression $(gy_x + \frac{1}{2}y'^2_x)\,dx$, taken between the limits $a$ and $x$, is

$$\underline{\int_a^x} (gy_x + \tfrac{1}{2}y'^2_x)\,dx = \tfrac{1}{2}g(y_x + y_a)(x-a) + \frac{(y_x - y_a)^2}{2(x-a)} - \frac{g^2}{24}(x-a)^3. \tag{87}$$

**13.** Besides the principal supplementary differential equation, the Calculus of Variations conducts also to another important relation called usually the Equation of Limits, which may be explained and investigated as follows.

Returning to the general differential equation of the first order

$$0 = f(x, y_x, z_x, y'_x, z'_x), \tag{8}$$

let us now imagine that after finding two functions $y_x, z_x$ which satisfy this original equation (8) and also the principal supplementary differential equation (44) or (52) and the initial and final conditions (12) and (13), and after calculating thus the principal final value $z_{a+\Delta a}$ of $z_x$ which corresponds to a given final value $y_{a+\Delta a}$ of $y_x$ and to given initial values $y_a$, $z_a$ of the same functions $y_x, z_x$, we then change these two functions as before to others of the form $y_x + \epsilon\eta_x$ and $z_x + \epsilon\zeta_x$; but that at the same time we change also the variable $x$ itself to a function of the form $x + \epsilon\xi_x$, and therefore the differential coefficients $y'_x$, $z'_x$ to the quotients

$$\frac{y'_x + \epsilon\eta'_x}{1 + \epsilon\xi'_x}, \quad \frac{z'_x + \epsilon\zeta'_x}{1 + \epsilon\xi'_x};$$

and that, by making all these changes in the original differential equation (8), we form a new but analogous differential equation and then oblige the functions $\xi_x$, $\eta_x$, $\zeta_x$ to satisfy this new equation, namely,

$$0 = f\left(x + \epsilon\xi_x, y_x + \epsilon\eta_x, z_x + \epsilon\zeta_x, \frac{y'_x + \epsilon\eta'_x}{1 + \epsilon\xi'_x}, \frac{z'_x + \epsilon\zeta'_x}{1 + \epsilon\xi'_x}\right). \tag{88}$$

We shall then have, instead of (37), the following equation deduced from the combination of (8) and (88):

$$0 = \frac{\delta f}{\delta x}\xi_x + \frac{\delta f}{\delta y_x}\eta_x + \frac{\delta f}{\delta z_x}\zeta_x + \frac{\delta f}{\delta y'_x}(\eta'_x - y'_x\xi'_x) + \frac{\delta f}{\delta z'_x}(\zeta'_x - z'_x\xi'_x) + \epsilon E_x, \tag{89}$$

$E_x$ being a factor which we need not at present determine; and since by the differential equation (54) we have

$$\frac{\delta f}{\delta x} = -y'_x\frac{\delta f}{\delta y_x} - z'_x\frac{\delta f}{\delta z_x} - y''_x\frac{\delta f}{\delta y'_x} - z''_x\frac{\delta f}{\delta z'_x}, \tag{90}$$

it will follow that

$$0=\frac{\delta f}{\delta z'_x}(\zeta'_x-z'_x\xi'_x-z''_x\xi_x)+\frac{\delta f}{\delta y'_x}(\eta'_x-y'_x\xi'_x-y''_x\xi_x)$$

$$+\frac{\delta f}{\delta z_x}(\zeta_x-z'_x\xi_x)+\frac{\delta f}{\delta y_x}(\eta_x-y'_x\xi_x)+\epsilon E_x. \tag{91}$$

Multiplying this differential equation (91) by the factor $\lambda_x$ of the 6th article, which satisfies the two conditions

$$\lambda_x\frac{\delta f}{\delta z_x}=\left(\lambda_x\frac{\delta f}{\delta z'_x}\right)', \quad \lambda_x\frac{\delta f}{\delta y_x}=\left(\lambda_x\frac{\delta f}{\delta y'_x}\right)', \tag{92}$$

we find this transformed differential equation

$$0=\left(\lambda_x\left\{\frac{\delta f}{\delta z'_x}(\zeta_x-z'_x\xi_x)+\frac{\delta f}{\delta y'_x}(\eta_x-y'_x\xi_x)\right\}\right)'+\epsilon\lambda_x E_x, \tag{93}$$

which gives, by integration,

$$0=\Delta\left(\lambda_a\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}\right)+\epsilon\int_a^{a+\Delta a}\lambda_x E_x dx, \tag{94}$$

$f_a$ denoting as in (57) the initial function $f(a, y_a, z_a, y'_a, z'_a)$ and $\Delta$ still implying that we are to take the finite difference corresponding to the transition from the initial to the final values. If then, without obliging $\xi_a$, $\eta_a$, $\zeta_a$, $\xi_{a+\Delta a}$ and $\eta_{a+\Delta a}$ to vanish, we suppose (as in general we may) that $\zeta_{a+\Delta a}$ is developed in a series of the form (26) according to positive and integer powers of the small constant multiplier $\epsilon$, we find for the first and principal term $\zeta^{(0)}_{a+\Delta a}$ of this series the equation

$$\frac{\delta f_{a+\Delta a}}{\delta z'_{a+\Delta a}}(\zeta^{(0)}_{a+\Delta}-z'_{a+\Delta a}\xi_{a+\Delta a})+\frac{\delta f_{a+\Delta a}}{\delta y'_{a+\Delta a}}(\eta_{a+\Delta a}-y'_{a+\Delta a}\xi_{a+\Delta a})$$

$$=\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}e^{-\int_a^{a+\Delta a}L_x dx}; \tag{95}$$

in which $L_x$ has the same meaning as in the 6th article, namely,

$$L_x=\frac{\lambda'_x}{\lambda_x}=\frac{\frac{\delta f}{\delta z_x}-\left(\frac{\delta f}{\delta z'_x}\right)'}{\frac{\delta f}{\delta z'_x}}=\frac{\frac{\delta f}{\delta y_x}-\left(\frac{\delta f}{\delta y'_x}\right)'}{\frac{\delta f}{\delta y'_x}}. \tag{96}$$

This equation (95) may be simplified when the function $f$ has the form

$$f=\phi(x, y_x, y'_x)-z'_x; \tag{76}$$

for it then becomes

$$-\zeta^{(0)}_{a+\Delta a}+\phi_{a+\Delta a}\xi_{a+\Delta a}+\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}(\eta_{a+\Delta a}-y'_{a+\Delta a}\xi_{a+\Delta a})$$

$$=-\zeta_a+\phi_a\xi_a+\frac{\delta\phi_a}{\delta y'_a}(\eta_a-y'_a\xi_a), \tag{97}$$

that is,
$$\zeta^{(0)}_{a+\Delta a}-\zeta_a=\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}\eta_{a+\Delta a}-\frac{\delta\phi_a}{\delta y'_a}\eta_a$$
$$+\left(\phi_{a+\Delta a}-y'_{a+\Delta a}\frac{\delta\phi_{a+\Delta a}}{\delta y'_{a+\Delta a}}\right)\xi_{a+\Delta a}-\left(\phi_a-y'_a\frac{\delta\phi_a}{\delta y'_a}\right)\xi_a. \tag{98}$$

For example, when the function $\phi$ has the form (62), so that
$$\phi=gy_x+\tfrac{1}{2}y'^2_x,\quad \phi_a=gy_a+\tfrac{1}{2}y'^2_a,\quad \phi_{a+\Delta a}=gy_{a+\Delta a}+\tfrac{1}{2}y'^2_{a+\Delta a}, \tag{99}$$
then the formula (98) becomes
$$\zeta^{(0)}_{a+\Delta a}-\zeta_a=y_{a+\Delta a}\eta_{a+\Delta a}-y'_a\eta_a+(gy_{a+\Delta a}-\tfrac{1}{2}y'^2_{a+\Delta a})\xi_{a+\Delta a}-(gy_a-\tfrac{1}{2}y'^2_a)\xi_a. \tag{100}$$
In general, if we write $x$ instead of $a+\Delta a$, the formula (95) becomes
$$\frac{\delta f_x}{\delta z'_x}(\zeta^{(0)}_x-z'_x\xi_x)+\frac{\delta f_x}{\delta y'_x}(\eta_x-y'_x\xi_x)=\left\{\frac{\delta f_a}{\delta z'_a}(\zeta_a-z'_a\xi_a)+\frac{\delta f_a}{\delta y'_a}(\eta_a-y'_a\xi_a)\right\}e^{-\int_a^x L_x dx}; \tag{101}$$
and the formula (98) becomes
$$\zeta^{(0)}_x-\zeta_a=\frac{\delta\phi_x}{\delta y'_x}\eta_x-\frac{\delta\phi_a}{\delta y'_a}\eta_a+\left(\phi_x-y'_x\frac{\delta\phi_x}{\delta y'_x}\right)\xi_x-\left(\phi_a-y'_a\frac{\delta\phi_a}{\delta y'_a}\right)\xi_a. \tag{102}$$

Thus, although the differential equation (88) is not sufficient of itself to determine the forms of the three functions $\xi_x$, $\eta_x$, $\zeta_x$, nor even to determine rigorously, in general, the final value $\zeta_x$ of one of these three functions when only the final values $\xi_x$, $\eta_x$ of the other two functions and the initial values $\xi_a$, $\eta_a$, $\zeta_a$ of all three are given; (because the term $\epsilon\lambda_x E_x$ in the transformed differential equation (93) is not in general immediately integrable;) yet, when the functions $y_x$ and $z_x$ satisfy the principal supplementary equation (44) as well as the original equation (8), we see that then the differential equation (88) is in general sufficient to determine the most important part or term $\zeta^{(0)}_x$ of the final value of the function $\zeta_x$, or the limit to which that final value tends while the small multiplier $\epsilon$ tends to 0, whatever arbitrary forms may be assumed for the three functions $\xi_x$, $\eta_x$, $\zeta_x$ consistently with that one differential equation (88) and with the three given initial and the two given final values, $\xi_a$, $\eta_a$, $\zeta_a$, and $\xi_x$, $\eta_x$.

**14.** Now, consistently with these five given values, initial and final, and with the differential equation of the first order (88), we may in general oblige the three functions $\xi_x$, $\eta_x$, $\zeta_x$ to satisfy other conditions; for example, we may in general oblige them to satisfy any two assumed supplementary differential equations of the second order; because a system of three differential equations between four variables $x$, $\xi_x$, $\eta_x$, $\zeta_x$ conducts in general to a system of three integral equations between those four variables involving five arbitrary constants, when one of the three differential equations is of the first order and the two other differential equations are both of the second order. Among all the supplementary differential equations which might be thus assumed, there is one which deserves special attention; namely that which is formed from the principal supplementary differential equation of the second order (44) by not only changing $x$, $y_x$, $z_x$ and $y'_x$, $z'_x$ to $x+\epsilon\xi_x$, $y_x+\epsilon\eta_x$, $z_x+\epsilon\zeta_x$ and $\dfrac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}$, $\dfrac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}$, but also changing, in like manner, $y''_x$ and $z''_x$ to
$$\frac{\left(\dfrac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x}\quad\text{and}\quad\frac{\left(\dfrac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x};$$

and which may therefore be thus written,

$$0=\psi\left(x+\epsilon\xi_x, y_x+\epsilon\eta_x, z_x+\epsilon\zeta_x, \frac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}, \frac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}, \frac{\left(\dfrac{y'_x+\epsilon\eta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x}, \frac{\left(\dfrac{z'_x+\epsilon\zeta'_x}{1+\epsilon\xi'_x}\right)'}{1+\epsilon\xi'_x}\right), \tag{103}$$

if the equation (44), from which it is formed, be written thus,

$$0=\psi(x, y_x, z_x, y'_x, z'_x, y''_x, z''_x). \tag{104}$$

For, although the two equations (88) and (103) are not alone sufficient to determine completely the forms of the three functions $\xi_x$, $\eta_x$, $\zeta_x$ even in conjunction with the five given initial and final values $\xi_a$, $\eta_a$, $\zeta_a$, $\xi_x$ and $\eta_x$, but require another supplementary differential equation of the second order, which might, for example, be assumed as follows,

$$\xi''_x=0, \tag{105}$$

for such complete determination; yet it is easy to see that, as the two equations (8) and (44) conducted to a principal integral relation of the form

$$0=F(a, y_a, z_a, x, y_x, z_x), \tag{58}$$

so the two similar equations (88) and (103) must conduct to a similar integral relation

$$0=F(a+\epsilon\xi_a, y_a+\epsilon\eta_a, z_a+\epsilon\zeta_a, x+\epsilon\xi_x, y_x+\epsilon\eta_x, z_x+\epsilon\zeta_x), \tag{106}$$

in which the form of the function $F$ is the same. Combining these two relations (58) and (106), we find a new equation to determine the limit $\zeta_x^{(0)}$ to which the final value of the function $\zeta_x$ tends, while the small multiplier $\epsilon$ tends to 0 and while the final values of the two other functions $\xi_x, \eta_x$ and the three initial values $\xi_a, \eta_a, \zeta_a$ of the same three functions $\xi_x, \eta_x, \zeta_x$ remain unchanged but arbitrary; namely, the equation

$$0=\frac{\delta F}{\delta a}\xi_a+\frac{\delta F}{\delta y_a}\eta_a+\frac{\delta F}{\delta z_a}\zeta_a+\frac{\delta F}{\delta x}\xi_x+\frac{\delta F}{\delta y_x}\eta_x+\frac{\delta F}{\delta z_x}\zeta_x^{(0)}. \tag{107}$$

The value of the limit $\zeta_x^{(0)}$ thus obtained from a combination of the two differential equations (88) and (103) must agree with the value of the same limit obtained in the 13th article from the equation (88) alone; and this agreement must exist independently of the five arbitrary values $\xi_a$, $\eta_a$, $\zeta_a$, $\xi_x$, $\eta_x$; comparing, therefore, the coefficients which multiply these five arbitrary values in the two expressions of $\zeta_x^{(0)}$, deduced from the two equations (107) and (101), we find these three relations:

$$\left.\begin{aligned}
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta a}&=\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\left\{y'_a\frac{\delta f_a}{\delta y'_a}+z'_a\frac{\delta f_a}{\delta z'_a}\right\}e^{-\int_a^x L_x dx},\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta y_a}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_a}{\delta y'_a}e^{-\int_a^x L_x dx},\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta z_a}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_a}{\delta z'_a}e^{-\int_a^x L_x dx},
\end{aligned}\right\} \tag{108}$$

and these two others:

$$\left.\begin{aligned}
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta x}&=-\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\left\{y'_x\frac{\delta f_x}{\delta y'_x}+z'_x\frac{\delta f_x}{\delta z'_x}\right\},\\
\left(\frac{\delta F}{\delta z_x}\right)^{-1}\frac{\delta F}{\delta y_x}&=\left(\frac{\delta f_x}{\delta z'_x}\right)^{-1}\frac{\delta f_x}{\delta y'_x}.
\end{aligned}\right\} \tag{109}$$

In that extensive class of cases, in which the function $f$ has the form (76) and in which therefore the function $F$ may be put under the form (81), the five relations (108) and (109) may be reduced to the four following:

$$\frac{\delta\Phi}{\delta a}=y'_a\frac{\delta\phi_a}{\delta y'_a}-\phi_a,\quad \frac{\delta\Phi}{\delta y_a}=-\frac{\delta\phi_a}{\delta y'_a}, \tag{110}$$

and

$$\frac{\delta\Phi}{\delta x}=-y'_x\frac{\delta\phi_x}{\delta y'_x}+\phi_x,\quad \frac{\delta\Phi}{\delta y_x}=\frac{\delta\phi_x}{\delta y'_x}. \tag{111}$$

In general, the three relations (108) conduct, by elimination, to the two following:

$$\left.\begin{aligned}\left(\frac{\delta F}{\delta z_a}\right)^{-1}\frac{\delta F}{\delta a}&=-\left(\frac{\delta f_a}{\delta z'_a}\right)^{-1}\left\{y'_a\frac{\delta f_a}{\delta y'_a}+z'_a\frac{\delta f_a}{\delta z'_a}\right\},\\ \left(\frac{\delta F}{\delta z_a}\right)^{-1}\frac{\delta F}{\delta y_a}&=\left(\frac{\delta f_a}{\delta z'_a}\right)^{-1}\frac{\delta f_a}{\delta y'_a}.\end{aligned}\right\} \tag{112}$$

**15.** To illustrate these general relations, let us resume the example of the 10th article, in which

$$0=f_x=gy_x+\tfrac{1}{2}y_x'^2-z'_x, \tag{61}$$

$$0=f_a=gy_a+\tfrac{1}{2}y_a'^2-z'_a, \tag{71}$$

and

$$0=F=z_x-z_a-\tfrac{1}{2}g(y_x+y_a)(x-a)-\frac{(y_x-y_a)^2}{2(x-a)}+\tfrac{1}{24}g^2(x-a)^3. \tag{72}$$

In this example

$$\left.\begin{aligned}&\frac{\delta f_x}{\delta y'_x}=y'_x,\quad \frac{\delta f_x}{\delta z'_x}=-1,\quad \frac{\delta f_a}{\delta y'_a}=y'_a,\quad \frac{\delta f_a}{\delta z'_a}=-1,\\ &\frac{\delta F}{\delta x}=-\tfrac{1}{2}g(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2(x-a)^2,\\ &\frac{\delta F}{\delta y_x}=-\tfrac{1}{2}g(x-a)-\frac{y_x-y_a}{x-a},\quad \frac{\delta F}{\delta z_x}=1,\\ &\frac{\delta F}{\delta a}=\tfrac{1}{2}g(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2(x-a)^2,\\ &\frac{\delta F}{\delta y_a}=-\tfrac{1}{2}g(x-a)+\frac{y_x-y_a}{x-a},\quad \frac{\delta F}{\delta z_a}=-1;\end{aligned}\right\} \tag{113}$$

the general relations (112) become, therefore,

$$\left.\begin{aligned}&-\tfrac{1}{2}g(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2(x-a)^2=y_a'^2-z'_a,\\ &\tfrac{1}{2}g(x-a)-\frac{y_x-y_a}{x-a}=-y'_a;\end{aligned}\right\} \tag{114}$$

to which forms the two first of the three relations (108) also reduce themselves, while the third of those relations (108) becomes identical, because, in this example, the function $L_x$ vanishes by (40); and the two general relations (109) become

$$\left.\begin{aligned} -\tfrac{1}{2}g(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2(x-a)^2=y_x'^2-z_x', \\ -\tfrac{1}{2}g(x-a)-\frac{y_x-y_a}{x-a}=-y_x'. \end{aligned}\right\} \tag{115}$$

And accordingly we may verify the existence of these four relations (114) and (115) in the present example by substituting for $y_x$, $y_x'$ their expressions (83), deduced by integration from the principal supplementary differential equation of the second order (67), and at the same time substituting for $z_x'$, $z_a'$ their values deduced from the original differential equation of the first order (62) and its initial form (71).

The same example enables us to illustrate the four simpler but less general relations (110) and (111); namely, by making, according to the 11th article,

$$\left.\begin{aligned} \phi_x=gy_x+\tfrac{1}{2}y_x'^2, \quad \phi_a=gy_a+\tfrac{1}{2}y_a'^2, \\ \Phi=\tfrac{1}{2}g(y_x+y_a)(x-a)+\frac{(y_x-y_a)^2}{2(x-a)}-\tfrac{1}{24}g^2(x-a)^3; \end{aligned}\right\} \tag{116}$$

and therefore

$$\left.\begin{aligned} \frac{\delta\phi_x}{\delta y_x'}=y_x', \quad \frac{\delta\phi_a}{\delta y_a'}=y_a', \\ \frac{\delta\Phi}{\delta x}=\tfrac{1}{2}g(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2(x-a)^2, \\ \frac{\delta\Phi}{\delta y_x}=\tfrac{1}{2}g(x-a)+\frac{y_x-y_a}{x-a}, \\ \frac{\delta\Phi}{\delta a}=-\tfrac{1}{2}g(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2(x-a)^2, \\ \frac{\delta\Phi}{\delta y_a}=\tfrac{1}{2}g(x-a)-\frac{y_x-y_a}{x-a}; \end{aligned}\right\} \tag{117}$$

for the two relations (110) thus become

$$\left.\begin{aligned} -\tfrac{1}{2}g(y_x+y_a)+\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2+\tfrac{1}{8}g^2(x-a)^2=\tfrac{1}{2}y_a'^2-gy_a, \\ \tfrac{1}{2}g(x-a)-\frac{y_x-y_a}{x-a}=-y_a'; \end{aligned}\right\} \tag{118}$$

and the two relations (111) become

$$\left.\begin{aligned} \tfrac{1}{2}g(y_x+y_a)-\frac{1}{2}\left(\frac{y_x-y_a}{x-a}\right)^2-\tfrac{1}{8}g^2(x-a)^2=-\tfrac{1}{2}y_x'^2+gy_x, \\ \tfrac{1}{2}g(x-a)+\frac{y_x-y_a}{x-a}=y_x'; \end{aligned}\right\} \tag{119}$$

and these relations (118) and (119), which agree with those marked (114) and (115), are satisfied, like them, by the expressions (83) for $y_x$ and $y_x'$.

Reciprocally, it is important to observe that those former expressions are included in these recent relations; in such a manner that they might have been obtained from the equations (110) and (111), if the form of the function $\Phi$ had been known as well as the forms of $\phi_x$ and $\phi_a$; or from the more general equations (112) and (109), if we had known the form of the function $F$ as well as the forms of $f_x$ and $f_a$. For, having thus obtained the relations

$$\left.\begin{aligned} y'_a &= \frac{y_x - y_a}{x-a} - \tfrac{1}{2}g\,(x-a), \\ y'_x &= \frac{y_x - y_a}{x-a} + \tfrac{1}{2}g\,(x-a), \end{aligned}\right\} \qquad (120)$$

we might thence have easily deduced the expressions

$$\left.\begin{aligned} y_x &= y_a + y'_a(x-a) + \tfrac{1}{2}g\,(x-a)^2, \\ y'_x &= y'_a + g\,(x-a), \end{aligned}\right\} \qquad (83)$$

which we have otherwise deduced before, by integration, from the principal supplementary differential equation

$$0 = -g + y''_x. \qquad (67)$$

40·2

## CHAPTER I.

### *General Theory of the Principal Integral of any Total Differential Equation of the First Order, but not of the First Degree, between three or more Variables.*

**1.** Let $x_1, x_2, \ldots x_n$ denote any $n$ sought functions of any one variable $x$, the number $n$ being supposed to be greater than unity; let $x_1', x_2', \ldots x_n'$ denote, according to the well-known notation of Lagrange, the $n$ derived functions corresponding, of the first order, or the first differential coefficients

$$\frac{dx_1}{dx}, \frac{dx_2}{dx}, \ldots \frac{dx_n}{dx}$$

of the $n$ sought functions respectively; let

$$f(x, x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n')$$

denote any known function of $x, x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n'$; and therefore let the equation

$$0 = f(x, x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n') \tag{A}$$

represent any proposed total differential equation of the first order, restricting the forms of the $n$ sought functions $x_1, \ldots x_n$ and assisting to determine those forms by establishing a known relation between those $n$ functions themselves, their $n$ derived functions of the first order and the independent variable $x$. Since the number $n$ of the sought functions $x_1, \ldots x_n$ has been supposed to be greater than unity, the one equation (A) is not in general sufficient to determine the forms of all and we may on the contrary assume any $n-1$ supplementary equations, differential or not differential, to connect these $n$ sought functions, in combination with that given equation of the first order (A).

Among all the supplementary equations which might be thus assumed, we shall select as deserving of special attention, for reasons that will hereafter appear, the following $n-1$ equations, which are in general of the second order, and we shall call them the *principal supplementaries* of the original equation (A):

$$\frac{f'(x_1) - \{f'(x_1')\}'}{f'(x_1')} = \frac{f'(x_2) - \{f'(x_2')\}'}{f'(x_2')} = \ldots = \frac{f'(x_n) - \{f'(x_n')\}'}{f'(x_n')}. \tag{B}$$

In this notation, which is borrowed from Lagrange, the $2n$ symbols

$$f'(x_1), f'(x_2), \ldots f'(x_n), \quad f'(x_1'), f'(x_2'), \ldots f'(x_n')$$

denote respectively the $2n$ partial derivatives, or partial differential coefficients, of the first order of the known function $f$, taken with respect to $x_1, x_2, \ldots x_n, x_1', x_2', \ldots x_n'$; and the $n$ symbols

$$\{f'(x_1')\}', \{f'(x_2')\}', \ldots \{f'(x_n')\}'$$

denote the total derivatives, or total differential coefficients, of the first order of the functions $f'(x_1'), f'(x_2'), \ldots f'(x_n')$ respectively, considered as depending on the independent variable $x$, not only so far as that variable enters into them explicitly by entering into the known composition of the function $f$, but also so far as it enters into them implicitly by entering into the unknown composition of the $n$ sought functions $x_1, x_2, \ldots x_n$ and of their derivatives $x_1', x_2', \ldots x_n'$; so that, according to Lagrange's analogous notations for partial and total derivatives of the second order,* the symbols $\{f'(x_1')\}', \ldots \{f'(x_n')\}'$ are equivalent, respectively, to the following more developed expressions:

$$f'{,}'(x, x_1') + x_1' f'{,}'(x_1, x_1') + x_2' f'{,}'(x_2, x_1') + \ldots + x_n' f'{,}'(x_n, x_1')$$
$$+ x_1'' f''(x_1') + x_2'' f'{,}'(x_1', x_2') + \ldots + x_n'' f'{,}'(x_1', x_n'),$$

..................................................................

$$f'{,}'(x, x_n') + x_1' f'{,}'(x_1, x_n') + x_2' f'{,}'(x_2, x_n') + \ldots + x_n' f'{,}'(x_n, x_n')$$
$$+ x_1'' f'{,}'(x_1', x_n') + x_2'' f'{,}'(x_2', x_n') + \ldots + x_n'' f''(x_n').$$

The object of the present Chapter is to establish a general method for rigorously expressing, and for at least approximately calculating, the integrals of all such systems of original and principal supplementary equations as the system (A) and (B) for any number of variables $x, x_1, \ldots x_n$ and for any form of the given function $f$, (some particular exceptions being set aside); but, before proceeding to the establishment of such a method, it may be proper to mention some of the chief reasons, (connected with the Calculus of Variations,) for the selection of the system of equations (B) as the supplementary system to be combined with the original equation (A).

**2.** Such reasons may be drawn from the comparison of that original equation (A) with another equation of like form, obtained by slightly altering as follows the original values of the variables $x, x_1, \ldots x_n$. If we represent by $\xi, \xi_1, \xi_2, \ldots \xi_n$ any functions of $x$ and by $\epsilon$ any small multiplier independent of $x$, we may then consider

$$x_1 + \epsilon\xi_1, \quad x_2 + \epsilon\xi_2, \ldots x_n + \epsilon\xi_n$$

as $n$ functions of $x + \epsilon\xi$, which do not much differ in their forms, or in their laws of functional dependence on $x + \epsilon\xi$, from the $n$ former functions $x_1, x_2, \ldots x_n$, considered as depending on $x$; and if we wish that these $n$ new functions $x_1 + \epsilon\xi_1$, &c. of $x + \epsilon\xi$ should be connected with each other by a differential relation of exactly the same form as that original differential relation, which was previously given to connect the $n$ old functions $x_1$, &c. of $x$, we must then establish this new differential equation of the first order, analogous to and formed from the original equation (A):

$$0 = f\left(x + \epsilon\xi, x_1 + \epsilon\xi_1, \ldots x_n + \epsilon\xi_n, \frac{x_1' + \epsilon\xi_1'}{1 + \epsilon\xi'}, \ldots \frac{x_n' + \epsilon\xi_n'}{1 + \epsilon\xi'}\right); \tag{C}$$

in which $\xi', \xi_1', \ldots \xi_n'$ are the first derivatives or differential coefficients of $\xi, \xi_1, \ldots \xi_n$, considered as functions of $x$; and consequently

$$\frac{x_1' + \epsilon\xi_1'}{1 + \epsilon\xi'}, \ldots \frac{x_n' + \epsilon\xi_n'}{1 + \epsilon\xi'}$$

are the first derivatives or differential coefficients of $x_1 + \epsilon\xi_1, \ldots x_n + \epsilon\xi_n$, considered as functions of $x + \epsilon\xi$. Developing this new differential equation (C) according to the ascending powers of the small multiplier $\epsilon$, and suppressing that part of the development which vanishes on account of the original differential equation (A), and finally dividing by $\epsilon$, we find

$$0 = \xi f'(x) + \xi_1 f'(x_1) + \ldots + \xi_n f'(x_n)$$
$$+ (\xi_1' - x_1'\xi') f'(x_1') + \ldots + (\xi_n' - x_n'\xi') f'(x_n') + \epsilon E; \tag{D}$$

* [Lagrange, *Théorie des fonctions analytiques* (1797).]

the coefficient $E$ denoting, for abridgement, a development of which the actual calculation is not necessary for our present purpose, because it disappears when we pass (as we shall shortly do) to the limit at which $\epsilon$ vanishes. The part independent of $\epsilon$ in the equation (D) may be transformed, by observing that the total derivative of the original equation (A) gives this relation:

$$0 = f'(x) + x_1' f'(x_1) + \ldots + x_n' f'(x_n) + x_1'' f'(x_1') + \ldots + x_n'' f'(x_n'); \tag{E}$$

for thus we find, by eliminating $f'(x)$ and by observing that $\xi_1' - x_1'\xi' - x_1''\xi = (\xi_1 - x_1'\xi)'$, &c.,

$$\begin{aligned} 0 = {} & (\xi_1 - x_1'\xi) f'(x_1) + \ldots + (\xi_n - x_n'\xi) f'(x_n) \\ & + (\xi_1 - x_1'\xi)' f'(x_1') + \ldots + (\xi_n - x_n'\xi)' f'(x_n') + \epsilon E. \end{aligned} \tag{F}$$

Let $\lambda$ denote a function of $x$, so chosen that when we multiply by it the sum of the terms $(\xi_n - x_n'\xi) f'(x_n)$ and $(\xi_n - x_n'\xi)' f'(x_n')$, the product shall be an exact derivative, independently of the forms of the functions $\xi$ and $\xi_n$; that is, let $\lambda$ satisfy the following differential equation of the first order,

$$\lambda f'(x_n) = \{\lambda f'(x_n')\}'. \tag{G}$$

The differential equation (F) may then be thus transformed,

$$\begin{aligned} 0 = {} & \{(\xi_1 - x_1'\xi)\lambda f'(x_1') + \ldots + (\xi_n - x_n'\xi)\lambda f'(x_n')\}' \\ & + (\xi_1 - x_1'\xi)(\lambda f'(x_1) - \{\lambda f'(x_1')\}') + \ldots \\ & + (\xi_{n-1} - x_{n-1}'\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x_{n-1}')\}') + \epsilon\lambda E; \end{aligned} \tag{H}$$

and it gives, in the notation of finite differences and of definite integrals,

$$\begin{aligned} 0 = {} & \Delta\{(\xi_1 - x_1'\xi)\lambda f'(x_1') + \ldots + (\xi_n - x_n'\xi)\lambda f'(x_n')\} \\ & + \int_a^x (\xi_1 - x_1'\xi)(\lambda f'(x_1) - \{\lambda f'(x_1')\}')\, dx + \ldots \\ & + \int_a^x (\xi_{n-1} - x_{n-1}'\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x_{n-1}')\}')\, dx + \epsilon \int_a^x \lambda E\, dx; \end{aligned} \tag{I}$$

the sign of a finite difference $\Delta$ implying here that we are to subtract the initial from the final value of the function to which it is prefixed, so that, the initial value of $x$ being supposed to be $a$, the symbol $\Delta F(x)$ is equivalent in this notation to $F(x) - F(a)$. Now if we take successively smaller and smaller values of the multiplier $\epsilon$, that is, values nearer and nearer to 0, so as to make $\epsilon$ tend to 0 as its limit, the product $\epsilon \int_a^x \lambda E\, dx$ will also tend to 0 as its limit, and thus the definite integral $\int_a^x \lambda E\, dx$ will disappear from the limiting or ultimate form of the equation (I), as being multiplied by an ultimately evanescent factor $\epsilon$. But when the term $\epsilon \int_a^x \lambda E\, dx$ is suppressed in the equation (I), the function $\xi_n$ enters into that equation by its initial and final values only and no trace of the intermediate values (or form) of this function $\xi_n$ remains; to accomplish which removal of the effects of all but the extreme values of $\xi_n$ from the equation (I) was (as will easily be perceived) the thing aimed at in establishing the foregoing relation (G) for the determination of the multiplier $\lambda$. As yet, however, the equation (I) involves not merely the extreme values, but also the intermediate forms, of the other $n$ new functions $\xi, \xi_1, \ldots \xi_{n-1}$, because it involves the definite integrals

$$\left.\begin{array}{l}\int_a^x (\xi_1 - x'_1\xi)(\lambda f'(x_1) - \{\lambda f'(x'_1)\}')\,dx, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \int_a^x (\xi_{n-1} - x'_{n-1}\xi)(\lambda f'(x_{n-1}) - \{\lambda f'(x'_{n-1})\}')\,dx;\end{array}\right\} \quad \text{(J)}$$

it is therefore evidently an object of particular interest to make these $n-1$ definite integrals vanish, without restricting the forms of the $n$ functions $\xi, \xi_1, \ldots \xi_{n-1}$, and for this purpose to make the coefficients of

$$\xi_1 - x'_1\xi, \ldots \xi_{n-1} - x'_{n-1}\xi$$

under the signs of integration vanish, by establishing the following $n-1$ relations between the $n$ original functions $x_1, \ldots x_n$, the multiplier $\lambda$ and the independent variable $x$:

$$\left.\begin{array}{l}\lambda f'(x_1) = \{\lambda f'(x'_1)\}', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda f'(x_{n-1}) = \{\lambda f'(x'_{n-1})\}';\end{array}\right\} \quad \text{(K)}$$

which can in general be done, consistently with the original given relation (A) and with the assumed equation (C). And if we eliminate $\lambda$ and its derivative $\lambda'$, or rather the ratio of the latter to the former, between the $n$ equations (G) and (K), we obtain the $n-1$ supplementary equations (B); which were accordingly selected as possessing this remarkable character, among others, of causing the effects of the forms of the $n+1$ new functions $\xi, \xi_1, \ldots \xi_n$ to disappear from the ultimate state of the integral of the varied equation (C), and of reducing this ultimate state of that integral (I) to the form of a linear relation between the final and initial values of these new functions, namely,

$$0 = \Delta\{(\xi_1 - x'_1\xi)\lambda f'(x'_1) + \ldots + (\xi_n - x'_n\xi)\lambda f'(x'_n)\}; \quad \text{(L)}$$

a relation which may also be thus written,

$$\left.\begin{array}{l}0 = (\xi_1 - x'_1\xi)\lambda f'(x'_1) + \ldots + (\xi_n - x'_n\xi)\lambda f'(x'_n) \\ \quad - (\alpha_1 - a'_1\alpha)\gamma f'(a'_1) - \ldots - (\alpha_n - a'_n\alpha)\gamma f'(a'_n),\end{array}\right\} \quad \text{(M)}$$

if we employ the symbols

$$a_1, \ldots a_n, \quad a'_1, \ldots a'_n, \quad \alpha, \alpha_1, \ldots \alpha_n, \quad \alpha', \alpha'_1, \ldots \alpha'_n, \quad \gamma$$

to denote the initial values of the functions

$$x_1, \ldots x_n, \quad x'_1, \ldots x'_n, \quad \xi, \xi_1, \ldots \xi_n, \quad \xi', \xi'_1, \ldots \xi'_n, \quad \lambda,$$

(corresponding to the initial value $a$ of $x$,) and employ also the symbols $f'(a'_1), \ldots f'(a'_n)$ to denote the corresponding initial values of $f'(x'_1), \ldots f'(x'_n)$.

**3.** In the important but particular case, in which the original differential equation (A), or the given function $f$, is of the first order with respect to the differential coefficients $x'_1, \ldots x'_n$, so that the expressions of the partial derivatives $f'(x'_1), \ldots f'(x'_n)$ do not contain those coefficients $x'_1, \ldots x'_n$, the supplementary equations (B) reduce themselves to the first order; and a reduction of the same sort takes place, with respect to all or some of these equations, in some other cases of exception, which are of less importance. But, in general, the $n-1$ supplementary differential equations (B) are of the 2nd order and compose, when combined with the total derivative (E) of the original differential equation of the 1st order (A), a system of $n$ total differential equations of the 2nd order, of which the complete integrals must involve $2n$ arbitrary constants, (besides the arbitrary initial value $a$ of the independent variable $x$); and since we may choose for these $2n$ constants the $2n$ initial values $a_1, \dots a_n, a'_1, \dots a'_n$, we may represent the $n$ integrals of the equations (B) and (E) as follows:

$$\left.\begin{aligned} x_1 &= \phi_1(x, a, a_1, \dots a_n, a'_1, \dots a'_n), \\ &\dots\dots \\ x_n &= \phi_n(x, a, a_1, \dots a_n, a'_1, \dots a'_n); \end{aligned}\right\} \tag{N}$$

the forms of the $n$ functions $\phi_1, \dots \phi_n$ remaining, as yet, unknown. Besides, the original differential equation (A) gives this initial condition:

$$0 = f(a, a_1, \dots a_n, a'_1, \dots a'_n); \tag{O}$$

so that the constants $a_1, \dots a_n, a'_1, \dots a'_n$ are not any longer all arbitrary (when $a$ is considered as known), but must be so assumed as to satisfy this condition (O), if we wish to represent, by the equations (N), no longer the integrals of the derived system (B) and (E) but the integrals of the original system (A) and (B). And if we knew these integrals of that original system, that is, if we knew the forms of the $n$ functions $\phi_1, \dots \phi_n$ in the $n$ equations (N), as well as the form of the function $f$ in the initial condition (O), we could in general eliminate the $n$ constants $a'_1, \dots a'_n$ between these $n+1$ equations (N) and (O) and so arrive at a relation which would involve only $x, x_1, \dots x_n$ and $a, a_1, \dots a_n$ and which would be thus denoted:

$$0 = \psi(x, x_1, \dots x_n, a, a_1, \dots a_n); \tag{P}$$

but not in general at two (or more) distinct relations of this kind; since, after the assumption of the $n$ initial values $a_1, \dots a_n$ of the $n$ functions $x_1, \dots x_n$ (corresponding to any assumed initial value of the independent variable $x$), there would still remain $n-1$ arbitrary constants to dispose of, and consequently the final values of any $n-1$ of the same $n$ functions (corresponding to any final value of the same independent variable $x$) would still admit, in general, of being assumed at pleasure. It is sufficient, for the present, to concede the possibility of such an elimination, and to perceive the general existence of a determinate relation such as (P), (that is, a relation between the final and initial values of the independent variable $x$ and of the $n$ functions $x_1, \dots x_n$ as a consequence of the differential equations (A) and (B)); for it will soon be shown that, instead of our being obliged to integrate first those differential equations and then to deduce the relation (P) by elimination from the integrals thus found, we may, on the contrary, with advantage, seek first by independent processes to discover the relation (P), and, when it has once in any way been found, may then deduce from it the whole system of integrals (N).

**4.** The equation (M) between the extreme values of the $n+1$ functions $\xi, \xi_1, \dots \xi_n$, introduced in the 2nd article, was obtained as a limiting form of the integral of the single differential equation (C), combined with the relations (A) and (B) between the $n$ functions $x_1, \dots x_n$; and therefore this equation (M) will still be true, after the introduction of any new equations between the same $n+1$ functions $\xi, \xi_1, \dots \xi_n$, if we combine these new equations with the equation (C) and with the system (A) and (B). If, then, by the introduction of any such new or supplementary relations between the functions $\xi, \xi_1, \ldots \xi_n$ we can obtain, in any new way, a new limiting linear relation between the extreme values of those functions, not visibly coincident with the relation (M) but of the form

$$\left.\begin{aligned} 0 = X\xi + X_1\xi_1 + \ldots + X_n\xi_n \\ + A\alpha + A_1\alpha_1 + \ldots + A_n\alpha_n, \end{aligned}\right\} \tag{Q}$$

in which the coefficients $X, X_1, \ldots X_n, A, A_1, \ldots A_n$ are independent of $\xi, \xi_1, \ldots \xi_n$; if also, by the nature of this new process, it is permitted to assume at pleasure any $2n+1$ of the $2n+2$ extreme values $\alpha, \alpha_1, \ldots \alpha_n, \xi, \xi_1, \ldots \xi_n$, provided that the limiting state of the remaining extreme value (corresponding to the limit $\epsilon = 0$) is then determined so as to satisfy the new relation (Q); (as, by the nature of the process which conducted to the former relation (M), it was permitted to assume at pleasure any $2n+1$ of the same $2n+2$ extreme values, provided that the limiting state of the remaining value was determined so as to satisfy that former relation); we shall be able to conclude that these two limiting relations (M) and (Q), though differing in appearance, must in reality coincide with each other, in such a manner that the $2n+2$ coefficients of the one must be proportional to those of the other, and that thus, by the introduction of a new multiplier $L$, we shall have the $2n+2$ equations following:

$$\left.\begin{aligned} X &= -L\lambda\{x_1' f'(x_1') + \ldots + x_n' f'(x_n')\}, \\ X_1 &= L\lambda f'(x_1'), \ \ldots \ X_n = L\lambda f'(x_n'), \\ A &= L\gamma\{a_1' f'(a_1') + \ldots + a_n' f'(a_n')\}, \\ A_1 &= -L\gamma f'(a_1'), \ldots A_n = -L\gamma f'(a_n'). \end{aligned}\right\} \tag{R}$$

**5.** Now among the various supplementary relations which might be assumed to connect the $n+1$ functions $\xi, \xi_1, \ldots \xi_n$, those relations which are derived from the principal supplementary differential equations of the second order (B) by changing

$$x, x_1, \ldots x_n, \quad x_1', \ldots x_n', \quad x_1'', \ldots x_n''$$

to

$$x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \ldots\ x_n+\epsilon\xi_n,\ \frac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'},\ \ldots\ \frac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \ldots\ \frac{\left(\dfrac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}$$

are deserving of special mention, namely the $n-1$ new supplementary equations, which may be thus denoted,

$$\left.\begin{aligned} 0 &= \overset{1}{f}\left(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \ldots\ x_n+\epsilon\xi_n,\ \frac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'},\ \ldots\ \frac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \ldots\ \frac{\left(\dfrac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}\right), \\ &\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= \overset{n-1}{f}\left(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \ldots\ x_n+\epsilon\xi_n,\ \frac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'},\ \ldots\ \frac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'},\ \frac{\left(\dfrac{x_1'+\epsilon\xi_1'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'},\ \ldots\ \frac{\left(\dfrac{x_n'+\epsilon\xi_n'}{1+\epsilon\xi'}\right)'}{1+\epsilon\xi'}\right), \end{aligned}\right\} \tag{S}$$

if the $n-1$ former principal supplementary equations (B) be denoted as follows:

$$\left.\begin{aligned} 0 &= \overset{1}{f}(x, x_1, \ldots x_n, x_1', \ldots x_n', x_1'', \ldots x_n''), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= \overset{n-1}{f}(x, x_1, \ldots x_n, x_1', \ldots x_n', x_1'', \ldots x_n''); \end{aligned}\right\} \tag{T}$$

and these $n-1$ equations (S), when combined with the equation (C), conduct to a limiting linear relation of the desired form (Q) between the $2n+2$ extreme values $\alpha, \alpha_1, \dots \alpha_n, \xi, \xi_1, \dots \xi_n$, leaving $2n+1$ of them arbitrary. For, although the $n$ differential equations (C) and (S), even along with any finite relation of arbitrary constants or initial data, are not in general sufficient to determine completely the forms of the $n+1$ functions $\xi, \xi_1, \dots \xi_n$ considered as depending on $x$, yet these $n$ differential equations are in general sufficient to determine completely the forms of the $n$ functions $x_1+\epsilon\xi_1, \dots x_n+\epsilon\xi_n$ considered as depending on $x+\epsilon\xi$, if the $2n$ initial constants $a_1+\epsilon\alpha_1, \dots a_n+\epsilon\alpha_n, \dfrac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'}, \dots \dfrac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}$, (as well as $a+\epsilon\alpha$,) be known or even any $2n-1$ of these $2n$ constants, because the equation (C) conducts to an initial condition connecting them; and thus we find $n$ integral equations of the forms (N), namely,

$$\left.\begin{array}{l} x_1+\epsilon\xi_1=\phi_1\left(x+\epsilon\xi,\ a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \dots a_n+\epsilon\alpha_n,\ \dfrac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'},\ \dots \dfrac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n+\epsilon\xi_n=\phi_n(x+\epsilon\xi, \dots\dots \qquad\qquad\qquad\qquad\qquad\qquad), \end{array}\right\} \tag{U}$$

together with this initial condition of the form (O)

$$0=f\left(a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \dots a_n+\epsilon\alpha_n,\ \frac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'},\ \dots \frac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'}\right); \tag{V}$$

which $n+1$ equations (U) and (V) conduct, by elimination of the $n$ constants

$$\frac{a_1'+\epsilon\alpha_1'}{1+\epsilon\alpha'}, \dots \frac{a_n'+\epsilon\alpha_n'}{1+\epsilon\alpha'},$$

to this following relation

$$0=\psi(x+\epsilon\xi,\ x_1+\epsilon\xi_1,\ \dots x_n+\epsilon\xi_n,\ a+\epsilon\alpha,\ a_1+\epsilon\alpha_1,\ \dots a_n+\epsilon\alpha_n), \tag{W}$$

the form of the function $\psi$ in this relation being the same as in the final relation (P); so that, developing this new relation (W) according to the ascending powers of $\epsilon$, suppressing the terms which vanish on account of the old relation (P), dividing across by $\epsilon$ and finally letting $\epsilon$ tend to 0, we have the equation

$$\left.\begin{array}{l} 0=\xi\psi'(x)+\xi_1\psi'(x_1)+\xi_2\psi'(x_2)+\dots+\xi_n\psi'(x_n) \\ \quad +\alpha\psi'(a)+\alpha_1\psi'(a_1)+\alpha_2\psi'(a_2)+\dots+\alpha_n\psi'(a_n). \end{array}\right\} \tag{X}$$

[Here there are 19 pages of manuscript missing.]

11. The reduced equation ($G^2$) may be put under the form

$$z_{2n-1}'=V_1z_1'+V_2z_2'+\dots+V_{2n-2}z_{2n-2}', \tag{A$^3$}$$

in which the $2n-2$ coefficients $V_1, V_2, \dots V_{2n-2}$ are to be considered as known functions of the $2n-1$ variables $z_1, z_2, \dots z_{2n-1}$ and of these alone. If we denote by the symbols $\overset{1}{z}_1, \overset{1}{z}_2, \dots \overset{1}{z}_{2n-3}$ any $2n-3$ assumed functions of these $2n-1$ variables and treat these functions as auxiliary variables, we may in general conceive that the $2n-3$ former variables $z_1, z_2, \dots z_{2n-3}$ are expressed, reciprocally, as $2n-3$ functions of these $2n-3$ new variables $\overset{1}{z}_1, \overset{1}{z}_2, \dots \overset{1}{z}_{2n-3}$ and of the two other old variables $z_{2n-2}, z_{2n-1}$; the partial derivatives, or differential coefficients, of the first order of these $2n-3$ new functions may be denoted by the symbols

$$z'_1(\overset{1}{z}_1),\ z'_1(\overset{1}{z}_2),\ \ldots\ z'_1(\overset{1}{z}_{2n-3}),\ z'_1(z_{2n-2}),\ z'_1(z_{2n-1}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z'_{2n-3}(\overset{1}{z}_1),\ z'_{2n-3}(\overset{1}{z}_2),\ \ldots\ z'_{2n-3}(\overset{1}{z}_{2n-3}),\ z'_{2n-3}(z_{2n-2}),\ z'_{2n-3}(z_{2n-1});$$

and their total derivatives of the same order may in like manner be denoted as follows:

$$\left.\begin{aligned} z'_1 &= z'_1(\overset{1}{z}_1)\overset{1}{z}{}'_1 + \ldots + z'_1(\overset{1}{z}_{2n-3})\overset{1}{z}{}'_{2n-3} + z'_1(z_{2n-2})z'_{2n-2} + z'_1(z_{2n-1})z'_{2n-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z'_{2n-3} &= z'_{2n-3}(\overset{1}{z}_1)\overset{1}{z}{}'_1 + \ldots + z'_{2n-3}(\overset{1}{z}_{2n-3})\overset{1}{z}{}'_{2n-3} + z'_{2n-3}(z_{2n-2})z'_{2n-2} + z'_{2n-3}(z_{2n-1})z'_{2n-1}. \end{aligned}\right\} \tag{B$^3$}$$

By the substitution of these expressions, the equation ($A^3$) takes the form

$$0 = \overset{1}{Z}_1\overset{1}{z}{}'_1 + \overset{1}{Z}_2\overset{1}{z}{}'_2 + \ldots + \overset{1}{Z}_{2n-3}\overset{1}{z}{}'_{2n-3} + \overset{1}{Z}_{2n-2}z'_{2n-2} + \overset{1}{Z}_{2n-1}z'_{2n-1}; \tag{C$^3$}$$

in which the coefficients $\overset{1}{Z}_1, \overset{1}{Z}_2, \ldots \overset{1}{Z}_{2n-1}$ have the following values:

$$\left.\begin{aligned} \overset{1}{Z}_1 &= V_1 z'_1(\overset{1}{z}_1) + \ldots + V_{2n-3}z'_{2n-3}(\overset{1}{z}_1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{1}{Z}_{2n-3} &= V_1 z'_1(\overset{1}{z}_{2n-3}) + \ldots + V_{2n-3}z'_{2n-3}(\overset{1}{z}_{2n-3}), \end{aligned}\right\} \tag{D$^3$}$$

$$\left.\begin{aligned} \overset{1}{Z}_{2n-2} &= V_1 z'_1(z_{2n-2}) + \ldots + V_{2n-3}z'_{2n-3}(z_{2n-2}) + V_{2n-2}, \\ \overset{1}{Z}_{2n-1} &= V_1 z'_1(z_{2n-1}) + \ldots + V_{2n-3}z'_{2n-3}(z_{2n-1}) - 1. \end{aligned}\right\} \tag{E$^3$}$$

This equation ($C^3$) will admit of being put under the simpler form

$$\overset{1}{z}{}'_{2n-3} = \overset{1}{V}_1\overset{1}{z}{}'_1 + \overset{1}{V}_2\overset{1}{z}{}'_2 + \ldots + \overset{1}{V}_{2n-4}\overset{1}{z}{}'_{2n-4} + \overset{1}{V}_{2n-1}z'_{2n-1}, \tag{F$^3$}$$

in which the $2n-4$ coefficients $\overset{1}{V}_1, \overset{1}{V}_2, \ldots \overset{1}{V}_{2n-4}$ will be known functions of the $2n-3$ new variables $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-3}$ and of $z_{2n-1}$, not involving $z_{2n-2}$, if we can so choose the new variables as to satisfy the $2n-3$ conditions following:

$$\overset{1}{Z}_{2n-2} = 0; \tag{G$^3$}$$

$$\frac{\overset{1}{Z}{}'_1(z_{2n-2})}{\overset{1}{Z}_1} = \frac{\overset{1}{Z}{}'_2(z_{2n-2})}{\overset{1}{Z}_2} = \ldots = \frac{\overset{1}{Z}{}'_{2n-3}(z_{2n-2})}{\overset{1}{Z}_{2n-3}}. \tag{H$^3$}$$

The expressions ($D^3$) give

$$\begin{aligned} \overset{1}{Z}{}'_i(z_{2n-2}) = {} & V_1 z''_1(\overset{1}{z}_i, z_{2n-2}) + \ldots + V_{2n-3}z''_{2n-3}(\overset{1}{z}_i, z_{2n-2}) \\ & + \{V'_1(z_1)\,.\,z'_1(z_{2n-2}) + \ldots + V'_1(z_{2n-3})\,.\,z'_{2n-3}(z_{2n-2}) + V'_1(z_{2n-2})\}\,z'_1(\overset{1}{z}_i) \\ & + \ldots\ldots \\ & + \{V'_{2n-3}(z_1)\,.\,z'_1(z_{2n-2}) + \ldots + V'_{2n-3}(z_{2n-3})\,.\,z'_{2n-3}(z_{2n-2}) + V'_{2n-3}(z_{2n-2})\}\,z'_{2n-3}(\overset{1}{z}_i); \end{aligned} \tag{I$^3$}$$

the index $i$ being here any one of the integer numbers 1, 2, ... $2n-3$; and the first expression ($E^3$) gives

$$\begin{aligned}\overset{1}{Z}{}'_{2n-2}(\overset{1}{z}_i) &= V_1 \overset{1}{z}{}_1''(z_i, z_{2n-2}) + \dots + V_{2n-3}\overset{1}{z}{}''_{2n-3}(z_i, z_{2n-2}) \\ &+ \{V_1'(z_1) \,.\, z_1'(z_{2n-2}) + \dots + V'_{2n-3}(z_1) \,.\, z'_{2n-3}(z_{2n-2}) + V'_{2n-2}(z_1)\} z_1'(\overset{1}{z}_i) \\ &+ \dots\dots \\ &+ \{V_1'(z_{2n-3}) \,.\, z_1'(z_{2n-2}) + \dots + V'_{2n-3}(z_{2n-3}) \,.\, z'_{2n-3}(z_{2n-2}) + V'_{2n-2}(z_{2n-3})\} z'_{2n-3}(\overset{1}{z}_i);\end{aligned} \tag{K$^3$}$$

so that we have the equation

$$\overset{1}{Z}{}_i'(z_{2n-2}) - \overset{1}{Z}{}'_{2n-2}(\overset{1}{z}_i) = W_1 z_1'(\overset{1}{z}_i) + \dots + W_{2n-3} z'_{2n-3}(\overset{1}{z}_i), \tag{L$^3$}$$

in which

$$\begin{aligned}W_k &= \{V_k'(z_1) - V_1'(z_k)\} z_1'(z_{2n-2}) \\ &+ \{V_k'(z_2) - V_2'(z_k)\} z_2'(z_{2n-2}) \\ &+ \dots\dots \\ &+ \{V_k'(z_{2n-3}) - V'_{2n-3}(z_k)\} z'_{2n-3}(z_{2n-2}) \\ &+ V_k'(z_{2n-2}) - V'_{2n-2}(z_k),\end{aligned} \tag{M$^3$}$$

the index $k$ (like $i$) denoting any integer from 1 to $2n-3$ inclusive. And since, by ($D^3$), we have

$$\overset{1}{Z}_i = V_1 z_1'(\overset{1}{z}_i) + \dots + V_{2n-3} z'_{2n-3}(\overset{1}{z}_i), \tag{N$^3$}$$

we see that we shall satisfy the $2n-3$ conditions ($G^3$) and ($H^3$) if we can so select the $2n-3$ auxiliary variables $\overset{1}{z}_1, \dots \overset{1}{z}_{2n-3}$ as to satisfy all the following conditions:

$$0 = V_1 z_1'(z_{2n-2}) + \dots + V_{2n-3} z'_{2n-3}(z_{2n-2}) + V_{2n-2}, \tag{O$^3$}$$

and

$$\frac{W_1}{V_1} = \frac{W_2}{V_2} = \dots = \frac{W_{2n-3}}{V_{2n-3}}. \tag{P$^3$}$$

In this manner we are led to endeavour to integrate a new auxiliary system of total differential equations of the 1st order, namely, the following system of $2n-3$ such equations between the $2n-2$ variables $z_1, z_2, \dots z_{2n-2}$:

$$0 = V_1 z_1' + V_2 z_2' + \dots + V_{2n-3} z'_{2n-3} + V_{2n-2}; \tag{Q$^3$}$$

$$\left.\begin{aligned}&\frac{1}{V_1}\{V_1'(z_1) \,.\, z_1' + V_2'(z_1) \,.\, z_2' + \dots + V'_{2n-3}(z_1) \,.\, z'_{2n-3} + V'_{2n-2}(z_1) - V_1'\} \\ &= \frac{1}{V_2}\{V_1'(z_2) \,.\, z_1' + V_2'(z_2) \,.\, z_2' + \dots + V'_{2n-3}(z_2) \,.\, z'_{2n-3} + V'_{2n-2}(z_2) - V_2'\} \\ &= \dots\dots \\ &= \frac{1}{V_{2n-3}}\{V_1'(z_{2n-3}) \,.\, z_1' + V_2'(z_{2n-3}) \,.\, z_2' + \dots + V'_{2n-3}(z_{2n-3}) \,.\, z'_{2n-3} + V'_{2n-2}(z_{2n-3}) - V'_{2n-3}\};\end{aligned}\right\} \tag{R$^3$}$$

in which $z_{2n-1}$ is treated as constant and $z_{2n-2}$ is taken for the independent variable, so that the total derivatives $V_1', \dots V'_{2n-3}$ are equivalent to the following more developed expressions:

$$\left.\begin{aligned}V_1' &= V_1'(z_1) \,.\, z_1' + \dots + V_1'(z_{2n-3}) \,.\, z'_{2n-3} + V_1'(z_{2n-2}), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ V'_{2n-3} &= V'_{2n-3}(z_1) \,.\, z_1' + \dots + V'_{2n-3}(z_{2n-3}) \,.\, z'_{2n-3} + V'_{2n-3}(z_{2n-2}).\end{aligned}\right\} \tag{S$^3$}$$

For, if we can integrate these $2n-3$ equations and thereby discover expressions for $z_1, z_2, \ldots z_{2n-3}$ as functions of the independent variable $z_{2n-2}$, involving also the variable $z_{2n-1}$, which has been treated as constant in the integration, and involving $2n-3$ arbitrary constants, we have only to treat these $2n-3$ new constants as variable and substitute them for the $2n-3$ auxiliary variables $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-3}$, which have hitherto been left undetermined; since thus we shall accomplish the desired reduction of the differential equation ($A^3$) to the form ($F^3$), in which the coefficient of $z'_{2n-2}$ is null, the coefficient of $\overset{1}{z}'_{2n-3}$ is unity and the coefficients of $\overset{1}{z}'_1, \overset{1}{z}'_2, \ldots \overset{1}{z}'_{2n-4}$ are independent of $z_{2n-2}$. The same reasoning shows that if we can integrate the following new system of $2n-5$ total differential equations of the 1st order:

$$0=\overset{1}{V}_1\overset{1}{z}'_1+\overset{1}{V}_2\overset{1}{z}'_2+\ldots+\overset{1}{V}_{2n-5}\overset{1}{z}'_{2n-5}+\overset{1}{V}_{2n-4}; \qquad (T^3)$$

$$\frac{1}{\overset{1}{V}_1}\{\overset{1}{V}'_1(z_1).\overset{1}{z}'_1+\ldots+\overset{1}{V}'_{2n-5}(z_1).\overset{1}{z}'_{2n-5}+\overset{1}{V}'_{2n-4}(z_1)-\overset{1}{V}'_1\}$$
$$=\ldots\ldots$$
$$=\frac{1}{\overset{1}{V}_{2n-5}}\{\overset{1}{V}'_1(z_{2n-5}).\overset{1}{z}'_1+\ldots+\overset{1}{V}'_{2n-5}(z_{2n-5}).\overset{1}{z}'_{2n-5}+\overset{1}{V}'_{2n-4}(z_{2n-5})-\overset{1}{V}'_{2n-5}\}; \qquad (U^3)$$

in which $\overset{1}{z}_{2n-3}$ and $z_{2n-1}$ are treated as constant and $\overset{1}{z}_{2n-4}$ is taken for the independent variable; and if we can thus discover $2n-5$ expressions for $\overset{1}{z}_1, \overset{1}{z}_2, \ldots \overset{1}{z}_{2n-5}$ as functions of $\overset{1}{z}_{2n-4}$, involving also in general $\overset{1}{z}_{2n-3}$ and $z_{2n-1}$ and $2n-5$ arbitrary constants; then, by treating these latter constants as $2n-5$ new auxiliary variables $\overset{2}{z}_1, \overset{2}{z}_2, \ldots \overset{2}{z}_{2n-5}$, we can reduce the equation ($F^3$) to this new and simpler form:

$$\overset{2}{z}'_{2n-5}=\overset{2}{V}_1\overset{2}{z}'_1+\overset{2}{V}_2\overset{2}{z}'_2+\ldots+\overset{2}{V}_{2n-6}\overset{2}{z}'_{2n-6}+\overset{2}{V}_{2n-3}\overset{1}{z}'_{2n-3}+\overset{2}{V}_{2n-1}z'_{2n-1}; \qquad (V^3)$$

in which the $2n-6$ coefficients $\overset{2}{V}_1, \overset{2}{V}_2, \ldots \overset{2}{V}_{2n-6}$ will be known functions of $\overset{2}{z}_1, \overset{2}{z}_2, \ldots \overset{2}{z}_{2n-5}, \overset{1}{z}_{2n-3}$ and $z_{2n-1}$, $\overset{2}{V}_{2n-3}$ will be a known function of the same variables and of $\overset{1}{z}_{2n-4}$, and $\overset{2}{V}_{2n-1}$ will involve in general the variable $z_{2n-2}$ along with all the former.

By successive reductions of this sort, depending on the integrations of several successive systems of fewer and fewer total differential equations of the first order between fewer and fewer variables; namely, on the integration of a system of $2n-7$ equations between $2n-6$ variables, a system of $2n-9$ equations between $2n-8$ variables, and so on till we come to the integration of a single total differential equation between only two variables; it is in general possible, at least in theory, to reduce the differential equation ($A^3$) to the form:

$$\overset{n-1}{z}'_1 = \overset{n-1}{V}_3\ \overset{n-2}{z}'_3 + \overset{n-1}{V}_5\ \overset{n-3}{z}'_5 + \ldots + \overset{n-1}{V}_{2n-3}\overset{1}{z}'_{2n-3} + \overset{n-1}{V}_{2n-1}z'_{2n-1}; \qquad (W^3)$$

in which the coefficient $\overset{n-1}{V}_3$ is, in general, a known function of the $n$ variables $\overset{n-1}{z}_1, \overset{n-2}{z}_3, \overset{n-3}{z}_5, \ldots \overset{2}{z}_{2n-5}, \overset{1}{z}_{2n-3}, z_{2n-1}$, involving also another variable which, according to the same analogy of notation, is to be denoted by the symbol $\overset{n-2}{z}_2$; $\overset{n-1}{V}_5$ involves in general the same variables as $\overset{n-1}{V}_3$ but involves also an additional variable $\overset{n-3}{z}_4$; and thus new variables $\overset{n-4}{z}_6, \overset{n-5}{z}_8$, &c. are introduced successively as we pass to new coefficients $\overset{n-1}{V}_7$, $\overset{n-1}{V}_9$, &c. till we come to the last coefficient $\overset{n-1}{V}_{2n-1}$ which is a known function of all the $2n-1$ variables of the new system, namely,

$$\overset{n-1}{z}_1, \overset{n-2}{z}_2, \overset{n-2}{z}_3, \overset{n-3}{z}_4, \overset{n-3}{z}_5, \dots \overset{2}{z}_{2n-6}, \overset{2}{z}_{2n-5}, \overset{1}{z}_{2n-4}, \overset{1}{z}_{2n-3}, z_{2n-2}, z_{2n-1}.$$

And now, at last, we see that the sought integral of the total differential equation ($A^3$) of the 1st order and 1st degree between $2n-1$ variables may in general be represented by the following system of $n$ equations, involving one arbitrary function of $n-1$ variables:

$$\overset{n-1}{z}_1 = \chi(\overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}); \tag{$X^3$}$$

$$\overset{n-1}{V}_3 = \chi'(\overset{n-2}{z}_3), \overset{n-1}{V}_5 = \chi'(\overset{n-3}{z}_5), \dots \overset{n-1}{V}_{2n-1} = \chi'(z_{2n-1}); \tag{$Y^3$}$$

in which the coefficients $\overset{n-1}{V}_3$, $\overset{n-1}{V}_5$, ... $\overset{n-1}{V}_{2n-1}$ are the known functions just now described, and in which the $2n-3$ new variables $\overset{n-1}{z}_1, \overset{n-2}{z}_2, \overset{n-2}{z}_3, \overset{n-3}{z}_4, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-4}, \overset{1}{z}_{2n-3}$ are themselves known functions of the $2n-1$ old variables $z_1, z_2, \dots z_{2n-1}$, which entered into the equation ($A^3$); while the function $\chi$ of the $n-1$ variables $\overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ remains entirely arbitrary. We see also that the equation ($A^3$) may be integrated by another important (though only particular) system of $n$ equations, namely, the following:

$$\overset{n-1}{z}_1 = e_1, \overset{n-2}{z}_3 = e_2, \overset{n-3}{z}_5 = e_3, \dots \overset{1}{z}_{2n-3} = e_{n-1}, z_{2n-1} = e_n; \tag{$Z^3$}$$

in which the $n$ expressions $\overset{n-1}{z}_1, \overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ are equated to $n$ arbitrary constants $e_1, e_2, e_3, \dots e_{n-1}, e_n$, instead of the first being treated as an arbitrary function of the rest. And to apply all this research respecting the integration of the equation ($A^3$) [or ($G^2$)] to the problem of integrating any partial differential equation of the 1st order

$$0 = F(\phi, x_1, \dots x_n, y_1, \dots y_n); \tag{$X^1$}$$

we have only to consider $\overset{n-1}{z}_1, \overset{n-2}{z}_3, \overset{n-3}{z}_5, \dots \overset{1}{z}_{2n-3}, z_{2n-1}$ and $\overset{n-1}{V}_3$, $\overset{n-1}{V}_5$, ... $\overset{n-1}{V}_{2n-1}$ as equal to $2n-1$ known functions of the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$; the forms of these functions being supposed to be discovered by combining the processes of the present and of the preceding article. For thus we shall either have, at once, the general integral of the proposed partial differential equation, with an arbitrary function $\chi$ of $n-1$ variables, by eliminating, or by conceiving eliminated, the $n$ partial differential coefficients $y_1, \dots y_n$ of the sought function $\phi$ between the proposed equation ($X^1$) itself and the general integral system ($X^3$), ($Y^3$), which is properly the method of Pfaff; or we shall have, if we prefer it, a simpler but only particular integral of the same partial differential equation ($X^1$), involving no arbitrary function but involving $n$ arbitrary constants, by eliminating the same $n$ partial differential coefficients $y_1, \dots y_n$ between the equations ($X^1$) and ($Z^3$); and then from this particular integral the general one can be easily deduced.

**12.** In order to apply the general method, explained in the two last articles, to the case where the relation

$$0 = F(\phi, x_1, \dots x_n, y_1, \dots y_n), \tag{$X^1$}$$

between the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$, is supposed to result by elimination of $n$ other variables, such as $u_1, \dots u_n$, from a system of $n+1$ given equations of the following forms

$$0 = f(\phi, x_1, \dots x_n, u_1, \dots u_n), \tag{A$^4$}$$

and

$$\left.\begin{aligned} y_1 &= \frac{f'(u_1)}{u_1 f'(u_1) + \dots + u_n f'(u_n)}, \\ &\dots\dots\dots \\ y_n &= \frac{f'(u_n)}{u_1 f'(u_1) + \dots + u_n f'(u_n)}; \end{aligned}\right\} \tag{B$^4$}$$

we must endeavour to deduce from these equations expressions for the $2n+1$ partial derivatives, or partial differential coefficients,

$$F'(\phi),\ F'(x_1),\ \dots F'(x_n),\ F'(y_1),\ \dots F'(y_n);$$

or at least for their $2n$ ratios, because these ratios enter into the formulae ($V^2$), ($W^2$). By the nature of these $2n+1$ partial derivatives, they are equal or proportional to the coefficients of that limiting and linear relation which connects the otherwise arbitrary increments that may be attributed to the $2n+1$ variables $\phi, x_1, \dots x_n, y_1, \dots y_n$, consistently with the relation ($X^1$), when these increments are made smaller and smaller; in such a manner, that if we represent these increments respectively by $\epsilon\xi, \epsilon\xi_1, \dots \epsilon\xi_n, \epsilon\eta_1, \dots \epsilon\eta_n$, $\epsilon$ being any small multiplier, and if, after developing the equation

$$0 = F(\phi + \epsilon\xi, x_1 + \epsilon\xi_1, \dots x_n + \epsilon\xi_n, y_1 + \epsilon\eta_1, \dots y_n + \epsilon\eta_n) \tag{C$^4$}$$

according to the ascending powers of $\epsilon$ and suppressing the part of the development which vanishes by ($X^1$), we then divide by $\epsilon$ and finally make $\epsilon$ tend to zero, we shall thus be conducted to the limiting and linear relation following, between $\xi, \xi_1, \dots \xi_n, \eta_1, \dots \eta_n$:

$$0 = \xi F'(\phi) + \xi_1 F'(x_1) + \dots + \xi_n F'(x_n) + \eta_1 F'(y_1) + \dots + \eta_n F'(y_n). \tag{D$^4$}$$

The equation ($A^4$) conducts in like manner to this other limiting and linear relation,

$$0 = \xi f'(\phi) + \xi_1 f'(x_1) + \dots + \xi_n f'(x_n) + v_1 f'(u_1) + \dots + v_n f'(u_n), \tag{E$^4$}$$

if, besides changing $\phi, x_1, \dots x_n$ to $\phi + \epsilon\xi, x_1 + \epsilon\xi_1, \dots x_n + \epsilon\xi_n$, we change also $u_1, \dots u_n$ to $u_1 + \epsilon v_1, \dots u_n + \epsilon v_n$ and make $\epsilon$ tend to 0 as before; and, by making the same changes in the $n$ equations ($B^4$), we should obtain $n$ other limiting and linear relations between $\xi, \xi_1, \dots \xi_n$, $v_1, \dots v_n$ and $\eta_1, \dots \eta_n$, which, when combined with the relation ($E^4$), would conduct by elimination to a new limiting and linear relation between $\xi, \xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, not involving $v_1, \dots v_n$; but we may simplify this elimination by observing that the equations ($B^4$) give

$$1 = y_1 u_1 + \dots + y_n u_n, \tag{F$^4$}$$

and therefore at the limit (corresponding to the limiting value 0 of $\epsilon$)

$$0 = \eta_1 u_1 + \dots + \eta_n u_n + v_1 y_1 + \dots + v_n y_n, \tag{G$^4$}$$

while the limiting relation ($E^4$) gives, on account of the equations ($B^4$),

$$0 = \xi f'(\phi) + \xi_1 f'(x_1) + \dots + \xi_n f'(x_n) + (v_1 y_1 + \dots + v_n y_n)\{u_1 f'(u_1) + \dots + u_n f'(u_n)\}. \tag{H$^4$}$$

We may therefore eliminate all the $n$ variables $v_1, \dots v_n$ together by eliminating the one combination $v_1 y_1 + \dots + v_n y_n$ between the two relations ($G^4$) and ($H^4$); and thus we obtain

$$\left.\begin{aligned} 0 = &-\{\xi f'(\phi) + \xi_1 f'(x_1) + \dots + \xi_n f'(x_n)\} \\ &+ (\eta_1 u_1 + \dots + \eta_n u_n)\{u_1 f'(u_1) + \dots + u_n f'(u_n)\}. \end{aligned}\right\} \tag{I$^4$}$$

Comparing this with the relation ($D^4$), we find, in general, the following expressions for the sought partial differential coefficients of the function $F$:

$$F'(\phi) = -Mf'(\phi), \quad F'(x_1) = -Mf'(x_1), \ldots F'(x_n) = -Mf'(x_n), \tag{K$^4$}$$

and

$$\left.\begin{aligned} F'(y_1) &= Mu_1\{u_1 f'(u_1) + \ldots + u_n f'(u_n)\}, \\ &\ldots\ldots\ldots \\ F'(y_n) &= Mu_n\{u_1 f'(u_1) + \ldots + u_n f'(u_n)\}; \end{aligned}\right\} \tag{L$^4$}$$

the multiplier $M$ remaining indeed still undetermined but disappearing from the expressions of the ratios of the coefficients $F'(\phi)$, &c., which alone are required for our purpose. For, since the expressions ($L^4$) give by ($F^4$)

$$M = \frac{y_1 F'(y_1) + \ldots + y_n F'(y_n)}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}, \tag{M$^4$}$$

and therefore

$$\left.\begin{aligned} &\frac{F'(y_1)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)} = u_1, \\ &\ldots\ldots\ldots \\ &\frac{F'(y_n)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)} = u_n; \end{aligned}\right\} \tag{N$^4$}$$

and also, by ($K^4$),

$$\left.\begin{aligned} &\frac{F'(\phi)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)} = -\frac{f'(\phi)}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}, \\ &\frac{F'(x_1)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)} = -\frac{f'(x_1)}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}, \\ &\ldots\ldots\ldots \\ &\frac{F'(x_n)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)} = -\frac{f'(x_n)}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}; \end{aligned}\right\} \tag{O$^4$}$$

we can now easily eliminate all the partial differential coefficients of the function $F$ from the auxiliary system of total differential equations ($V^2$) and ($W^2$), and may write these equations as follows:

$$x_1' = u_1 \phi', \ldots x_n' = u_n \phi'; \tag{P$^4$}$$

and

$$\left.\begin{aligned} y_1' &= \frac{\{f'(x_1) + y_1 f'(\phi)\}\phi'}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}, \\ &\ldots\ldots \\ y_n' &= \frac{\{f'(x_n) + y_n f'(\phi)\}\phi'}{u_1 f'(u_1) + \ldots + u_n f'(u_n)}. \end{aligned}\right\} \tag{Q$^4$}$$

It results from the general theory explained in the 10th article that the equations of this system ($P^4$) and ($Q^4$) must be capable of being reduced to $2n-1$ distinct differential equations between $2n$ variables; and, accordingly, we may consider any one of the $2n$ equations of this system as being a consequence of the $2n-1$ other equations of the same system, because the equations ($A^4$) and ($B^4$) give this differential relation, analogous to ($I^4$) and deduced by a similar process:

$$\begin{aligned} 0 = f'(\phi) \cdot \phi' + f'(x_1) \cdot x_1' + \ldots + f'(x_n) \cdot x_n' \\ - (u_1 y_1' + \ldots + u_n y_n')\{u_1 f'(u_1) + \ldots + u_n f'(u_n)\}; \end{aligned} \tag{R$^4$}$$

while the same $n+1$ equations, ($A^4$) and ($B^4$), enable us to consider any $n+1$ of the $3n+1$ variables $\phi, x_1, \ldots x_n, y_1, \ldots y_n, u_1, \ldots u_n$, for example, the $n+1$ variables $u_1, \ldots u_n$ and $y_n$, as known functions of the $2n$ other variables $\phi, x_1, \ldots x_n, y_1, \ldots y_{n-1}$. If, then, omitting any one of the $2n$ equations ($P^4$), ($Q^4$), for example, the last of these equations, we can integrate the rest as a system of $2n-1$ total differential equations of the first order between $2n$ variables, such as $\phi, x_1, \ldots x_n, y_1, \ldots y_{n-1}$; and if we then change the $2n-1$ arbitrary constants of this integration, $c_1, \ldots c_{2n-1}$, to so many auxiliary variables $z_1, \ldots z_{2n-1}$, and differentiate them as such; we shall be able, by what was proved in the tenth article, to transform the partial differential equation of the 1st order ($X^1$), in which $y_1, \ldots y_n$ are the partial differential coefficients of $\phi$ taken with respect to $x_1, \ldots x_n$ and which results by elimination from the system ($A^4$) and ($B^4$), into a total differential equation, such as ($G^2$), of the 1st order and 1st degree between the $2n-1$ auxiliary variables $z_1, \ldots z_{2n-1}$. But when, to simplify this preliminary reduction, we select, as we are at liberty to do, the variable $\phi$ itself for that hitherto undetermined and independent variable $x$, on which all the rest are conceived to depend in these total differential equations and with respect to which the total derivatives or total differential coefficients, $\phi', x_1', \ldots x_n'$, $y_1', \ldots y_n'$, have all been imagined to be taken; and when, in consequence, we change $\phi$ to $x$ and $\phi'$ to 1, as at the end of the 10th article; the equations ($P^4$) then become:

$$x_1' = u_1, \ldots x_n' = u_n; \qquad (S^4)$$

the equation ($A^4$) reduces itself to the original equation (A) of the 1st article; the equations ($B^4$) transform themselves to ($D^1$); and the equations ($Q^4$) to ($O^1$); which former equations, (A), ($D^1$) and ($O^1$), had been found in the 7th article to compose a system equivalent to the system of the original equation (A) and its principal supplementaries (B); whereas it was for the very purpose of accomplishing or dispensing with the integration of that system of many total differential equations that we were led to desire to integrate the one partial differential equation. Since, then, the preliminary reduction required by the general method of Pfaff (for the transformation of any partial differential equation of the first order between $n+1$ variables to a total differential equation of the 1st order and 1st degree between $2n-1$ other variables) conducts us back, in the research of a principal integral, to that very system of total differential equations with which we originally set out, and requires that those equations should previously be integrated as an auxiliary system, it appears impossible to derive any aid from this method of that eminent mathematician towards completing the solution of the special problem of the present Chapter.

**13.** Yet some interesting consequences result from the foregoing discussion; and especially the existence of a very intimate connexion between the *general integral* of any proposed partial differential equation of the first order with any number of variables, (a few particular forms being excepted,) and the *principal integral* of a certain total differential equation of the same order, involving the same number of variables; which connexion, if it fails to enable us to discover the latter integral through the former, yet at least allows us to deduce conversely the former from the latter; and gives thereby a new importance to the theory of principal integrals. For it is easy now to perceive (from the investigations of the last article) that in order to integrate any proposed partial differential equation of the 1st order

$$0 = F(\phi, x_1, \ldots x_n, y_1, \ldots y_n); \qquad (X^1)$$

in which $F$ denotes a known function of $\phi, x_1, \ldots x_n, y_1, \ldots y_n$, while $\phi$ denotes a sought function of the $n$ independent variables $x_1, \ldots x_n$ and $y_1, \ldots y_n$ denote its partial differential coefficients of the 1st order

$$y_1 = \phi'(x_1), \ldots y_n = \phi'(x_n); \tag{Y$^1$}$$

we may in general proceed as follows. We may change the previously dependent variable or function $\phi$ to a new and independent variable $x$, on which the previously independent variables $x_1, \ldots x_n$ are now to be imagined to depend; so that $x_1, \ldots x_n$ are now to be considered as functions of $x$, of which the 1st derived functions or differential coefficients may be denoted by $x_1', \ldots x_n'$. These $n$ derived functions are next to be conceived as being connected with the functions $x_1, \ldots x_n$ themselves and with the independent variable $x$ by a differential relation of the form

$$0 = f(x, x_1, \ldots x_n, x_1', \ldots x_n'), \tag{A}$$

obtained by eliminating $y_1, \ldots y_n$ between the $n+1$ equations following,

$$0 = F(x, x_1, \ldots x_n, y_1, \ldots y_n), \tag{P$^1$}$$

and

$$\left.\begin{aligned} x_1' &= \frac{F'(y_1)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' &= \frac{F'(y_n)}{y_1 F'(y_1) + \ldots + y_n F'(y_n)}; \end{aligned}\right\} \tag{Y$^2$}$$

and then we are to find, if we can, the *principal integral* of the total differential equation (A), which will (by the theory of such principal integrals, explained in the early articles of the present chapter) be of the form

$$0 = \psi(x, x_1, \ldots x_n, a, a_1, \ldots a_n), \tag{P}$$

$a_1, \ldots a_n$ being the initial values of the variables $x_1, \ldots x_n$ corresponding to the initial value $a$ of $x$. This *principal integral* (P) of the total differential equation (A), when it is changed by restoring $\phi$ for $x$ to the form

$$0 = \psi(\phi, x_1, \ldots x_n, a, a_1, \ldots a_n), \tag{U$^1$}$$

is itself (by what has been already proved) a *particular integral* of the proposed partial differential equation (X$^1$); and since it contains $n$ arbitrary constants $a_1, \ldots a_n$ (besides the initial value $a$ of $x$, which may be treated as $=0$ or any other assumed and absolute number) we may immediately deduce from it the *general integral* of the same proposed equation (X$^1$) by the process mentioned at the end of the 9th article; namely, by treating these $n$ arbitrary constants as so many connected variables, of which any one, for example $a_1$, may be considered as an arbitrary function of the rest, and by then eliminating, or conceiving eliminated, the $n$ constants $a_1, \ldots a_n$, thus rendered variable, between the particular integral (U$^1$) and the $n$ equations following:

$$\left.\begin{aligned} a_1 &= \chi(a_2, a_3, \ldots a_n), \\ \chi'(a_2) &= -\frac{\psi'(a_2)}{\psi'(a_1)}, \ldots \chi'(a_n) = -\frac{\psi'(a_n)}{\psi'(a_1)}. \end{aligned}\right\} \tag{T$^4$}$$

It results also from what has been shown in former articles that by eliminating $x_1', \dots x_n'$ between the following equations

$$\left.\begin{aligned} y_1 &= \frac{f'(x_1')}{x_1'f'(x_1')+\dots+x_n'f'(x_n')}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= \frac{f'(x_n')}{x_1'f'(x_1')+\dots+x_n'f'(x_n')}, \end{aligned}\right\} \qquad (\mathrm{D}^1)$$

with the help of the relation (A), we should (in general) return to the equation ($\mathrm{P}^1$); or, putting $\phi$ for $x$, to the proposed partial differential equation ($\mathrm{X}^1$): so that the supposition made respecting that equation at the beginning of the 12th article, namely, that it results by elimination from relations of the forms ($\mathrm{A}^4$) and ($\mathrm{B}^4$), is generally permitted and ought not to be considered as restricting to any particular class the form of the function $F$, or of the partial differential equation. However, like most (and perhaps all) other general methods, the method of integration proposed in the present article is subject to some particular exceptions, among which the most important is the case where the proposed partial differential equation ($\mathrm{X}^1$), or the proposed function $F$, is *linear* with respect to all the $n$ partial differential coefficients $y_1, \dots y_n$ of the sought function $\phi$; so that the $n$ derivatives $F'(y_1), \dots F'(y_n)$ and also the combination $y_1F'(y_1)+\dots+y_nF'(y_n)$ are all, either immediately or at least in virtue of the proposed equation $F=0$, expressed as known functions of the sought function $\phi$ (or $x$) and of the $n$ independent variables $x_1, \dots x_n$, not involving $y_1, \dots y_n$. For then the variables $y_1, \dots y_n$ disappear from each of the $n$ equations ($\mathrm{V}^2$), and therefore we cannot deduce from the system ($\mathrm{X}^1$), ($\mathrm{Y}^2$), in this particular case, (although we can in general,) any one determinate relation, such as (A), between $x, x_1, \dots x_n, x_1', \dots x_n'$, to the exclusion of all other such relations; since in the present particular case we have many different relations of that form: whereas the determinateness or uniqueness of the relation thus deduced is essential to the success of our method. On the other hand, in this particular and simple case, when the proposed partial differential equation ($\mathrm{X}^1$) is *linear*, we know from the researches of Lagrange that a particular and comparatively simple method may be applied, in which the equations ($\mathrm{V}^2$) are still useful as auxiliary relations; and which consists in integrating those relations ($\mathrm{V}^2$) as an auxiliary system of $n$ total differential equations between the $n+1$ variables $x, x_1, \dots x_n$, and in then treating any one of the $n$ arbitrary constants of this particular and auxiliary integration as an arbitrary function of the rest.

**14.** It may serve to illustrate still more fully the intimate connexion which exists between the theory of the general integral of a partial differential equation and that of the principal integral of a total differential system, and thus to exhibit more plainly the meaning and utility of the latter, if we suppose for a moment that this latter theory is complete and avail ourselves of its assistance to accomplish the several processes required for the completion of the former theory, as set forth in the method of Pfaff.

[Here the manuscript ends.]