368.

ON A PROBLEM OF GEOMETRICAL PERMUTATION.

[From the *Philosophical Magazine*, vol. XXX. (1865), pp. 370—372.]

IT is required to find in how many modes the nine points of inflexion of a cubic curve can be denoted by the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, in such wise that the twelve lines, each containing three points of inflexion, shall be in every case denoted by the same triads of figures, say by the triads

123, 147, 159, 168,

456, 258, 267, 249,

789, 369, 348, 357.

We may imagine the inflexions so denoted in one particular way, which may be called the primitive denotation; then in any other mode of denotation, a figure, for example 1, is either affixed to the inflexion to which it originally belonged, and it is then said to be *in loco*, or it is affixed to some other point of inflexion. This being so, the total number of modes is $= 432$; viz. this number is made up as follows:

9 figures *in loco*	1
3 „ „	60
1 figure „	243
0 „ „	128
	432

There is of course only one mode wherein the nine figures remain *in loco*. It may be seen without much difficulty that there is not any mode in which 8, 7, 6, 5, or 4 figures remain *in loco*. There is no mode in which only 2 figures remain *in loco*;

for any two inflexions are in a line with a third inflexion; and if the figures which belong to the first two inflexions are *in loco*, then the figure belonging to the third inflexion will be *in loco;* that is, there will be 3 figures *in loco.* The only remaining modes are therefore those which have 3 figures, 1 figure, or 0 figure *in loco.*

First, if three figures are *in loco*, these, as just seen, will be the figures which belong to three inflexions in a line. Suppose the figures are 1, 2, 3; then the inflexion originally denoted, say by the figure 4, may be denoted by any one of the remaining figures 5, 6, 7, 8, 9; but when the figure is once fixed upon, then the remaining inflexions can be denoted only in one manner. Hence when the figures 1, 2, 3 remain *in loco* there are 5 modes; and consequently the number of modes wherein 3 figures remain *in loco* is $5 \times 12, = 60$.

Next, if only a single figure, suppose 1, remains *in loco*, the triads which belong to the figure 1 are 123, 147, 159, 168; and there is 1 mode in which we simultaneously interchange all the pairs (2, 3), (4, 7), (5, 9), (6, 8). (Observe that the triads 123, 147, 159, 168 here denote the same lines respectively as in the primitive denotation, the figure 1 remains *in loco*, but the figures belonging to the other two inflexions on each of the four lines are interchanged.) There are, besides this, 2 modes in which the figures (2, 3), but not any other two figures, are interchanged; similarly 2 modes in which the figures (4, 7), 2 modes in which the figures (5, 9), 2 modes in which the figures (6, 8), but in each case no other two figures, are interchanged; this gives in all $1 + 2 + 2 + 2 + 2, = 9$ modes. There are besides, the figure 1 still remaining *in loco*, 18 modes where there are no two figures (2, 3), (4, 7), (5, 9), or (6, 8) which are interchanged: viz. the figure 2 may be made to denote any one of the inflexions originally denoted by 4, 5, 6, 7, 8, or 9. Suppose the inflexion originally denoted by 4; 3 will then denote the inflexion originally denoted by 7: it will be found that of three of the remaining six inflexions, any one may be denoted by the figure 4, and that the scheme of denotation can then in each case be completed in one way only. This gives $6 \times 3, = 18$, as above, for the number of the modes in question; and we have then $9 + 18, = 27$, for the number of the modes in which the figure 1 remains *in loco;* and $9 \times 27, = 243$, for the number of modes in which some one figure remains *in loco.*

Finally, if no figure remains *in loco*, the figure 1 will then denote some one of the inflexions originally denoted by 2, 3, 4, 5, 6, 7, 8, 9. Suppose it to denote that originally denoted by 2; 2 cannot then denote the inflexion originally denoted by 1, for if it did, 3 would remain *in loco:* 2 must therefore denote the inflexion originally denoted by 3, or else some one of the inflexions originally denoted by 4, 5, 6, 7, 8, 9. It appears, on examination, that in the first case there are 4 ways of completing the scheme, and in each of the latter cases 2 ways; there are therefore in all $1 \times 4 + 6 \times 2, = 16$ ways; that is, 16 modes in which (no figure remaining *in loco*) the figure 1 is used to denote the inflexion originally denoted by 2; and therefore $8 \times 16, = 128$ modes, for which no figure remains *in loco.* This completes the investigation of the numbers 1, 60, 243, and 128, which together make up the total number 432 of the modes of denotation of the nine inflexions.