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ON THE WAVE SURFACE.

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SOME very beautiful results in relation to the Wave Surface have been recently obtained by Herr Zech, in the Memoirs "Die Eigenschaften der Wellenflächen der zweiaxigen Krystalle mittels der höhern Geometrie abgeleitet," *Crelle*, t. LII. pp. 243—254 (1856), and "Die Krümmungslinien der Wellenfläche zweiaxiger Krystalle," *Crelle*, t. LIV. pp. 72—77 (1857). According to the former of Fresnel's two modes of generation, the wave surface is the envelope of a plane whose perpendicular distance v is a certain given function of the direction cosines l, m, n. For the same system of direction cosines, there are in fact two values of the perpendicular distance: call these v and w, and let the corresponding planes (parallel of course to each other) be called P and Q. Then the entire system of the planes P and Q envelope the wave surface, viz. the planes P may be considered as enveloping one sheet and the planes Q the other sheet of the surface. But if instead of considering the entire system of planes we consider only the planes P and the parallel planes Q, for which the perpendicular on the plane P has a given constant value v, then the planes P will envelope a developable F, and the planes Q will envelope a developable G, these two developables being, it is to be observed, distinct surfaces, not sheets of one and the same surface. Or, what is the same thing, the planes P for which the perpendicular distance has a given constant value v will envelope a developable F, and the planes P for which the perpendicular distance of the parallel planes Q has a given constant value w, will envelope a developable G. And it is obvious that the developables F and G touch the wave surface along curves. The equation of the developable F contains of course the arbitrary parameter v, and the equation of the developable G contains in like manner the arbitrary parameter w, so that we in fact have two series of developables

F and G respectively touching the wave surface along two series of curves. And it is shown in the second of the Memoirs above referred to that these curves are the *curves of curvature* of the wave surface.

The developable F is obtained as the envelope of the plane P, whose equation is

$$lx + my + nz = v,$$

where v has a given constant value and l, m, n are parameters which vary, subject to the two conditions

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0.$$

And in like manner the developable G is obtained as the envelope of the plane Q, whose equation is

$$lx + my + nz = \frac{abc}{w} \sqrt{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)},$$

where w has a given constant value and l, m, n are parameters which vary, subject to the two conditions

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{a^2 - w^2} + \frac{m^2}{b^2 - w^2} + \frac{n^2}{c^2 - w^2} = 0.$$

{It is hardly necessary to remark that if in the last-mentioned system of equations, the parameter w is also treated as variable, we obtain the wave surface: in fact v^2, w^2 being the roots of the equation

$$\frac{l^2}{a^2 - \theta} + \frac{m^2}{b^2 - \theta} + \frac{n^2}{c^2 - \theta} = 0,$$

we have, attending to the condition $l^2 + m^2 + n^2 = 1$, the identical equation

$$l^2 (b^2 - \theta)(c^2 - \theta) + m^2 (c^2 - \theta)(a^2 - \theta) + n^2 (a^2 - \theta)(b^2 - \theta) = (v^2 - \theta)(w^2 - \theta),$$

and thence

$$vw = abc \sqrt{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)},$$

which shows that the system of equations for the plane Q (w being treated as variable) is in fact identical with the system of equations for the plane P.}

The form of the equations of the planes P and Q respectively shows that each of these planes is parallel to a tangent plane of the cone

$$\frac{x^2}{a^2 - v^2} + \frac{y^2}{b^2 - v^2} + \frac{z^2}{c^2 - v^2} = 0,$$

or in other words, that the planes P and Q are respectively tangents to the conic or infinitely thin surface of the second order, which is the second of the last-mentioned cone by the plane at infinity. Moreover it is obvious from the same equations that the plane P is a tangent plane of the sphere whose equation is

$$x^2 + y^2 + z^2 = v^2,$$

and that the plane Q is a tangent plane of the ellipsoid whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 = \frac{a^2b^2c^2}{v^2}.$$

Hence each of the developables F and G is the envelope of a plane which is the common tangent plane of two surfaces of the second order; such developables are in general of the eighth order, see my paper "On the Developable Surfaces which arise from Two Surfaces of the Second Order," *Camb. and Dubl. Math. Journ.*, t. v. pp. 46—57 (1850), [84], and it will be presently seen that this is in fact the order of the developables F and G respectively.

The before-mentioned cone

$$\frac{x^2}{a^2 - v^2} + \frac{y^2}{b^2 - v^2} + \frac{z^2}{c^2 - v^2} = 0,$$

(Zech's cone K) is a cone having for its focal lines the optic axes (or normals to the circular sections) of the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ (Zech's ellipsoid E) which is used in the theory of the Wave Surface in the place of Fresnel's Surface of Elasticity. The complementary cone

$$(a^2 - v^2)\, x^2 + (b^2 - v^2)\, y^2 + (c^2 - v^2)\, z^2 = 0,$$

(Zech's cone C) meets the last-mentioned ellipsoid in a curve lying on the sphere whose equation is $x^2 + y^2 + z^2 = \frac{1}{v^2}$, a property which may be considered as affording the geometrical construction of the magnitude v, by means of the cone K. The ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = \frac{a^2b^2c^2}{v^2}$ is obviously an ellipsoid similar to the ellipsoid E, and the value of v being determined as above, the sphere $x^2+y^2+z^2=v^2$ and the ellipsoid $a^2x^2+b^2y^2+c^2z^2=\frac{a^2b^2c^2}{v^2}$ (which are used in the preceding geometrical construction of the developables F and G respectively) may be considered as given by construction.

To find the equation of the developable F, we have, from the equations

$$\begin{aligned} lx + my + nz &= v, \\ l^2 + m^2 + n^2 &= 1, \\ \frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} &= 0, \end{aligned}$$

treating them by the method of arbitrary multipliers in the usual manner, we obtain

$$x+\left(\rho+\frac{\theta}{a^2-v^2}\right) l=0,$$

$$y+\left(\rho+\frac{\theta}{b^2-v^2}\right) m=0,$$

$$z+\left(\rho+\frac{\theta}{c^2-v^2}\right) n=0,$$

equations which give in the first instance $v+\rho=0$ or $\rho=-v$, and then substituting this value for ρ,

$$l=\frac{x(v^2-a^2)}{v(v^2-a^2)+\theta},\quad m=\frac{y(v^2-b^2)}{v(v^2-b^2)+\theta},\quad z=\frac{z(v^2-c^2)}{v(v^2-c^2)+\theta},$$

and thence

$$\frac{x^2(v^2-a^2)}{v(v^2-a^2)+\theta}+\frac{y^2(v^2-b^2)}{v(v^2-b^2)+\theta}+\frac{z^2(v^2-c^2)}{v(v^2-c^2)+\theta}=v,$$

$$\frac{x^2(v^2-a^2)}{[v(v^2-a^2)+\theta]^2}+\frac{y^2(v^2-b^2)}{[v(v-b^2)+\theta]^2}+\frac{z^2(v^2-c^2)}{[v(v^2-c^2)+\theta]^2}=0,$$

the latter of which is the derived equation of the former with respect to the parameter θ, hence writing the former equation under the form

$$\begin{aligned}&\{\theta+v(v^2-a^2)\}\{\theta+v(v^2-b^2)\}\{\theta+v(v^2-c^2)\}\\ &\quad-\frac{x^2}{v^2}v(v^2-a^2)\ \{\theta+v(v^2-b^2)\}\{\theta+v(v^2-c^2)\}\\ &\quad-\&c.=0,\end{aligned}$$

or what is the same thing

$$(A,\ B,\ C,\ D)(\theta,\ 1)^3=0,$$

where

$A=3v^2,$

$B=(v^2-x^2)(v^2-a^2)+(v^2-y^2)(v^2-b^2)+(v^2-z^2)(v^2-c^2),$

$C=(v^2-y^2-z^2)(v^2-b^2)(v^2-c^2)+(v^2-z^2-x^2)(v^2-c^2)(v^2-a^2)+(v^2-x^2-y^2)(v^2-a^2)(v^2-b^2),$

$D=3(v^2-x^2-y^2-z^2)(v^2-a^2)(v^2-b^2)(v^2-c^2),$

the equation of the developable F is

$$(AD-BC)^2-4(AC-B^2)(BD-C^2)=0.$$

The investigation for the developable G is very similar to the preceding. Write for shortness

$$\Lambda^2=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2},$$

then the system of equations is

$$lx+my+nz-\frac{abc}{w}\Lambda=0,$$

$$l^2+m^2+n^2=1,$$

$$\frac{l^2}{a^2-w^2}+\frac{m^2}{b^2-w^2}+\frac{n^2}{c^2-w^2}=0,$$

and we then have

$$x - \frac{abc}{wa^2\Lambda}\,l + \left(\rho + \frac{\theta}{a^2-w^2}\right) l = 0,$$

$$y - \frac{abc}{wb^2\Lambda}\,m + \left(\rho + \frac{\theta}{b^2-w^2}\right) m = 0,$$

$$z - \frac{abc}{wc^2\Lambda}\,n + \left(\rho + \frac{\theta}{c^2-w^2}\right) n = 0,$$

equations which give $\rho = 0$, and substituting this value, we obtain

$$l = \frac{x\,(w^2-a^2)}{\dfrac{abc}{wa^2\Lambda}(w^2-a^2)+\theta},$$

$$m = \frac{y\,(w^2-b^2)}{\dfrac{abc}{wb^2\Lambda}(w^2-b^2)+\theta},$$

$$n = \frac{z\,(w^2-c^2)}{\dfrac{abc}{wc^2\Lambda}(w^2-c^2)+\theta}.$$

Substituting these values in the equations

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = \Lambda^2,$$

$$\frac{l^2}{w^2-a^2} + \frac{m^2}{w^2-b^2} + \frac{n^2}{w^2-c^2} = 0,$$

we find

$$\frac{\dfrac{x^2}{a^2}(w^2-a^2)^2}{\left\{\dfrac{abc}{wa^2}(w^2-a^2)+\Lambda\theta\right\}^2} + \frac{\dfrac{y^2}{b^2}(w^2-b^2)^2}{\left\{\dfrac{abc}{wb^2}(w^2-b^2)+\Lambda\theta\right\}^2} + \frac{\dfrac{z^2}{c^2}(w^2-c^2)^2}{\left\{\dfrac{abc}{wc^2}(w^2-c^2)+\Lambda\theta\right\}^2} = 1,$$

$$\frac{x^2\,(w^2-a^2)}{\left\{\dfrac{abc}{wa^2}(w^2-a^2)+\Lambda\theta\right\}^2} + \frac{y^2\,(w^2-b^2)}{\left\{\dfrac{abc}{wb^2}(w^2-b^2)+\Lambda\theta\right\}^2} + \frac{z^2\,(w^2-c^2)}{\left\{\dfrac{abc}{wc^2}(w^2-c^2)+\Lambda\theta\right\}^2} = 0,$$

and multiplying the first equation by $\dfrac{abc}{w}$ and the second equation by $\Lambda\theta$ and adding, we obtain

$$\frac{x^2\,(w^2-a^2)}{\dfrac{abc}{wa^2}(w^2-a^2)+\Lambda\theta} + \frac{y^2\,(w^2-b^2)}{\dfrac{abc}{wb^2}(w^2-b^2)+\Lambda\theta} + \frac{z^2\,(w^2-c^2)}{\dfrac{abc}{wc^2}(w^2-c^2)+\Lambda\theta} = \frac{abc}{w},$$

of which equation the latter of the foregoing two equations is the derived equation with respect to the parameter $\Lambda\theta$.

Comparing this with the foregoing equation

$$\frac{x^2(v^2-a^2)}{v(v^2-a^2)+\theta}+\frac{y^2(v^2-b^2)}{v(v^2-c^2)+\theta}+\frac{z^2(v^2-c^2)}{v(v^2-c^2)+\theta}=v,$$

we see that it is deduced from it by writing

$$a^2x^2,\ b^2y^2,\ c^2z^2,\ \frac{w^2}{a^2}-1,\ \frac{w^2}{b^2}-1,\ \frac{w^2}{c^2}-1,\ \frac{abc}{w^2},\ \Lambda\theta$$

in the place of

$$x^2,\quad y^2,\quad z^2,\ v^2-a^2,\ v^2-b^2,\ v^2-c^2,\ v\ ,\quad \theta$$

respectively, and the equation of the developable G is therefore

$$(A'D'-B'C')^2-4(A'C'-B'^2)(B'D'-C'^2)=0,$$

where A', B', C', D' are what A, B, C, D become when

$$x^2,\ \ y^2,\ \ z^2,\ v^2-a^2,\ v^2-b^2,\ v^2-c^2,\ \ v\ ,$$

are replaced by

$$a^2x^2,\ b^2y^2,\ c^2z^2,\ \frac{w^2}{a^2}-1,\ \frac{w^2}{b^2}-1,\ \frac{w^2}{c^2}-1,\ \frac{abc}{w}.$$

The equations of the developables F and G, although radically distinct from each other, are consequently similar in form, and each is at once deducible from the other.

2, *Stone Buildings, W.C.*, 9*th March*, 1858.