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NOTE ON THE WAVE SURFACE.

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In the paper "On the Wave Surface" in the last Number of the *Journal*, [277], I stated that it was shown in the second of Zech's Memoirs that the curves of contact of the developables F and G with the wave surface were *the curves of curvature* of the wave surface. I had not examined the demonstration of this theorem, and I had overlooked the author's note "Die Krümmungslinien der Wellenfläche zweiaxiger Krystalle, &c.," *Crelle*, t. LIV. p. 94 (Feb. 1858), where he points out that in a phrase which he quotes, he had assumed without demonstration a theorem which was in fact erroneous, and that he retracted all that he had said in No. 11 of his Memoir (the portion which contains the theorem as to the curves of curvature of the wave surface). M. Bertrand in a note in the *Comptes Rendus*, t. XLVII. pp. 817—819 (Nov. 1858), after referring to my paper, remarks that the theorem as to the curves of curvature of the wave surface appeared to him so remarkable that he hastened to investigate a proof of it, but that he very soon discovered that the theorem was unfortunately erroneous; and he proceeds to show why the theorem cannot be true. M. Bertrand's demonstration is as follows:

THEOREM I. If from a point O, we let fall perpendiculars on the tangent planes of a surface, the locus of their feet is a new surface. Let P be a point of this surface corresponding to the point M of the first surface, the normal at P passes through the middle point of OM.

THEOREM II. If the curve of curvature of a surface is such that the tangent planes at the several points of the curve are equidistant from a point O, the curve of curvature is situate on a sphere having the point O for its centre.

For suppose that we let fall from the point O perpendiculars on the tangent planes to the surface at the several points of the curve of curvature in question. The locus of the feet will be a spherical curve, a normal at any point P of the curve will it is clear pass through the point O; besides, in virtue of the first theorem, another normal will pass through the middle point of the radius OM drawn to the corresponding point M of the surface; the tangent to the curve which is the locus of the points P is therefore perpendicular to the plane MOP, and consequently to the line MP. But when a developable surface is circumscribed about a sphere, the perpendiculars let fall from the centre of the sphere on the tangent planes of the developable surface have their feet on the generating lines; and consequently the curve which is the locus of the points P is situate on the developable surface (viz., the developable surface enveloped by the tangent planes at the several points of the curve of curvature of the given surface) and cuts the generating lines at right angles. But the curve, the locus of the point M, being by hypothesis a curve of curvature of the given surface, also cuts at right angles the generating lines of the developable surface, the two curves are therefore equidistant curves on the developable surface, viz. MP is constant, and since by hypothesis OP is constant, OM is also constant, which proves the second theorem.

This being premised, it is to be recollected that the wave surface may be generated in two different ways; 1°. it is the locus of the extremities of the central perpendiculars to the diametral sections of an ellipsoid E, equal to the axes of these sections; 2°. it is the envelope of planes parallel to the diametral sections of a second ellipsoid E' at distances inversely proportional to the axes of the sections. To obtain all the tangent planes of the wave surface situate at a distance h from the centre, it is necessary to find in the ellipsoid the diametral sections which have an axis equal to $\frac{1}{h}$; for this, we may cut the ellipsoid by a concentric sphere of the radius $\frac{1}{h}$, and draw tangent planes to the cone having for its vertex the centre of the ellipsoid and passing through the curve of intersection with the sphere. The tangent planes of the wave surface respectively parallel to the tangent planes of the cone will be at the distance h from the centre, and, if they touched the wave surface along a curve of curvature, it would follow from the foregoing theorem II. that their points of contact would be all at the same distance from the centre; but the distance of the centre of the wave surface from the point of contact of a tangent plane is inversely proportional to the perpendicular let fall from the centre of the ellipsoid E' on the tangent plane at the corresponding point of this ellipsoid: it follows that the curve of intersection of an ellipsoid with a concentric sphere is not such that the tangent planes of the ellipsoid at the several points of the curve are at the same distance from the centre, and consequently the tangent planes which were under consideration do *not* determine by their points of contact a curve of curvature of the wave surface.

I remark in relation to M. Bertrand's theorem I. that it is immediately connected with a well-known theorem which occurs in optics—viz., if rays proceeding from a point are reflected at a surface, and if from the radiant point to the surface and thence along the reflected ray, we measure off a constant distance, the surface which is the locus of the point so obtained (the secondary focal surface of Dandelin and Quetelet) is an orthogonal trajectory of the reflected rays. In fact, if O be the radiant point, and OM' the incident ray, and if from M' we measure off on the reflected ray a distance $= - OM'$, that is, on the reflected ray produced backwards, a distance $M'P = OM'$, then the whole distance from O is $OM' - OM' = 0$, and the surface which is the locus of the point P is consequently an orthogonal trajectory to the reflected rays. But the point P may, it is clear, be constructed as follows: viz., on the tangent plane at M' let fall the perpendicular OP', and produce it to a point P such that $OP' = P'P$. And if we produce OM' to M so that $OM' = M'M$, then it is clear that the locus of M is a surface similar and similarly situated with the original surface, but of double the magnitude, and that OP is the perpendicular from O upon the tangent plane at M of the last-mentioned surface. And, by what precedes, the line PM' from P to the middle point of OM is a normal of the surface which is the locus of P: the corresponding theorem *in plano* is in fact actually given by Dandelin.

31*st Dec.*, 1858.