NOTES AND REFERENCES.

302, 305. The theory of curves in space was proposed as the subject of the Prize-question of the Steiner Foundation by the Academy of Sciences of Berlin in the year 1881 "irgend eine auf die Theorie der höheren algebraischen Raumcurven sich beziehende Frage von wesentlicher Bedeutung vollstandig zu erledigen," and the prize was divided between the two memoirs

Halphen, "Mémoire sur la classification des courbes gauches algébriques." *Jour. École Polyt.* Cah. LII. (1882), pp. 1—200, and

Nöther, "Zur Grundlegung der Theorie der algebraischen Raumcurven." *Abh. der Akad. zu Berlin vom Jahre* 1882, pp. 1 to 120; both treating of the classification of curves in space.

We have also the valuable memoir

Valentiner, "Zur Theorie der Raumcurven," *Acta Mathematica*, t. II., 1883, pp. 136—230, which relates less directly to the question of classification.

The three authors all refer to these papers in the *Comptes Rendus*, and make considerable use of my conception of the monoid surface. It would be out of place to attempt to give any account here of these memoirs: I only refer to such remarks or theorems contained in them as stand in immediate connection with the remarks which follow.

The question of classification is much simplified by excluding from consideration the curves with singular points (that is actual double points and stationary points), and this is in fact done both by Halphen and Nöther and in the present Note. The curves considered are thus curves with only apparent double points (adps.) viz. for a curve of the order d (I use Halphen's letters) with h apparent double points, taking an arbitrary point as vertex, the cone through the curve is a cone of the order d, with h nodal lines, each of these meeting the curve in two (real or imaginary) non-coincident points. Such a curve is the partial intersection of the cone in question say $U, =(x, y, z)^d, =0$ with a monoid surface $w, =\dfrac{(x, y, z)^{k+1}}{(x, y, z)^k}, =\dfrac{Q}{P}$, where the inferior cone $P, =(x, y, z)^k, =0$, of the monoid surface, and the superior cone $Q, =(x, y, z)^{k+1}, =0$, of the monoid surface each of them pass through all the h

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nodal lines of the cone, and besides through θ lines of the cone: the complete intersection of the cone and monoid surface is thus made up of the curve once, the h nodal lines each twice, and the θ lines each once; and if as before the order of the curve is $=d$, then we thus have $(k+1)\,d = d + 2h + \theta$, viz. we must have $kd = 2h + \theta$, as the condition to be satisfied in order that the curve of the order d may be the partial intersection of the cone and monoid surface.

In my papers in the *Comptes Rendus* I endeavoured to find, and Halphen and Nöther both endeavour to find, the surfaces of lowest order which have the curve of order d for their complete or partial intersection. This (although, as will presently appear, the theory may be considered in a more complete form) is an important and interesting question; but upon further reflection it appears to me that it is a question beside that which first presents itself and ought to be in the first instance considered, viz. this is the question of the classification of curves in space according to the foregoing representation of any such curve as the partial intersection of a cone and monoid surface. Supposing it effected, and a kind of curve completely defined according to this mode of representation, then there arises the further question to which I have referred (Salmon's *Solid Geometry*, Ed. 3, (1874), p. 285, and Ed. 4, (1882), p. 281), viz. we may have passing through any given curve a *complete* system of surfaces, that is a system $U=0$, $V=0$, $W=0, \ldots$ where these functions are not connected by any such equation as $U = NV + PW + \ldots$, and where every other surface which passes through the curve is expressible in the form $MU + NV + PW + \ldots = 0$. It is not easy to prove (but as to this see Hilbert "Zur Theorie der algebraischen Gebilde," *Göttingen Nachrichten*, 1888, p. 454), but it may be safely assumed that for a curve of any given order whatever, the number of equations in such a complete system is finite, and we have thus the representation of a curve in space by means of a complete system of surfaces passing through it. Obviously the curve is here the partial (or if the system consists of only two surfaces then the complete) intersection say of the two surfaces $U=0$, $V=0$ of lowest order passing through it, which is the question above referred to.

Reverting to the representation by the cone and monoid surface, Halphen gives the capital theorem, that if we have any particular inferior cone $P=0$ passing through the curve, then we may without loss of generality take the equation of the monoid surface to be $w = \frac{Q}{P}$: viz. if instead hereof the equation of the monoid surface is taken to be $w = \frac{Q'}{P'}$, then this equation in virtue of the equation $U=0$ of the cone is always reducible to the first mentioned form $w = \frac{Q}{P}$; that is in virtue of the equation $U=0$, we have $w = \frac{Q'}{P'} = \frac{Q}{P}$, or what is the same thing, $\frac{Q'}{P'} = \frac{Q}{P}$ in virtue of $U=0$, that is $Q'P - QP' = MU$, where M is a rational and integral function $(x, y, z)^\lambda$ of the degree λ, $= k + n + 1 - d$, if k be the degree of P' and n that of P.

It thus appears that if n be the order of the cone of lowest order which passes through the h nodal lines of the cone $U=0$, then we have always functions Q, P

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of the orders $n+1$, n respectively, such that the equation of the monoid surface is $w=\frac{Q}{P}$. Or what is the same thing, we have always a monoid surface of the order $n+1$: we thus arrive at the notion of Halphen's characteristic n.

Instead of the foregoing equation $kd=2h+\theta$, we thus have $nd=2h+\theta$, and for given values of d, h there is thus a minimum value of n (viz. nd must be at least $=2h$); there is also a maximum value of n, viz. this is the least value for which $\frac{1}{2}n(n+3)$ is $=$ or $<h$, for with such a value of n there is always through the h nodal lines a cone of the order n.

For a given value of d, we have $h=$ at most $\frac{1}{2}(d-1)(d-2)$, and Halphen shows that h must be at least $=[\frac{1}{4}(d-1)^2]$, if we denote in this manner the integral part of the expression within the brackets. And then, h having any value between these limits, for any given values of d, h we have by what precedes a certain number of values of n.

We thus have *primâ facie* curves in space of the several forms (d, h, n): but it may very well be, and in fact Halphen finds that when d is $=9$ or upwards, then for certain values of h, n as above, there is not any curve (d, h, n): thus $d=9$, $h=17$, the values of n are $n=4$ or 5, but there is not any curve $d=9$, $h=17$, for either of these values of n; or say the curves (9, 17, 4) and (9, 17, 5) are non-existent.

And I notice further that in certain cases for which Halphen finds a curve (d, h, n) such curve does not exist except for special configurations of the h nodal lines not determined by the mere definition of n as the order of the cone of lowest order which passes through the h nodal lines: for instance $d=9$, $h=16$, $n=4$ for which Halphen gives a curve, I find that it is not enough that the 16 nodal lines are situate on a quartic cone, but that they must be the 16 lines of intersection of two quartic cones.

I remark moreover that Halphen does not carry out the foregoing principle of classification according to the values of (d, h, n): thus $d=9$, $h=22$, the values of n are 6 and 5; viz. the 22 nodal lines are in general on a sextic cone but they may be on a quintic cone; the curves (9, 22, 6) and (9, 22, 5) exist each of them, but he gives only the former of the two forms. The form (9, 22, 6) has a capacity 36 (depends upon 36 constants) but (9, 22, 5) a capacity 35 only, and I assume that Halphen considered it as a particular case of (9, 22, 6), (there is it seems to me a want of precision in his definition of a family)—but I consider that this is an abandonment of the principle—the two curves differ *ipso facto* in that in the first form the 22 nodal lines are not, in the second form they are, on a quintic curve. In Nöther's theory the characteristic n does not present itself.

Resuming the general theory, and considering d, h, n as given, we start from the cone $U=0$ of the order d, with h nodal lines lying in a cone of the order n: we take $P=0$ a cone of the order n passing through the h nodal lines, and besides meeting the cone $U=0$ in θ lines; $nd=2h+\theta$, (where θ may be $=0$). And we then have $Q=0$ a cone of the order $n+1$ passing through the h lines and the θ lines;

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and this being so we have $w=\dfrac{Q}{P}$ for the equation of the monoid surface, and consequently $U=0$ and $w=\dfrac{Q}{P}$ for the equations of the curve, viz. the cone $U=0$ and the monoid surface of the order $n+1$ meet in the h lines each twice, in the θ lines, and in the curve of the order d; $(n+1)\,d=2h+\theta+d$. Observe here that the cone $Q=0$ as a cone of the order $n+1$ subjected only to the conditions of passing through the h lines and the θ lines has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$; this number should be $=3$ at least, for if it were $=2$, we should have $Q=(x+\beta y+\gamma z)\,P$ (since $P=0$ is a cone of the next inferior order through the same $h+\theta$ lines), and thus the curve would be a plane curve. Observe further that the cone $U=0$, quà cone of the order d with h nodal lines has in general a capacity $=\frac{1}{2}d(d+3)-h$; the cone $P=0$, by what precedes may be regarded as determinate, and the cone $Q=0$ as just appearing has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$; there is a term $+1$ for the implicit constant factor in the function Q, and we thus find for the capacity of the curve the expression $\frac{1}{2}d(d+3)-h+1+\frac{1}{2}(n+1)(n+4)-h-\theta$, viz. this is $=\frac{1}{2}d(d+3)+\frac{1}{2}(n^2+5n)+3-nd$, $=\frac{1}{2}(d-n)^2+\frac{1}{2}(3d+5n)+3$, which putting for a moment $d-n=\alpha$ is $=\frac{1}{2}\alpha^2+\frac{1}{2}(8d-5\alpha)+3$, $=4d+\frac{1}{2}(\alpha-2)(\alpha-3)$; hence restoring for α its value, we find capacity of curve $=4d+\frac{1}{2}(d-2-n)(d-3-n)$: in particular if $n=d-2$ or $d-3$, the capacity is $=4d$.

We are thus able in the case where $\frac{1}{2}(n+1)(n+4)-h-\theta=3$ or more, say $\frac{1}{2}n(n+5)$ = or > $h+\theta+1$, actually to construct the equation of a curve (d, h, n), having in the case where $n=d-2$ or $d-3$ a capacity $=4d$: the conditions in question for any given value of d, are satisfied by the considerable number of curves which form Halphen's "premier groupe."

For instance $d=9$, then the complete table of the values of h, n, θ is

d	h	n	θ	Cap.
9	16	4	4	38
		5	13	0
	17	4	2	0
		5	11	0
	18	4	0	36
		5	9	36
	19	5	7	36
	20	5	5	36
	21	5	3	36
		6	12	0
	22	5	1	35
		6	10	36†
	23	6	8	36†
	24	6	6	36†
	25	6	4	36†
	26	6	2	36†
	27	6	0	36†
	28	7	7	36†

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and the conditions are satisfied for those values of (d, h, n) against which I have set the capacity 36†. I do not explain the remaining figures of the column of capacities, but remark only that 0 means that the curve is non-existent, and that 35 refers to the curve (9, 22, 5) which is alluded to above as not specified by Halphen.

It is important to remark that if the above-mentioned condition $\frac{1}{2}n(n+5) =$ or $> h+\theta+1$, or restoring it to the original form $\frac{1}{2}(n+1)(n+4)-h-\theta = 3$ at least, is *not* satisfied, then it by no means follows, and it is not in general the case, that the curve is non-existent: I have said only that the cone $Q=0$ has in general a capacity $=\frac{1}{2}(n+1)(n+4)-h-\theta$, but the configuration of the $h+\theta$ lines may be such as not to impose on the cone $Q=0$ which passes through them so many as $h+\theta$ conditions, and the capacity of the cone may thus be greater than $\frac{1}{2}(n+1)(n+4)-h-\theta$, and may thus be $=3$ at least; moreover supposing that in such a case the curve exists, the capacity of the cone $U=0$ instead of being $=\frac{1}{2}d(d+3)-h$, may very well have, and presumably has, a greater value, and the reasoning by which the capacity of the curve was found to be $=4d+\frac{1}{2}(d-2-n)(d-3-n)$ ceases to be applicable. The theory, as depending upon special configurations of the h lines and the θ lines, is a complicated and difficult one, and I do not attempt to enter upon it.

In conclusion I wish to refer to an important theorem given by Valentiner and also by Halphen and Nöther. Considering in connexion with the curve of the order d, a surface of the order m, then since the capacity hereof (or number of constants contained in its equation) is $=\frac{1}{6}(m+1)(m+2)(m+3)-1$ or $\frac{1}{6}m(m^2+6m+11)$, it is obvious that if this be greater than md, the surface can be made to pass through more than md points of the curve, and thus that the curve will lie upon a surface of the order m. But the condition which has really to be satisfied in order that the curve may lie upon a surface of the order m is a less stringent one: if p be the deficiency of the curve, $=\frac{1}{2}(d-1)(d-2)-h$, if as before the curve is without actual singularities, and h be the number of its apparent double points, then the condition is $\frac{1}{6}m(m^2+6m+11)$ greater than $md-p$, viz. the surface of the order m being made to pass through $md+1-p$ points assumed at pleasure on the curve will *ipso facto* pass through p determinate points of the curve, that is in all through $md+1$ points of the curve, or it will contain the curve. The theorem is true subject only to the limitation $m =$ or $> d-2$. The most simple form of statement is perhaps that given by Valentiner, p. 194 (changing only his letters), viz. if m be $=$ or $> d-2$, the intersections of a surface of the order m with a curve of the order d with h apparent double points are determined by means of

$$dm-\tfrac{1}{2}(d-1)(d-2)+h\,(=dm-p)$$

of these intersections.

312. The generalisation which is here given of Euler's theorem $S+F=E+2$, is a first step towards the theory developed in Listing's Memoir "Census räumlicher Complexe oder Verallgemeinerung des Euler'schen Satzes von den Polyedern." Göttingen Abh. t. x. (1862).

320. The transcendent $i\,\mathrm{gd}(-iu)$, with a pure imaginary argument is the function $\log\tan(\frac{1}{4}\pi+\frac{1}{2}u)$ (hyperbolic logarithm) tabulated by Legendre, *Exer. de Calcul Intégral,*

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t. II. (1816), Table IV. and *Traité des Fonctions Elliptiques*, t. II. (1826), Table IV. at intervals of 30′ from 0° to 90°, to twelve decimals and fifth differences. But the march of the function is somewhat disguised by the argument being taken in degrees and minutes and the function in abstract number. I have in the paper "On the orthomorphosis of the circle into the parabola," *Quart. Math. Jour.* vol. XX. (1885), pp. 213—220, see p. 220, given the table (at intervals of 1° to seven decimals) exhibiting the argument and the function each of them in degrees and minutes and also in abstract number.

335. Besides the 13 numbers mentioned by Gauss it appears by the paper, Perott, "Sur la formation des déterminants irreguliers," *Crelle*, t. XCV. (1883), pp. 232—236, that in the first thousand the determinants -468 and -931 are irregular.

341. Consider the equation of a curve as given in the form $y-f(x)=0$; then in the notation of Reciprocants $(t=y',\ a=\frac{1}{2}y'',\ b=\frac{1}{6}y''',\ c=\frac{1}{24}y'''',\ d=\frac{1}{120}y''''')$, where the accents denote differentiation in regard to x) the equation of the conic of five-pointic contact at the point (x, y) of the curve is

$$\begin{aligned}&a^4(X-x)^2\\&+a^2b(X-x)\{Y-y-t(X-x)\}\\&+(ac-b^2)\quad\{Y-y-t(X-x)\}^2\\&-a^3\qquad\quad\ \{Y-y-t(X-x)\}=0,\end{aligned}$$

which I verify as follows: writing $X=x+\theta$, we have

$$Y=y+t\theta+a\theta^2+b\theta^3+c\theta^4+d\theta^5,$$

and thence

$$Y-y-t(X-x)=\quad a\theta^2+b\theta^3+c\theta^4+d\theta^5.$$

Substituting these values and developing as far as θ^5 we find

$$\begin{aligned}&a^4\theta^2\\&+a^2b\theta(a\theta^2+b\theta^3+c\theta^4)\\&+(ac-b^2)(a^2\theta^4+2ab\theta^5)\\&-a^3(a\theta^2+b\theta^3+c\theta^4+d\theta^5)=0,\end{aligned}$$

viz. this is

$$0\theta^2+0\theta^3+0\theta^4-a(a^2d-3abc+2b^3)\theta^5=0.$$

The equation is thus satisfied as far as θ^4, showing that the conic is a conic of 5-pointic contact; and it will be satisfied as far as θ^5 if only $a(a^2d-3abc+2b^3)=0$. The value $a=0$ belongs to an inflexion, and reduces the equation of the conic to $\{Y-y-t(X-x)\}^2=0$, viz. this is the stationary tangent taken twice, which is in an improper sense a conic of six-pointic contact: the other factor determines a sextactic point, viz. we have $a^2d-3abc+2b^3=0$ as the condition of a sextactic point.

We might from this form, which belongs to the curve as given by the equation $y-f(x)=0$, pass to the form belonging to the curve as given by the equation

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$U, =(x, y, z)^m, =0$, and thus obtain the form given in the memoir, and the process would I can well imagine be a more simple one, but it would certainly be very complicated: as an illustration take the simple case of an inflexion: the condition for this, for the equation $y-f(x)=0$ of the curve is $a=0$, that is $\frac{d^2y}{dx^2}=0$. Passing first to the form $U, =(x, y, 1)^m=0$, we have

$$\frac{dU}{dx}+\frac{dU}{dy}\frac{dy}{dx}=0,$$

and thence

$$\frac{d^2U}{dx^2}+2\frac{d^2U}{dxdy}\frac{dy}{dx}+\frac{d^2U}{dy^2}\left(\frac{dy}{dx}\right)^2+\frac{dU}{dy}\frac{d^2y}{dx^2}=0,$$

viz. substituting for $\frac{dy}{dx}$ its value from the first equation the condition $\frac{d^2y}{dx^2}=0$, becomes

$$\left(\frac{dU}{dy}\right)^2\frac{d^2U}{dx^2}-2\frac{dU}{dy}\frac{dU}{dx}\frac{d^2U}{dxdy}+\left(\frac{dU}{dx}\right)^2\frac{d^2U}{dy^2}=0,$$

and we can then make the further transformation to the form $U=(x, y, z)^m, =0$, and so obtain but not *very* easily the result $H(U)=0$: but in the transformations for the sextactic point, besides the differential coefficient a of the second order we have the coefficients b, c, d of the orders 3, 4 and 5 respectively; and the complication is thus very much greater.

343, 354, 374. The principal paper is 374; 354 is a mere résumé of this; and 343 relates to the higher singularity which first presented itself, and which is there shown to arise from the coalescence of a node and a cusp, but in 374 (where it is considered more fully) it is shown to be equivalent to a node, a cusp, a double tangent and an inflexion.

On the general subject, and founded on 374, we have

Smith, H. J. S., "On the Higher Singularities of Plane Curves," *Proc. Lond. Math. Soc.* vol. VI. (1875), pp. 153—182. The author refers to the two following enquiries:

(1) It is important to prove that the indices of singularity as defined by Professor Cayley satisfy the equations of Plücker; and that the "genus" or "deficiency" of the plane curve is correctly given by these indices.

(2) It is also of interest to examine whether any given singularity can be actually formed by the coalescence of the ordinary singularities to which it is regarded as equivalent: in other words whether a singularity of which the indices are δ, τ, κ, i and which is therefore regarded as equivalent to δ double points, τ double tangents, κ cusps and i inflexions possesses a penultimate form in which all these singularities exist distinct from one another but infinitely close together.

The paper relates chiefly to the first of these enquiries, the second being reserved for a further communication which was never made.

See also Halphen's "Étude sur les points singuliers des courbes algébriques planes," published as an Appendix, pp. 537—648, to the translation of Salmon's *Higher*

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Plane Curves, "Traité de Géométrie Analytique," par G. Salmon traduit par O. Chemin, 8vo. Paris, 1884, and the list of Memoirs given, p. 538.

347. I attach some importance to this short paper as giving my own general views of the subject to which it relates, and in particular as to the line of separation between finite and transcendental analysis.

378. I have printed this Report as it was in some measure in connexion therewith that the Royal Society of London undertook the very important work, their Catalogue of Scientific Papers. I do not remember by whom the Report was drafted but some of the recommendations contained in it are due to me. The Catalogue is on a more extensive plan than that recommended in the Report, inasmuch as it is not limited to Physics and Mathematics but extends to all branches of Natural Knowledge—but it is interesting to compare the extent of it with the estimate in the Report—vols. I. to VI. (1800 to 1863) contain together 5743 pages: vols. VII. and VIII. (1864 to 1873) contain together 2357 pages—the number of entries on a page is about = 30; and we thus have, 1800 to 1863, about 173,000 entries, and 1864 to 1873, about 71,000 entries.

END OF VOL. V.

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