IV.

ON THE APPLICATION TO DYNAMICS OF A GENERAL MATHEMATICAL METHOD PREVIOUSLY APPLIED TO OPTICS

[*British Association Report*, 1834, pp. 513–518.]

The method is founded on a combination of the principles of variations with those of partial differentials, and may suggest to analysts a separate branch of algebra, which may be called, perhaps, the *Calculus of Principal Functions*; because, in all the chief applications of algebra to physics and in a very extensive class of purely mathematical questions, it reduces the determination of many mutually connected functions to the search and study of one principal or central relation. In applying this method to Dynamics, (having previously applied it to Optics,) Professor Hamilton has discovered the existence of a principal function which, if its form were fully known, would give by its partial differential coefficients all the intermediate and all the final integrals of the known equations of motion.

Professor Hamilton is of opinion that the mathematical explanation of all the phenomena of matter distinct from the phenomena of life will ultimately be found to depend on the properties of systems of attracting and repelling points. And he thinks that those who do not adopt this opinion in all its extent must yet admit the properties of such systems to be more highly important in the present state of science than any other part of the application of mathematics to physics. He therefore accounts it the capital problem of Dynamics "to determine the $3n$ rectangular coordinates, or other marks of position, of a free system of n attracting or repelling points as functions of the time," involving also $6n$ initial constants which depend on the initial circumstances of the motion and involving besides n other constants called the masses, which measure, for a standard distance, the attractive or repulsive energies.

Denoting these n masses by $m_1, m_2, \dots m_n$ and their $3n$ rectangular coordinates by $x_1, y_1, z_1, \dots x_n, y_n, z_n$ and also the $3n$ component accelerations, or second differential coefficients of these coordinates taken with respect to the time, by $x_1'', y_1'', z_1'', \dots x_n'', y_n'', z_n''$, he adopts Lagrange's statement of this problem; namely, a formula of the following kind,

$$\Sigma m\,(x''\delta x + y''\delta y + z''\delta z) = \delta U, \tag{1.}$$

in which U is the sum of the products of the masses, taken two by two, and then multiplied by each other and by certain functions of their mutual distances such that their first derived functions express the laws of their mutual repulsion, being negative in the case of attraction. Thus, for the solar system, each product of two masses is to be multiplied by the reciprocal of their distance and the results are to be added in order to compose the function U.

Mr. Hamilton next multiplies this formula of Lagrange by the element of the time dt, and integrates from the time 0 to the time t, considering the time and its element as not subject at present to the variation δ. He denotes the initial values, or values at the time 0, of the coordinates x, y, z and of their first differential coefficients x', y', z' by a, b, c and a', b', c'; and thus he obtains from Lagrange's formula (1.) this other important formula

$$\Sigma m\,(x'\delta x + y'\delta y + z'\delta z - a'\delta a - b'\delta b - c'\delta c) = \delta S, \tag{2.}$$

S being the definite integral

$$S = \int_0^t \left\{ U + \Sigma \frac{m}{2}(x'^2 + y'^2 + z'^2) \right\} dt. \tag{3.}$$

If the known equations of motion, of the forms

$$m_i x_i'' = \frac{\delta U}{\delta x_i}, \quad m_i y_i'' = \frac{\delta U}{\delta y_i}, \quad m_i z_i'' = \frac{\delta U}{\delta z_i}, \tag{4.}$$

had been completely integrated, they would give the $3n$ coordinates x, y, z and therefore also S as a function of the time t, the masses $m_1, \ldots m_n$ and the $6n$ initial constants a, b, c, a', b', c'; so that, by eliminating the $3n$ initial components of velocities a', b', c', we should in general obtain a relation between the $7n+2$ quantities $S, t, m, x, y, z, a, b, c$ which would give S as a function of the time, the masses and the final and initial coordinates. We do not yet know the form of this last function, but we know its variation (2.) taken with respect to the $6n$ coordinates; and on account of the independence of their $6n$ variations we can resolve this expression (2.) into two groups containing each $3n$ equations: namely,

$$\frac{\delta S}{\delta x_i} = m_i x_i', \quad \frac{\delta S}{\delta y_i} = m_i y_i', \quad \frac{\delta S}{\delta z_i} = m_i z_i', \tag{5.}$$

and

$$\frac{\delta S}{\delta a_i} = -m_i a_i', \quad \frac{\delta S}{\delta b_i} = -m_i b_i', \quad \frac{\delta S}{\delta c_i} = -m_i c_i'; \tag{6.}$$

the first members being partial differential coefficients of the function S, which Mr. Hamilton calls the *Principal Function* of motion of the attracting or repelling system. He thinks that, if analysts had perceived this principal function S and these groups of equations (5.) and (6.), they must have perceived their importance. For the group (5.) expresses the $3n$ intermediate integrals of the known equations of motion (4.) under the form of $3n$ relations between the time t, the masses m, the varying coordinates x, y, z, the varying components of velocities x', y', z' and the $3n$ initial constants a, b, c; while the group (6.) expresses the $3n$ final integrals of the same known differential equations as $3n$ relations, with $6n$ initial and arbitrary constants a, b, c, a', b', c', between the time, the masses and the $3n$ varying coordinates. These $3n$ intermediate and $3n$ final integrals it was the problem of dynamics to discover. Mathematicians had found seven intermediate and none of the final integrals.

Professor Hamilton's solution of this long celebrated problem contains, indeed, one unknown function, namely, the *principal function* S, to the search and study of which he has reduced mathematical dynamics. This function must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations. Lagrange's function *states*, Mr. Hamilton's function would *solve* the problem. The one serves to form the *differential* equations of motion, the other would give their *integrals*. To assist in pursuing this new track and in discovering the form of this new function, Mr.

Hamilton remarks that it must satisfy the following partial differential equation of the first order and second degree, (the time being now made to vary,)

$$\frac{\delta S}{\delta t}+\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S}{\delta x}\right)^2+\left(\frac{\delta S}{\delta y}\right)^2+\left(\frac{\delta S}{\delta z}\right)^2\right\}=U; \tag{7.}$$

which may rigorously be thus transformed, by the help of the equations (5.),

$$S=S_1+\int_0^t\left(U-\frac{\delta S_1}{\delta t}-\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S_1}{\delta x}\right)^2+\left(\frac{\delta S_1}{\delta y}\right)^2+\left(\frac{\delta S_1}{\delta z}\right)^2\right\}\right)dt$$

$$+\int_0^t\Sigma\frac{1}{2m}\left\{\left(\frac{\delta S}{\delta x}-\frac{\delta S_1}{\delta x}\right)^2+\left(\frac{\delta S}{\delta y}-\frac{\delta S_1}{\delta y}\right)^2+\left(\frac{\delta S}{\delta z}-\frac{\delta S_1}{\delta z}\right)^2\right\}dt, \tag{8.}$$

S_1 being any arbitrary function of the same quantities t, m, x, y, z, a, b, c, supposed only to vanish (like S) at the origin of time. If this arbitrary function S_1 be so chosen as to be an approximate value of the sought function S, (and it is always easy so to choose it,) then the two definite integrals in the formula (8.) are small, but the second is in general much smaller than the first; it may, therefore, be neglected in passing to a second approximation, and in calculating the first definite integral the following approximate forms of the equations (6.) may be used,

$$\frac{\delta S_1}{\delta a}=-ma', \quad \frac{\delta S_1}{\delta b}=-mb', \quad \frac{\delta S_1}{\delta c}=-mc'. \tag{9.}$$

In this manner, a first approximation may be successively and indefinitely corrected. And for the practical perfection of the method nothing further seems to be required, except to make this process of correction more easy and rapid in its applications.

Professor Hamilton has written two Essays on this new method in Dynamics, and one of them is already printed in the second part of the *Philosophical Transactions* (of London) for 1834.* The method did not at first present itself to him under quite so simple a form. He used at first a *Characteristic Function* V, more closely analogous to that optical function which he had discovered, and had denoted by the same letter, in his *Theory of Systems of Rays*. In both optics and dynamics this function was the quantity called *Action*, considered as depending (chiefly) on the final and initial coordinates. But when this *Action-Function* was employed in dynamics it involved an auxiliary quantity H, namely, the known constant part in the expression of half the living force of a system; and many troublesome eliminations were required in consequence, which are avoided by the new form of the method.

Mr. Hamilton thinks it worth while, however, to point out briefly a new property of this constant H, which suggests a new manner of expressing the differential and integral equations of motion of an attracting or repelling system. It is often useful to express the $3n$ rectangular coordinates $x_1, y_1, z_1, \dots x_n, y_n, z_n$ as functions of $3n$ other marks of position, which may be thus denoted, $\eta_1, \eta_2, \dots \eta_{3n}$; and if $3n$ other new variables, $\varpi_1, \varpi_2, \dots \varpi_{3n}$, be introduced and defined as follows,

$$\varpi_i=\Sigma m\left(x'\frac{\delta x}{\delta\eta_i}+y'\frac{\delta y}{\delta\eta_i}+z'\frac{\delta z}{\delta\eta_i}\right), \tag{10.}$$

* [Pp. 103–161 of this volume.]

it is in general possible to express, reciprocally, the $6n$ variables x, y, z, x', y', z' as functions of these $6n$ new variables η, ϖ; it is, therefore, possible to express as such a function the quantity

$$H = \Sigma \frac{m}{2}(x'^2 + y'^2 + z'^2) - U \qquad (11.)$$

under the form

$$H = F(\varpi_1, \ldots \varpi_{3n}, \eta_1, \ldots \eta_{3n}) - U(\eta_1, \ldots \eta_{3n}), \qquad (12.)$$

in which the part F is rational, integer and homogeneous of the second dimension with respect to the variables ϖ. Now Mr. Hamilton has found that when the quantity H is expressed in this last way as a function of these $6n$ new variables, η, ϖ, its variation may be put under this form,

$$\delta H = \Sigma(\eta'\delta\varpi - \varpi'\delta\eta), \qquad (13.)$$

η', ϖ' denoting the first differential coefficients of these new variables η, ϖ, considered as functions of the time. The $3n$ differential equations of motion of the second order, (4.), between the rectangular coordinates and the time, for any attracting or repelling system, may therefore be generally transformed into twice that number of equations of the first order between these $6n$ variables and the time of the forms

$$\eta_i' = \frac{\delta H}{\delta \varpi_i}, \quad \varpi_i' = -\frac{\delta H}{\delta \eta_i}. \qquad (14.)$$

To integrate this system of equations is to assign from them $6n$ relations between the time t, the $6n$ variables η_i, ϖ_i and their $6n$ initial values which may be called e_i, p_i. Mr. Hamilton resolves the problem, under this more general form, by the same *principal function* S as before, regarding it however as depending now on the new marks η, e of final and initial positions of the various points of the system. For, putting in this new notation

$$S = \int_0^t \left(\Sigma\varpi\frac{\delta H}{\delta\varpi} - H\right)dt, \qquad (15.)$$

and considering the time as given, he finds now the formula of variation

$$\delta S = \Sigma(\varpi\delta\eta - p\delta e), \qquad (16.)$$

and therefore the $6n$ separate equations

$$\varpi_i = \frac{\delta S}{\delta \eta_i}, \quad p_i = -\frac{\delta S}{\delta e_i}, \qquad (17.)$$

which are forms for the sought relations.

Professor Hamilton thinks that these two formulae of variation, (13.) and (16.), namely,

$$\delta H = \Sigma(\eta'\delta\varpi - \varpi'\delta\eta) \qquad (A.)$$

and

$$\delta S = \Sigma(\varpi\delta\eta - p\delta e), \qquad (B.)$$

are worthy of attention as expressing, under concise and simple forms, the one the differential and the other the integral equations of motion of an attracting or repelling system. They may be extended to other problems of dynamics besides this capital problem. The expression H can always easily be found and the function S can be determined with indefinite accuracy by a method of successive approximation of the kind already explained.

The properties of his *Principal Function* are treated of more fully in his "Second Essay on a General Method in Dynamics*"; in which he has introduced several forms of a certain *Function of Elements*, connected with the Principal Function and with each other, and adapted to questions of perturbation; and has shown that for the perturbations of a ternary or multiple system with any laws of attraction or repulsion and with one predominant mass the differential equations of the varying elements of *all* the smaller masses may be expressed together, and as simply as in the usual way, by the coefficients of *one* disturbing function, (namely, the disturbing part of the whole expression H,) and may be integrated rigorously by a corollary of his general method.

* This essay will be found in the *Philosophical Transactions* for 1835. [Pp. 162–211 of this volume.]