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DEMONSTRATION OF A THEOREM IN FINITE DIFFERENCES.

THE formula (B) [of Sir J. W. F. Herschel's Memoir], substituting therein for A_x the value $\frac{1}{[x]}\nabla^{-n}0^x$, becomes

$$\nabla^{-n}0^x = -\frac{1}{[n-1]}\left\{\frac{[x+n]}{[x-1]}\frac{1}{[n+1]}2.3\ldots n\,\nabla - \frac{\overline{[x+n]}}{[x-2]}\frac{1}{[n+2]}3.4\ldots\overline{n+1}\,\nabla^2 - \&\text{c.}\right\}0^x\,;$$

or, as this may be written,

$$\nabla^{-n}0^x = \frac{[x+n]}{[n-1]}\left\{\qquad -\frac{1}{[1][x-1](n+1)}\nabla + \frac{1}{[2][x-2](n+2)}\nabla^2 - \&\text{c.}\right\}0^x\,;$$

or, inserting a first term which vanishes except in the case $x = 0$, and which is required in order that the formula may hold good for this particular value,

$$\nabla^{-n}0^x = \frac{[x+n]}{[n-1]}\left\{\frac{1}{[0][x]n}\nabla^0 - \frac{1}{[1][x-1](n+1)}\nabla + \frac{1}{[2][x-2](n+2)}\nabla^2 - \&\text{c.}\right\}0^x\,;$$

where the series on the right-hand side need only be continued up to the term containing $\nabla^x 0^x$, since the subsequent terms vanish. [In these formulæ and throughout the present paper $[x]$ is written to denote the factorial $[x]^x$ or $\Pi(x)$, and so in other cases.]

Now $\nabla^{-n}0^x$, or $\left(\dfrac{\Delta}{\log(1+\Delta)}\right)^{-n}0^x$, is equal to $[x] \times$ coef. t^x in $\left(\dfrac{e^t-1}{t}\right)^{-n}$, and so $\nabla^q 0^x$, or $\left(\dfrac{\Delta}{\log(1+\Delta)}\right)^q 0^x$, is equal to $[x] \times$ coef. t^x in $\left(\dfrac{e^t-1}{t}\right)^q$.

Hence, putting $R = \dfrac{e^t - 1}{t}$, the last-mentioned formula will be true if, as regards the term which contains t^x, we have

$$R^{-n} = \frac{[x+n]}{[n-1]} \left\{ \frac{1}{[0][x]\,n} R^0 - \frac{1}{[1][x-1](n+1)} R^1 + \frac{1}{[2][x-2](n+2)} R^2 - \&c. \right\},$$

the series on the right-hand side being continued up to the term in R^x. This formula is, in fact, true if R, instead of being restricted to denote $\dfrac{e^t - 1}{t}$, denotes any function whatever of the form $1 + bt + ct^2 + \&c.$, and it is true not only for the term in t^x, but for all the powers of t not higher than t^x. And, moreover, R^{-n} may denote any positive or negative integral or fractional power of R. In fact, the formula (assuming for a moment the truth of it) shows that the expansion of any power whatever of a series of the form in question, can be obtained by means of the expansions of the successive positive integer powers of the same series: the existence of such a formula (at least for negative powers) was indicated by Eisenstein, *Crelle*, t. XXXIX. p. 181 (1850), and the formula itself, in a slightly different form, was obtained in a very simple manner by Professor Sylvester in his paper, "Development of an idea of Eisenstein," *Quart. Math. Journ.* t. I. p. 199 (1855); the demonstration was in fact as follows, viz. writing

$$R^n = (1 + \overline{R-1})^n = 1 + \frac{n}{1}(R-1) + \frac{n \cdot n - 1}{1 \cdot 2}(R-1)^2 + \&c. :$$

if we attend only to the terms involving powers of t not higher than t^x, the series on the right-hand side needs only to be continued up to the term involving $(R-1)^x$, and the right side being thus converted into a rational and integral function of R, it may be developed in a series of powers of R (the highest power being of course R^x), and the coefficients of the several powers are finite series which admit of summation; this gives the required formula. But there is an easier method; the process shows that the series on the right-hand side, continued as above up to the term involving t^x, is, as regards n, a rational and integral function of the degree x; and by Lagrange's interpolation formula, any rational and integral function of n of the degree x, can be at once expressed in terms of the values corresponding to $x + 1$ particular values of n. The investigation will be as follows:—Let R denote a series of the form $1 + bt + ct^2 + \&c.$, and let R^n denote the development of the nth power of R, continued up to the term containing t^x, the terms involving higher powers of t being rejected: R^0, R^1, R^2, &c., and generally R^s, will in like manner denote the developments of these powers up to the term involving t^x, or what is the same thing, they will be the values of R^n, corresponding to $n = 0, 1, 2, \ldots s$. By what precedes R^n is a rational and integral function of n of the degree x, and it can therefore be expressed in terms of the values R^0, R^1, R^2, $\ldots R^x$, which correspond to $n = 0, 1, 2, \ldots x$. Let s have any one of the last-mentioned values, then the expression

$$\frac{n \cdot n - 1 \cdot n - 2 \ldots n - x}{n - s} \, \frac{1}{s \cdot s - 1 \ldots 2 \cdot 1 \cdot -1 \cdot -2 \ldots -(x-s)},$$

which, as regards n, is a rational and integral function of the degree x (the factor

$n - s$, which occurs in the numerator and in the denominator being of course omitted), vanishes for each of the values $n = 0, 1, 2, .. x$, except only for the value $n = s$, in which case it becomes equal to unity. The required formula is thus seen to be

$$R^n = \Sigma \left\{ \frac{n \cdot n - 1 \cdot n - 2 \ldots n - x}{n - s} \frac{1}{s \cdot s - 1 \ldots 2 \cdot 1 \cdot - 2 \ldots - (x - s)} R^s \right\},$$

where the summation extends to the several values $s = 0, 1, 2, \ldots x$; or, what is the same thing, it is

$$R^n = \Sigma \left\{ \frac{n \cdot n - 1 \cdot n - 2 .. n - x}{n - s} \frac{(-)^{x-s}}{1 . 2 \ldots s . 1 . 2 \ldots (x - s)} R^s \right\};$$

or changing the sign of n, it is

$$R^{-n} = \Sigma \left\{ \frac{n \cdot n + 1 \cdot n + 2 .. n + x}{n + s} \frac{(-)^s}{1 . 2 \ldots s . 1 . 2 \ldots x - s} R^s \right\},$$

or, as this may be written,

$$R^{-n} = \frac{[x + n]}{[n - 1]} \Sigma \left\{ \frac{(-)^s}{[s][x - s](n + s)} R^s \right\}$$

or substituting for s the values $0, 1, 2, .. x$, the formula is

$$R^{-n} = \frac{[x + n]}{[n - 1]} \left\{ \frac{1}{[0][x] \, n} R^0 - \frac{1}{[1][x - 1](n + 1)} R^1 + \frac{1}{[2][x - 2](n + 2)} R^2 - .. \right\}$$

continued up to the term involving R^x, which is the theorem in question.