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NOTE ON THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

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THE following investigation (which has been in my possession for a good many years) affords I think a simple explanation of the theory of the singular solutions of differential equations.

Let the primitive equation be

$$c^n + Pc^{n-1} + Qc^{n-2} + \ldots = 0,$$

where c is the arbitrary constant and $P, Q \ldots$ are any functions of x, y; then the differential equation is obtained by eliminating c from the foregoing equation and the derived equation

$$P'c^{n-1} + Q'c^{n-2} + \ldots = 0,$$

and the result may be represented by

$$F(P, Q, \ldots, P', Q', \ldots) = 0.$$

Assume now

$$c^n + Pc^{n-1} + Qc^{n-2} + \ldots = (c+X)(c+Y)(c+Z)\ldots,$$

then we have

$$P = X + Y + Z + \&c.,$$
$$Q = XY + XZ + YZ + \&c.,$$
$$\&c.,$$

and consequently

$$P' = X' + Y' + Z' + \&c.,$$
$$Q' = (Y + Z + \&c.)\, X' + \&c.,$$
$$\&c.,$$

and substituting these values in the function $F(P, Q, \ldots P', Q', \ldots)$ it is clear that we shall have $F(P, Q, \ldots, P', Q', \ldots) = UX'Y'Z'\ldots$ where U is a symmetrical function of X, Y, Z, &c., and therefore a function of $P, Q, \ldots$; and this equation will be identically true whatever values we attribute to $X', Y', Z', \ldots$, hence putting these quantities respectively equal to unity, we have

$$P' = n,$$
$$Q' = (n-1)P,$$
$$R' = (n-2)Q,$$
&c.,

and with these values

$$U = F(P, Q, \ldots, P', Q', \ldots),$$

that is $U = 0$ is the result obtained by eliminating c from the primitive equation and the equation

$$nc^{n-1} + (n-1)Pc^{n-2} + \ldots = 0,$$

which is the equation obtained by differentiating the primitive equation with respect to the arbitrary constant c: that is, $U = 0$ being the singular solution, the differential equation is

$$UX'Y'Z'\ldots = 0.$$

It is to be remarked that (P, Q, &c. being rational and integral functions) then if the roots X, Y, Z, &c. are also rational and integral functions, the differential equation contains U as a separable rational and integral factor, but if the roots are irrational then the differential equation does not really contain the rational and integral factor U, but $X'Y'Z'\ldots$ is here a rational fraction containing U in the denominator and $UX'Y'Z'\ldots$ is an indecomposable rational and integral function. This is easily verified *à posteriori* for a quadratic equation.

2, *Stone Buildings, W.C., 28th January*, 1858.