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ADDITION TO THE MEMOIR ON AN EXTENSION OF ARBOGAST'S METHOD OF DERIVATIONS.

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THE process explained in the foregoing Memoir is not easily workable, and I have in fact never used it. I have devised a new process theoretically less complete, and of a somewhat mixed character consisting, as it does, in the analytical reduction of the problem to the same problem for a smaller number of letters. I have however found it very convenient for obtaining the literal terms in the theory of the sextic, viz. where we have the seven letters (a, b, c, d, e, f, g); and I propose to explain the process by applying it to the determination of the terms of the discriminant, degree $= 6$, weight $= 18$.

Here writing a, b, c, d, e, f, g to denote the indices of these letters respectively in a term such as $a^\alpha b^\beta c^\gamma \ldots$ (that is, writing for convenience $a, b, c, \ldots$ instead of $\alpha, \beta, \gamma, \ldots$) we have

$$a + b + c + d + e + f + g = 6,$$
$$b + 2c + 3d + 4e + 5f + 6g = 18,$$

and thence

$$6a + 5b + 4c + 3d + 2e + f = 18.$$

I separate off from the others the first three letters a, b, c. The equation gives $a = 3$ at most. And then

if $a = 3$, then $5b + 4c + 3d + 2e + f = 18$, $b = 0$, $c = 0$;

if $a = 2$, $5b + 4c + 3d + 2e + f = 6$, $b = 1$ at most;

if $b = 1$, $4c + 3d + 2e + f = 1$, $c = 0$,

„ $b = 0$, $4c + 3d + 2e + f = 6$, $c = 1$ at most,

if $a = 1$, $5b + 4c + 3d + 2e + f = 12$, $b = 2$ at most;
 if $b = 2$, $4c + 3d + 2e + f = 2$, $c = 0$,
 „ $b = 1$, $4c + 3d + 2e + f = 7$, $c = 1$ at most,
 „ $b = 0$, $4c + 3d + 2e + f = 12$, $c = 3$ „ „
if $a = 0$, $5b + 4c + 3d + 2e + f = 18$, $b = 3$ at most;
 if $b = 3$, $4c + 3d + 2e + f = 3$, $c = 0$,
 „ $b = 2$, $4c + 3d + 2e + f = 8$, $c = 2$ at most,
 „ $b = 1$, $4c + 3d + 2e + f = 13$, $c = 3$ „ „
 „ $b = 0$, $4c + 3d + 2e + f = 18$, $c = 4$ „ „

Hence considering the several terms as arranged in alphabetical order and writing down only the factors in a, b, c, we obtain col. 1 of the following diagram;

Col. 1.	Col. 2.	Col. 3.	
a^3	$(d^3)^9$	1	1
a^2b	$(d^3)^8$	1	1
a^2b^0c	$(d^3)^7$	2	
c^0	$(d^4)^6$	5	7
$a\,b^2c^0$	$(d^3)^7$	2	
$b\,c^1$	$(d^3)^6$	3	
c^0	$(d^4)^5$	4	9
$a\,b^0c^3$	$(d^2)^6$	1	
c^2	$(d^3)^5$	3	
c^1	$(d^4)^4$	4	
c^0	$(d^5)^3$	3	11
$a^0b^3c^0$	$(d^3)^6$	3	3
b^2c^2	$(d^2)^6$	1	
c^1	$(d^3)^5$	3	
c^0	$(d^4)^4$	4	8
b^1c^3	$(d^2)^5$	1	
c^2	$(d^3)^4$	3	
c^1	$(d^4)^3$	3	
c^0	$(d^5)^2$	2	9
b^0c^4	$(d^2)^4$	2	
c^3	$(d^3)^3$	3	
c^2	$(d^4)^2$	2	
c^1	$(d^5)^1$	1	
c^0	$(d^6)^0$	1	9
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We then form col. 2 by annexing to each term a term $(d^\theta)^\phi$ which denotes the derivative ϕ of d^θ; the value of θ being such that the whole term may be of the proper degree 6, and the value of ϕ being such that the whole term may be of the proper weight 18. Thus

$$a^3(d^3)^9, \text{ degree is } 3 + 3, = 6; \text{ weight is } 0 + 9 + 9, = 18,$$
$$ab^2c^0(d^3)^7, \quad „ \quad 1 + 2 + 3, = 6; \quad „ \quad „ \quad 0 + 2 + 9 + 7, = 18,$$

viz. d^3 being of weight 9, then $(d^3)^9$ is of weight $9 + 9$, and $(d^3)^7$ of weight $9 + 7$.

The derivatives of any power of d are given by those of the indefinite power d^m, which, omitting therein the several powers of d, may be tabulated thus. I remark that the table is carried up to the column 21 in order that it may be applicable to the calculation of the literal terms of the degree 15 and weight 45 which belong to the highest invariant of the sextic.

SUBSIDIARY TABLE OF THE DERIVATIVES OF d^m.

	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44	48	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
1	1	e	f	g	eg	fg	g^2	eg^2	fg^2	g^3	eg^3	fg^3	g^4	eg^4	fg^4	g^5	eg^5	fg^5	g^6	eg^6	fg^6	g^7	1
2			e^2	ef	f^2	e^2g	efg	f^2g	e^2g^2	efg^2	f^2g^2	e^2g^3	efg^3	f^2g^3	e^2g^4	efg^4	f^2g^4	e^2g^5	efg^5	f^2g^5	e^2g^6	efg^6	2
3				e^3	e^2f	ef^2	f^3	e^2fg	ef^2g	f^3g	e^2fg^2	ef^2g^2	f^3g^2	e^2fg^3	ef^2g^3	f^3g^3	e^2fg^4	ef^2g^4	f^3g^4	e^2fg^5	ef^2g^5	f^3g^5	3
4					e^4	e^3f	e^3g	ef^3	f^4	e^3g^2	ef^3g	f^4g	e^3g^3	ef^3g^2	f^4g^2	e^3g^4	ef^3g^3	f^4g^3	e^3g^5	ef^3g^4	f^4g^4	e^3g^6	4
5						e^5	e^2f^2	e^4g	e^3fg	e^2f^2g	f^5	e^3fg^2	$e^2f^2g^2$	f^5g	e^3fg^3	$e^2f^2g^3$	f^5g^2	e^3fg^4	$e^2f^2g^4$	f^5g^3	e^3fg^5	$e^2f^2g^5$	5
6							e^4f	e^3f^2	e^2f^3	ef^4	e^4g^2	e^2f^3g	ef^4g	e^4g^3	$e^2f^3g^2$	ef^4g^2	e^4g^4	$e^2f^3g^3$	ef^4g^3	e^4g^5	$e^2f^3g^4$	ef^4g^4	6
7							e^6	e^5f	e^5g	e^4fg	e^3f^2g	ef^5	f^6	$e^3f^2g^2$	ef^5g	f^6g	$e^3f^2g^3$	ef^5g^2	f^6g^2	$e^3f^2g^4$	ef^5g^3	f^6g^3	7
8								e^7	e^4f^2	e^3f^3	e^2f^4	e^5g^2	e^4fg^2	e^2f^4g	f^7	e^4fg^3	$e^2f^4g^2$	f^7g	e^4fg^4	$e^2f^4g^3$	f^7g^2	e^4fg^5	8
9									e^6f	e^6g	e^5fg	e^4f^2g	e^3f^3g	ef^6	e^5g^3	$e^3f^3g^2$	ef^6g	e^5g^4	$e^3f^3g^3$	ef^6g^2	e^5g^5	$e^3f^3g^4$	9
10									e^8	e^5f^2	e^4f^3	e^3f^4	e^2f^5	e^5fg^2	$e^4f^2g^2$	e^2f^5g	f^8	$e^4f^2g^3$	$e^2f^5g^2$	f^8g	$e^4f^2g^4$	$e^2f^5g^3$	10
11										e^7f	e^7g	e^6fg	e^6g^2	e^4f^3g	e^3f^4g	ef^7	e^5fg^3	$e^3f^4g^2$	ef^7g	e^5fg^4	$e^3f^4g^3$	ef^7g^2	11
12										e^9	e^6f^2	e^5f^3	e^5f^2g	e^3f^5	e^2f^6	e^6g^3	$e^4f^3g^2$	e^2f^6g	f^9	$e^4f^3g^3$	$e^2f^6g^2$	f^9g	12
13											e^8f	e^8g	e^4f^4	e^7g^2	e^6fg^2	$e^5f^2g^2$	e^3f^5g	ef^8	e^6g^4	$e^3f^5g^2$	ef^8g	e^6g^5	13
14											e^{10}	e^7f^2	e^7fg	e^6f^2g	e^5f^3g	e^4f^4g	e^2f^7	e^6fg^3	$e^5f^2g^3$	e^2f^7g	f^{10}	$e^5f^2g^4$	14
15												e^9f	e^6f^3	e^5f^4	e^4f^5	e^3f^6	e^7g^3	$e^5f^3g^2$	$e^4f^4g^2$	ef^9	e^6fg^4	$e^4f^4g^3$	15
16												e^{11}	e^9g	e^8fg	e^8g^2	e^7fg^2	$e^6f^2g^2$	e^4f^5g	e^3f^6g	e^7g^4	$e^5f^3g^3$	$e^3f^6g^2$	16
17													e^8f^2	e^7f^3	e^7f^2g	e^6f^3g	e^5f^4g	e^3f^7	e^2f^8	$e^6f^2g^3$	$e^4f^5g^2$	e^2f^8g	17
18													$e^{10}f$	$e^{10}g$	e^6f^4	e^5f^5	e^4f^6	e^8g^3	e^7fg^3	$e^5f^4g^2$	e^3f^7g	ef^{10}	18
19													e^{12}	e^9f^2	e^9fg	e^9g^2	e^8fg^2	$e^7f^2g^2$	$e^6f^3g^2$	e^4f^6g	e^2f^9	e^7fg^4	19
20														$e^{11}f$	e^8f^3	e^8f^2g	e^7f^3g	e^6f^4g	e^5f^5g	e^3f^8	e^8g^4	$e^6f^3g^3$	20
21														e^{13}	$e^{11}g$	e^7f^4	e^6f^5	e^5f^6	e^4f^7	e^8fg^3	$e^7f^2g^3$	$e^5f^5g^2$	21
22															$e^{10}f^2$	$e^{10}fg$	$e^{10}g^2$	e^9fg^2	e^9g^3	$e^7f^3g^2$	$e^6f^4g^2$	e^4f^7g	22
23															$e^{12}f$	e^9f^3	e^9f^2g	e^8f^3g	$e^8f^2g^2$	e^6f^5g	e^5f^6g	e^3f^9	23
24															e^{14}	$e^{12}g$	e^8f^4	e^7f^5	e^7f^4g	e^5f^7	e^4f^8	e^9g^4	24
25																$e^{11}f^2$	$e^{11}fg$	$e^{11}g^2$	e^6f^6	$e^{10}g^3$	e^9fg^3	$e^8f^2g^3$	25
26																$e^{13}f$	$e^{10}f^3$	$e^{10}f^2g$	$e^{10}fg^2$	$e^9f^2g^2$	$e^8f^3g^2$	$e^7f^4g^2$	26
27																e^{15}	$e^{13}g$	e^9f^4	e^9f^3g	e^8f^4g	e^7f^5g	e^6f^6g	27
28																	$e^{12}f^2$	$e^{12}fg$	e^8f^5	e^7f^6	e^6f^7	e^5f^8	28
29																	$e^{14}f$	$e^{11}f^3$	$e^{12}g^2$	$e^{11}fg^2$	$e^{11}g^3$	$e^{10}fg^3$	29
30																	e^{16}	$e^{14}g$	$e^{11}f^2g$	$e^{10}f^3g$	$e^{10}f^2g^2$	$e^9f^3g^2$	30
31																		$e^{13}f^2$	$e^{10}f^4$	e^9f^5	e^9f^4g	e^8f^5g	31
32																		$e^{15}f$	$e^{13}fg$	$e^{13}g^2$	e^8f^6	e^7f^7	32
33																		e^{17}	$e^{12}f^3$	$e^{12}f^2g$	$e^{12}fg^2$	$e^{12}g^3$	33
34																			$e^{15}g$	$e^{11}f^4$	$e^{11}f^3g$	$e^{11}f^2g^2$	34
35																			$e^{14}f^2$	$e^{14}fg$	$e^{10}f^5$	$e^{10}f^4g$	35
36																			$e^{16}f$	$e^{13}f^3$	$e^{14}g^2$	e^9f^6	36
37																			e^{18}	$e^{16}g$	$e^{13}f^2g$	$e^{13}fg^2$	37
38																				$e^{15}f^2$	$e^{12}f^4$	$e^{12}f^3g$	38
39																				$e^{17}f$	$e^{15}fg$	$e^{11}f^5$	39
40																				e^{19}	$e^{14}f^3$	$e^{15}g^2$	40
41																					$e^{17}g$	$e^{14}f^2g$	41
42																					$e^{16}f^2$	$e^{13}f^4$	42
43																					$e^{18}f$	$e^{16}fg$	43
44																					e^{20}	$e^{15}f^3$	44
45																						$e^{18}g$	45
46																						$e^{17}f^2$	46
																						$e^{19}f$	47
																						e^{21}	48

This means for instance that the derivatives of d^2 are

0	1	2	3	4	5	6
d^2	de	df e^2	dg ef	eg f^2	fg	g^2

those of d^3 are

0	1	2	3	4	5	6	7	8	9
d^3	d^2e	d^2f de^2	d^2g def e^3	deg df^2 e^2f	dfg e^2g ef^2	dg^2 efg f^3	eg^2 f^2g	fg^2	g^3

and so on; viz. to form any column we affix to the terms of the corresponding column of the subsidiary table the proper power of d, so that the degree in all the letters may be 2, 3, &c. as the case may be, using from each column of the subsidiary table only the terms which are not of too high a degree; for instance for col. 4 of the derivatives of d^3, only the terms eg, f^2, e^2f of the complete column eg, f^2, e^2f, e^3.

It is to be observed that these tables of the derivatives of d^2, d^3, &c. do not need to be actually formed; any column which is wanted can be written down at once, *currente calamo*, from the column of the subsidiary table. And if we require only the number of terms, then these numbers are at once taken out from the Subsidiary Table. Thus for the numerical column of the diagram, the Subsidiary Table, col. 9, contains but 1 term g^3 of degree 3, and thus the number set opposite to $(d^3)^9$ is 1; so col. 8 contains but 1 term fg^2 of the degree 3, and so the number opposite to $(d^3)^8$ is 1; col. 7 contains 2 terms eg^2, f^2g of degree 3, and so the number opposite to $(d^3)^7$ is 2; and so on: we have in this way the numbers 1, 1, 2, 5, 2, 3, 4, &c. the partial sums of which are 1, 1, 7, 9, 11, 3, 8, 9, 9 giving the total sum 58: viz. this is the number of the terms degree 6, weight 18, which can be formed with the seven letters (a, b, c, d, e, f, g).

When the literal terms are required, it is proper in the first instance as a safeguard against accidental omissions in copying, to take the numbers in this manner; and we can then take out the terms themselves, viz. these are

a^3	g^3
a^2b	fg^2
a^2c	eg^2
	f^2g
a^2	d^2g^2
	$defg$
	df^3
	e^3g
	e^2f^2,

and so on, the whole number being = 58 as just mentioned.