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ON A SPECIAL SEXTIC DEVELOPABLE.

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THE present paper contains some investigations in relation to the special sextic developable or torse

$$(ae-4bd)^3-27(-ad^2-b^2e)^2=0,$$

considered Nos. 26 to 35 of my paper "On Certain Developable Surfaces," *Quarterly Mathematical Journal*, t. VI. (1864), pp. 108—126, [344].

The cuspidal curve is

$$ae-4bd=0,\quad ad^2+b^2e=0,$$

and the nodal curve is

$$ae+2bd=0,\quad ad^2-b^2e=0,$$

viz. to put this in evidence, the equation is to be written in the form

$$(ae+2bd)^2(ae-16bd)-27(ad^2-b^2e)^2=0.$$

The coordinates of a point on the cuspidal curve may be taken to be

$$a=2,\quad b=-t,\quad d=+t^3,\quad e=-2t^4,$$

and then if A, B, D, E are current coordinates, and α, β, δ, ϵ arbitrary coefficients, the equation of a plane through the tangent line is

$$\begin{vmatrix} A, & B, & D, & E \\ 2\,, & -t, & +t^3, & -2t^4 \\ . & -1, & +3t^2, & -8t^3 \\ \alpha\,, & \beta, & \delta\,, & \epsilon \end{vmatrix}=0,$$

which is

$$\left.\begin{aligned}(A\beta-B\alpha)&(-\ t^6)\\+(A\delta-D\alpha)&(-3t^4)\\+(A\epsilon-E\alpha)&(-\ t^3)\\+(B\delta-D\beta)&(-8t^3)\\+(B\epsilon-E\beta)&(-3t^2)\\+(D\epsilon-E\delta)&(-1\ )\end{aligned}\right\}=0,$$

or, what is the same thing,

$$\left.\begin{aligned}\alpha\,(\ \ \ Bt^6+3Dt^4+\ Et^3)\\+\beta\,(-\ At^6+8Dt^3+3Et^2)\\+\delta\,(-3At^4-8Bt^3+\ E\ )\\+\epsilon\,(-\ At^3-3Bt^2-\ D\ )\end{aligned}\right\}=0,$$

and by equating to zero the coefficients α, β, δ, ϵ, we have four equations which are easily seen to reduce themselves to two equations only, and which are in fact the equations of the tangent line, the equations of this line may therefore be taken to be

$$\left.\begin{aligned}At^3+3Bt^2+D=0\\3At^4+8Bt^3-E=0\end{aligned}\right\}.$$

The coordinates of a point in the nodal curve may be taken to be

$$a=\sqrt{2},\quad b=\tau,\quad d=-\tau^3,\quad e=\sqrt{2}\,\tau^4,$$

and substituting these values in the place of A, B, D, E in the equations of the tangent line, we have

$$\sqrt{2}\,t^3+3t^2\tau-\tau^3\qquad=0,$$

$$3\sqrt{2}\,t^4+8t^3\tau-\sqrt{2}\,\tau^4=0,$$

or, what is the same thing,

$$\tau^3-3t^2\tau\qquad+\sqrt{2}\,t^3=0,\ \text{i.e.}\ \{\tau+\sqrt{2}\,t\qquad\}\{\tau^2-\sqrt{2}\,t\tau-t^2\}=0,$$

$$\tau^4-4\sqrt{2}\,t^3\tau-3t^3\quad=0,\qquad\{\tau^2+\sqrt{2}\,t\tau-t^2\}\{\tau^2-\sqrt{2}\,t\tau-t^2\}=0,$$

so that the equations are satisfied by the values of τ given by the equation

$$\tau^2-\sqrt{2}\,t\tau-t^2=0,$$

that is, by the values

$$\tau=\frac{1\pm\sqrt{3}}{\sqrt{2}}\,t,$$

which belong to the points where the tangent line meets the nodal curve. Call these values τ_1 and τ_2; then considering a, b as current coordinates, the values of $a:b$

belonging to the point where the tangent line meets the cuspidal curve considered as *three* coincident points, and to the points where it meets the nodal curve, are given by the equation

$$(2b + at)^3 \{b\sqrt{2} - a\tau_1\} (b\sqrt{2} - a\tau_2\} = 0,$$

that is

$$(2b + at)^3 (2b^2 - 2abt - a^2t^2) = 0,$$

or say

$$(at + 2b)^3 (a^2t^2 + 2abt - 2b^2) = 0.$$

I proceed to find the intersections of the tangent with the Prohessian: for this purpose putting for a moment in the last-mentioned equation x for at and y for b, this is

$$(x + 2y)^3 (x^2 + 2xy - 2y^2) = 0,$$

or, if in the place of $(x+y)$ we write x, this is

$$(x+y)^3 (x^2 - 3y^2) = 0,$$

and the Hessian of this is easily found to be

$$(x+y)^4 (3x^2 + 8xy + 4y^2);$$

whence, replacing x by $(x+y)$, the Hessian of

$$(x + 2y)^3 (\ x^2 + \ 2xy - \ 2y^2),$$

is

$$(x + 2y)^4 (3x^2 + 14xy + 18y^2).$$

We have thus

$$3x^2 + 14xy + 18y^2 = 0;$$

that is

$$3x + \{7 \pm \sqrt{-5}\} y = 0,$$

or

$$3at + \{7 \pm \sqrt{-5}\} b = 0;$$

and therefore

$$\frac{b}{a} = \frac{-3}{7 \pm \sqrt{-5}} t = \frac{-3\{7 \mp \sqrt{-5}\}}{54} t = -\frac{7 \mp \sqrt{-5}}{18} t;$$

or putting

$$n_1 = -\frac{7 + \sqrt{-5}}{18},$$

$$n_2 = -\frac{7 - \sqrt{-5}}{18},$$

so that $n_1 + n_2 = -\frac{7}{9}$, $n_1n_2 = \frac{1}{6}$, and n_1, n_2 are the roots of the equation $18n^2 + 14n + 3 = 0$, then we have $\frac{b}{a} = n_1t$ or n_2t, say $\frac{b}{a} = n_1t$, or assuming $a = 1$, then $b = n_1t$.

But the equations of the tangent line being *ut suprâ*

$$at^3 + 3bt^2 + d = 0,$$

$$3at^4 + 8bt^3 - e = 0,$$

we have thus

$a = 1,$	$a = 1,$
$b = n_1 t,$	$b = n_2 t,$
$d = (-1 - 3n_1)\, t^3,$	$d = (-1 - 3n_2)\, t^3,$
$e = (\ \ 3 + 8n_1)\, t^4,$	$e = (\ \ 3 + 8n_2)\, t^4,$

as the coordinates of the required points, viz. the tangent line meets the Prohessian, in the point on the cuspidal edge considered as 6 points, in two points on the nodal curve and in the last-mentioned 2 points; $6 + 2 + 2 = 10$ the order of the Prohessian.

The foregoing equations give

$$ae - 6bd = 18n_1^2 + 14n_1 + 3 = 0,$$

$$\frac{ad^2}{b^2e} = \frac{(1 + 3n_1)^2}{n_1^2(8n_1 + 3)} = \tfrac{1}{81}(144\, n_1 - 23),$$

$$(\text{in virtue of } 18n_1^2 + 14n_1 + 3 = 0),$$

so that the two points in question are the intersections of the tangent line with the surface $ae - 6bd = 0$.

If we consider the intersection of this surface with the torse

$$(ae - 4bd)^3 - 27\,(-ad^2 - b^2e)^3 = 0,$$

the equation $ae - 6bd = 0$, gives

$$(ae - 4bd)^3 = (2bd)^3 = 8b^2d^2bd = \tfrac{3}{4}\, aeb^2d^2,$$

and thence

$$4ab^2d^2e - 81\,(ad^2 + b^2e)^2 = 0\,;$$

that is

$$81a^2d^4 - 158ab^2d^2e + 81b^4e^2 = 0,$$

an equation which should agree with

$$\frac{ad^2}{b^2e} = \tfrac{1}{81}\,(144\, n_1 - 23).$$

In fact writing

$$x = \tfrac{1}{81}\,(144\, n_1 - 23),$$

the equation $18n_1^2 + 14n_1 + 3 = 0$ is $(18n_1 + 7)^2 + 5 = 0$; that is, $(144\, n_1 + 56)^2 + 320 = 0$, but $144\, n_1 + 56 = 81x + 79$, or the equation becomes $(81x + 79)^2 + 320 = 0$, that is

$$\overline{81}^2x^2 + 81\,.\,158\, x + \overline{81}^2 = 0, \text{ or } 81x^2 + 158\, x + 81 = 0,$$

which is right.

Consider in like manner the intersection of the torse with the surface $ae-\lambda bd=0$, where λ is a given constant coefficient; we have

$$(ae-4bd)^3=(\lambda-4)^3 b^3d^3=\frac{(\lambda-4)^3}{\lambda}aeb^2d^2,$$

and therefore

$$(\lambda-4)^3 aeb^2d^2-27\lambda(ad^2+b^2e)^2=0,$$

that is

$$27\lambda a^2d^4+[54\lambda-(\lambda-4)^3]\,ab^2d^2e+27\lambda b^2e^4=0,$$

which gives

$$ad^2-\theta_1 b^2e=0, \text{ or } ad^2-\theta_2 b^2e=0,$$

if θ_1, θ_2 are the roots of

$$27\lambda\theta^2+[54\lambda-(\lambda-4)^3]\,\theta+27\lambda=0.$$

The surfaces $ae-\lambda bd=0$, $ad^2-\theta_1 b^2e=0$ have in common the two lines $(a=0,\ b=0)$ and $(d=0,\ e=0)$, and they intersect besides in a quartic curve. And so for the surfaces $ae-\lambda bd=0$, $ad^2-\theta_2 b^2e=0$. That is, the surface $ae-\lambda bd=0$ intersects the torse $(ae-4bd)^2-27(-ad^2-b^2e)^2=0$, in the line $a=0$, $b=0$ twice, in the line $d=0$, $e=0$ twice, and in two quartic (excubo-quartic) curves. The two quartic curves become identical, if

$$(54\lambda)^2=\{54\lambda-(\lambda-4)^3\}^2,$$

that is

$$\pm 54\lambda=54\lambda-(\lambda-4)^3,$$

and therefore, if either

$$(\lambda-4)^3 \qquad =0,$$

which gives the cuspidal curve; or else if

$$(\lambda-4)^3-108\lambda=0,$$

that is

$$\lambda^3-12\lambda^2-60\lambda-64=(\lambda+2)^2(\lambda-16)=0.$$

$(\lambda+2)^2=0$ or $\lambda=-2$ gives the nodal curve: $\lambda-16=0$ gives $ae-16bd=0$, a surface which intersects the developable in the line $a=0$, $b=0$ twice, in the line $d=0$, $e=0$ twice, and in the two coincident quartic (excubo-quartic) curves given by the equations $ae-16bd=0$, $ad^2-b^2e=0$. As a verification, I remark, that the surface $ad^2-b^2e=0$ combined with the developable gives

$$(ae-4bd)^3-27(ad^2+b^2e)^2=(ae-4bd)^3-108\,ab^2d^2e=0,$$

that is $(ae+2bd)^2(ae-16bd)=0$, or it meets the developable in its curve of intersection with $ae+2bd=0$ twice, and in its curve of intersection with $ae-16bd=0$; that is, in the line $a=0$, $b=0$ three times, in the line $d=0$, $e=0$ three times, in the nodal quartic $ae+2bd=0$, $ad^2-b^2e=0$ twice, and in the quartic $ae-16bd=0$, $ad^2-b^2e=0$ once; $3+3+8+4=18$, the order of the complete intersection.

Greenwich, January 4, 1864.

In my theory of the singularities of curves and torses, *Liouville*, t. x. (1845) pp. 245—250, [30], translated under the title "On Curves of Double Curvature and Developable Surfaces," *Cambridge and Dublin Mathematical Journal*, t. v. (1850), pp. 18—22, [83], I omitted to take account of a noteworthy singularity, viz. this is, the stationary tangent line; or when the system has three consecutive points in a line, or, what is the same thing, three consecutive planes through a line. I reproduce the theory with this addition as follows. We have

m, the order of the system, = order of the curve,

r, „ rank of the system, = class of curve, = order of torse,

n, „ class of the system, = class of torse.

α, „ number of stationary planes,

β „ „ stationary points,

ϑ „ „ stationary lines,

g „ „ lines in two planes,

h „ „ lines through two points,

x „ „ points in two lines,

y „ „ planes through two lines.

This being so, the section of the torse by an arbitrary plane is a plane curve for which

r is the order,

n „ class,

x „ number of nodes,

$m+\vartheta$ „ „ cusps,

g „ „ double tangents,

α „ „ inflexions;

and we have thence Plücker's six equations, which may be considered as included in the three equations

$$n = r(r-1) - 2x - 3(m+\vartheta),$$

$$\alpha = 3r(r-2) - 6x - 8(m+\vartheta),$$

$$r = n(n-1) - 2g - 3\alpha.$$

Similarly considering the cone standing on the curve and having an arbitrary point for vertex, then for this cone

m is the order,

r „ class,

h „ number of nodal lines,

β „ „ cuspidal lines,

y „ „ double tangent planes,

$n+\vartheta$ „ „ inflexions;

and we have Plücker's six equations, which may be considered as included in the three equations

$$m = r(r-1) - 2y - 3(n+\vartheta),$$
$$\beta = 3r(r-2) - 6y - 8(n+\vartheta),$$
$$r = m(m-1) - 2h - 3\beta.$$

These two systems constitute together a system of six equations between the ten quantities $m, r, n, \alpha, \beta, \vartheta, g, h, x, y$. Considering m, r, x, ϑ as arbitrary, the six equations determine the remaining quantities $n, \alpha, \beta, h, x, y$.

The curve

$$ae - 4bd + 3c^2 = 0, \quad ace + 2bcd - ad^2 - b^2e - c^3 = 0,$$

is a sextic curve, the edge of regression of the sextic torse

$$(ae - 4bd + 3c^2)^3 - 27(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0,$$

and we have in this case, as is well known,

$$m, r, n, \alpha, \beta, \vartheta, g, h, x, y$$
$$= 6, 6, 4, 0, 4, 0, 3, 6, 4, 6.$$

But putting as above $c=0$, then instead of the sextic curve we have the excubo-quartic curve $ae - 4bd = 0$, $ad^2 + b^2e = 0$, which is a curve having two stationary tangents, viz. these are the lines $(a=0, b=0)$ and $(d=0, e=0)$, which are in fact given along with the curve, by the foregoing equations $ae - 4bd = 0$, $ad^2 + b^2e = 0$. We have in this case $\vartheta = 2$, and the system is thus found to be

$$m, r, n, \alpha, \beta, \vartheta, g, h, x, y$$
$$= 4, 6, 4, 0, 0, 2, 3, 3, 4, 4,$$

it was in fact the consideration of this case which led me to take account of the new singularity of the stationary tangent lines.

I take the opportunity of referring to a most valuable and interesting paper by Schwarz, "De superficiebus in planum explicabilibus primorum septem ordinum," *Crelle*, t. LXIV. (1864), pp. 1—16. The author, after referring to my paper "On the developable derived from an equation of the fifth order," *Cambridge and Dublin Mathematical Journal*, t. V. (1850), pp. 152—159, [86], enters into the enquiry there suggested as to the means of ascertaining the degree of the 'planarity' of a developable surface. He starts from certain theorems derived from Riemann's theory of transcendental functions, viz.: If an algebraical (plane) curve of the order r has $\frac{1}{2}(r-1)(r-1) - \rho$ double points (nodes or cusps), then the coordinates of a point of the curve may be expressed rationally

If $\rho = 0$, that is, if the curve has the maximum number of double points, by a single parameter.

If $\rho = 1$, by a single parameter, and the square root of a cubic or quartic function of this parameter.

If $\rho = 2$, by a single parameter, and the square root of a quintic or sextic function of this parameter.

If $\rho > 2$, by a parameter ξ, and an algebraical function thereof η; where ξ, η are connected by an equation of the order $\frac{1}{2}(\rho + 3)$ or $\frac{1}{2}(\rho + 2)$ according as ρ is odd or even.

These principles establish a division of plane curves into algebraical classes; all plane curves (other than the generating lines) situate on a ruled surface, belong to the same algebraical class, and the surface itself belongs to the same class. Hence, if on a ruled surface there is either a right line which is not a generating line (this cannot be the case for developables) or a conic, or a cubic having a double point, or any other plane curve having the maximum number of double points, the surface belongs to the class for which $\rho = 0$; and in the case of a developable surface the equation of the tangent plane may be rationally expressed by means of a single parameter; that is, the degree of the planarity is $= 1$, or the surface is *planar*. This leads to the conclusion, that the developable surfaces or torses of the orders 4, 5, 6 and 7 are all of them planar.

The author points out that the 'special quintic developable' of my paper first above referred, (viz. that obtained by writing $b = 0$ in the equation of the sextic developable) is in fact the *general* developable of the fifth order, or quintic torse.

The foregoing theorem, that for a curve which has the maximum number of double points, the coordinates may be expressed rationally by a single parameter, admits of a very simple algebraical proof, as is shown in the paper by Clebsch "Ueber Curven deren coordinaten rationale Functionen eines Parameters sind," *Crelle*, t. LXIV. (1864), pp. 43—65. In another paper by the same author, "Ueber die Singularitäten algebraischer Curven," pp. 98—100, it is remarked that if in any plane curve we have m the order, n the class, δ the number of nodes, κ of cusps, τ of double tangents, ι of inflexions, then as a deduction from Riemann's principles, but also at once obtainable from Plücker's equations, we have

$$\tfrac{1}{2}(m-1)(m-2) - \delta - \kappa = \tfrac{1}{2}(n-1)(n-2) - \tau - \iota;$$

and moreover if from a given curve we derive in any manner another curve, such that to each tangent (or point) of the first curve there corresponds a *single* point (or tangent) of the second curve, then in the second curve the expression

$$\tfrac{1}{2}(m'-1)(m'-2) - \delta' - \kappa', = \tfrac{1}{2}(n'-1)(n'-2) - \tau' - \iota',$$

has the same value as in the first curve.

The like property exists for curves in space—viz. taking account as above of the new singularity of the stationary lines, then we have

$$\tfrac{1}{2}(m-1)(m-2)-h-\beta,$$
$$=\tfrac{1}{2}(r-1)(r-2)-y-n-\vartheta,$$
$$=\tfrac{1}{2}(r-1)(r-2)-x-m-\vartheta,$$
$$=\tfrac{1}{2}(n-1)(n-2)-g-\alpha,$$

which equations are in fact at once deducible from the above-mentioned system of six equations between the quantities m, r, n, α, β, ϑ, g, h, x, y, and may if we please be taken for equations of the system.

If from a given curve and torse we derive a second curve and torse, in such manner that to each point (or plane) of the first figure there corresponds a *single* plane (or point) of the second figure—then the corresponding expressions $\tfrac{1}{2}(m'-1)(m'-2)-h'-\beta'$, &c., have the same value for the second as for the first figure.

Cambridge, April 11, 1865.