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A THEOREM IN CONICS.

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THE following theorem is given in Todhunter's *Conic Sections*, [Ed. 7, p. 304], "If ellipses be inscribed in a triangle, each with one focus in a fixed straight line, the locus of the other focus is a conic section through the angular points of the triangle." A focus is the intersection of tangents to the conic from the circular points at infinity; and instead of the circular points at infinity we may substitute any two points whatever. This being so, let the equations of the sides of the triangle be $x = 0$, $y = 0$, $z = 0$, and let a pair of tangents to the curve from the points (α, β, γ), $(\alpha', \beta', \gamma')$ meet in the point (ξ, η, ζ), and the other pair of tangents from the same two points meet in the point (X, Y, Z). I find that we have the very simple relation

$$X\xi : Y\eta : Z\zeta = \alpha\alpha' : \beta\beta' : \gamma\gamma',$$

and consequently, when the locus of the point (ξ, η, ζ) is given, that of the point (X, Y, Z) is at once determined by substituting in the equation of the first-mentioned locus, in the place of ξ, η, ζ, the values $\frac{\alpha\alpha'}{\xi}, \frac{\beta\beta'}{\eta}, \frac{\gamma\gamma'}{\zeta}$, or as we may express it, the second locus is derived from the first by the method of reciprocal trilinear substitutions. And, in particular, when the first locus is a line, the second locus is a conic through the angular points of the triangle, which is Mr Todhunter's theorem. I have considered some of the properties of this substitution in a Memoir "Sur quelques transmutations des Courbes," *Liouville*, t. XIV. (1849), pp. 40—46 and t. XV. (1850), pp. 351—356, [80 and 81].

To demonstrate the theorem, I take for the equation of the conic

$$\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0,$$

and I write for shortness

$$\beta\zeta - \gamma\eta,\ \gamma\xi - \alpha\zeta,\ \alpha\eta - \beta\xi = A\ ,\ B\ ,\ C,$$
$$\beta'\zeta - \gamma'\eta,\ \gamma'\xi - \alpha'\zeta,\ \alpha'\eta - \beta'\xi = A',\ B',\ C',$$
$$\xi(\beta\gamma' - \beta'\gamma) + \quad \eta(\gamma\alpha' - \gamma'\alpha) + \quad \zeta(\alpha\beta' - \alpha'\beta) = \Delta,$$
$$\eta\zeta\alpha\alpha'(\beta\gamma' - \beta'\gamma) + \zeta\xi\beta\beta'(\gamma\alpha' - \gamma'\alpha) + \xi\eta\gamma\gamma'(\alpha\beta' - \alpha'\beta) = \square,$$

so that in fact

$$\Delta = \begin{vmatrix} \xi, & \eta, & \zeta \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad \square = \alpha\beta\gamma\alpha'\beta'\gamma'\xi\eta\zeta \begin{vmatrix} \frac{1}{\xi}, & \frac{1}{\eta}, & \frac{1}{\zeta} \\ \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \end{vmatrix},$$

and we have

$$BC' - B'C = \xi\Delta,\quad CA' - C'A = \eta\Delta,\quad AB' - A'B = \zeta\Delta,$$
$$\beta\gamma'BC' - \beta'\gamma B'C = \gamma\alpha'CA' - \gamma'\alpha C'A = \alpha\beta'AB' - \alpha'\beta A'B = \square.$$

The conditions in order that the conic

$$\surd(lx) + \surd(my) + \surd(nz) = 0$$

may touch the line through (α, β, γ) and (ξ, η, ζ) is

$$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0,$$

and the condition in order that it may touch the line through $(\alpha', \beta', \gamma')$ and (ξ, η, ζ) is

$$\frac{l}{A'} + \frac{m}{B'} + \frac{n}{C'} = 0,$$

and we thus have

$$l : m : n = \frac{1}{BC'} - \frac{1}{B'C} : \frac{1}{CA'} - \frac{1}{C'A} : \frac{1}{AB'} - \frac{1}{A'B},$$

or, what is the same thing,

$$l : m : n = \quad AA'\xi \quad : \quad BB'\eta \quad : \quad CC'\zeta,$$

which determine the constants in the equation of the conic.

Consider now the tangents to the conic from the point (α, β, γ); if the equation of the tangent is assumed to be

$$px + qy + rz = 0,$$

then we have

$$p\alpha + q\beta + r\gamma = 0,$$
$$\frac{l}{p} + \frac{m}{q} + \frac{n}{r} = 0,$$

and these equations are of course satisfied by $p : q : r = A : B : C$, since the line through (ξ, η, ζ) is a tangent. They are also satisfied by

$$p : q : r = \frac{l}{A\alpha} : \frac{m}{B\beta} : \frac{n}{C\gamma},$$

as is obvious by substitution, we have therefore

$$\frac{l}{A\alpha} x + \frac{m}{B\beta} y + \frac{n}{C\gamma} z = 0,$$

or more simply

$$\frac{A'\xi}{\alpha} x + \frac{B'\eta}{\beta} y + \frac{C'\zeta}{\gamma} z = 0,$$

for the equation of the other tangent through (α, β, γ), and we have in like manner

$$\frac{A\xi}{\alpha'} x + \frac{B\eta}{\beta'} y + \frac{C\zeta}{\gamma'} z = 0,$$

for the equation of the other tangent through $(\alpha', \beta', \gamma')$; the last-mentioned two lines intersect in the point X, Y, Z, that is we have

$$X\xi : Y\eta : Z\zeta = \frac{B'C}{\beta\gamma'} - \frac{BC'}{\beta'\gamma} : \frac{C'A}{\gamma\alpha'} - \frac{CA'}{\gamma'\alpha} : \frac{A'B}{\alpha\beta'} - \frac{AB'}{\alpha'\beta},$$

or attending to an above-mentioned equation, we have

$$X\xi : Y\eta : Z\zeta = \alpha\alpha' : \beta\beta' : \gamma\gamma',$$

which is the property in question. In the particular case, where the points (α, β, γ), $(\alpha', \beta', \gamma')$ are the foci, the theorem is an immediate consequence of the well-known proposition that the product of the perpendiculars let fall from the two foci upon any tangent of the conic is a constant.

2, *Stone Buildings, W.C.*, 17*th March*, 1860.