

828.

A MEMOIR ON SEMINVARIANTS.

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Introductory. Art. Nos. 1 to 3.

1. A VERY remarkable discovery in the Theory of Seminvariants has been recently made by Capt. MacMahon, viz. considering the equation

$$0 = 1 + b \frac{x}{1} + c \frac{x^2}{1 \cdot 2} + d \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.,$$

and its roots $\alpha, \beta, \gamma, \dots$, as defined by the identity

$$1 + b \frac{x}{1} + c \frac{x^2}{1 \cdot 2} + \&c. = 1 - \alpha x \cdot 1 - \beta x \cdot 1 - \gamma x \dots,$$

then, any symmetrical function of the roots being represented by a partition symbol in the usual manner, $1 = \Sigma \alpha$, $2 = \Sigma \alpha^2$, 11 or $1^2 = \Sigma \alpha \beta$, &c., the theorem is that any symmetric function represented by a non-unitary symbol (or symbol not containing a 1), say for shortness any non-unitary symmetric function, is a seminvariant in regard to the coefficients $1, b, c, d, e, \&c.$

We have for instance

$$2 = - (c - b^2),$$

$$3 = - \frac{1}{2} (d - 3bc + 2b^3),$$

$$4 = \frac{1}{6} (-e + 4bd + 3c^2 - 12bc + 6b^4),$$

where to verify that this is a seminvariant, observe that the value may be written

$$= \frac{1}{6} \{-(e - 4bd + 3c^2) + 6(c - b^2)^2\},$$

$$22 = \frac{1}{12}(e - 4bd + 3c^2),$$

&c.;

and observe further that the forms 2, 4 and 22, are connected by the identical relation

$$2 \cdot 2 = 4 + 2 \cdot 22.$$

2. We conclude that the theory of seminvariants is a part of that of symmetric functions. I take the opportunity of remarking that (the subject of the memoir being seminvariants) I use in general non-unitary symbols, even in cases where the restriction is unnecessary, and the symbols might have contained 1's: thus instead of $2 \cdot 2 = 4 + 2 \cdot 22$, the equation $1 \cdot 1 = 2 + 2 \cdot 11$ would have served equally well as an instance of an identical relation between symmetric functions; and so in general, in formulæ relating to symmetric functions, the symbols are not restricted to be non-unitary. I remark also that, for instance, instead of 4443322222, or $4^3 3^2 2^5$, I usually write 444332^5 , introducing the index only for the 2; the reason is only that the 2 is often repeated a large number of times, so that the abbreviation, which I dispense with for the higher numbers, becomes convenient for the 2.

3. Reckoning the coefficients 1, b, c, d, e, ... as being each of them of the degree 1, and of the weights 0, 1, 2, 3, 4, ... respectively, then any symmetric function is of a degree which is equal to the highest number, and of a weight which is equal to the sum of all the numbers, in the partition-symbol. And we frequently speak of the deg. weight: thus for the function 22, the deg. weight is = 2. 4.

Multiplication of Two Symmetric Functions. Art. Nos. 4 to 17.

4. We require a theory for the multiplication of two symmetric functions. We have for instance $3 \cdot 2 = 5 + 32$: for 3 denoting $\Sigma \alpha^3$, and 2 denoting $\Sigma \alpha^2$, the product contains the term α^5 , and the term $\alpha^3 \beta^2$, and it is thus = $\Sigma \alpha^5 + \Sigma \alpha^3 \beta^2$, which is = 5 + 32. But multiplying for instance 2 by itself, the product contains the term α^4 , and the term $\alpha^2 \beta^2$ twice, and it is thus = $\Sigma \alpha^4 + 2 \Sigma \alpha^2 \beta^2$, and we thus have the before-mentioned formula $2 \cdot 2 = 4 + 2 \cdot 22$.

And so, l, m being different

$$l \cdot m = (l + m) + lm;$$

but when $m = l$,

$$l \cdot l = (2l) + 2 \cdot ll.$$

And in general, for any symmetric function $lmnpqr \dots$, where the numbers are all of them different, if any two of the m's become equal, we must multiply the term by 2; if any three of them become equal, we must multiply the term by 6; and so in other cases, viz. if the term becomes $l^a m^b n^c \dots$, we must multiply it by $[\alpha]^a [\beta]^b [\gamma]^c \dots$

5. We may, taking in the first instance the numbers l, m, n, p, q, \dots to be all of them different, developpe an algorithm as follows :

$$\begin{array}{l|l} l & m \\ \hline m & \\ \hline \end{array} = \begin{array}{l} (l+m), \\ lm \end{array} \text{ that is, } l \cdot m = \begin{array}{l} (l+m) \\ + lm, \end{array}$$

$$\begin{array}{l|l} lm & n \\ \hline n & \\ \hline \end{array} = \begin{array}{l} (l+n)m, \\ l(m+n) \\ mn \end{array} \text{ that is, } lm \cdot n = \begin{array}{l} (l+n)m \\ + l(m+n) \\ + lmn, \end{array}$$

and so in other cases;

$$\begin{array}{l|l} lmn & p \\ \hline p & \\ \hline p & \\ \hline p & \\ \hline p & \\ \hline \end{array} = \begin{array}{l} (l+p)mn \\ l(m+p)n \\ lm(n+p) \\ lmn p, \end{array}$$

$$\begin{array}{l|l} lm & np \\ \hline np & \\ \hline pn & \\ \hline n & p \\ \hline p & n \\ \hline n & p \\ \hline p & n \\ \hline np & \\ \hline \end{array} = \begin{array}{l} (l+n)(m+p) \\ (l+p)(m+n) \\ (l+n)mp \\ (l+p)mn \\ l(m+n)p \\ l(m+p)n \\ lmn p, \end{array}$$

$$\begin{array}{l|l} lmn & pq \\ \hline pq & \\ \hline qp & \\ \hline p & q \\ \hline q & p \\ \hline pq & \\ \hline qp & \\ \hline p & q \\ \hline q & p \\ \hline p & q \\ \hline q & p \\ \hline p & q \\ \hline q & p \\ \hline pq & \\ \hline \end{array} = \begin{array}{l} (l+p)(m+q)n \\ (l+q)(m+p)n \\ (l+p)m(n+q) \\ (l+q)m(n+p) \\ l(m+p)(n+q) \\ l(m+q)(n+p) \\ (l+p)mnq \\ (l+q)mnp \\ l(m+p)q \\ l(m+q)p \\ lm(n+p)q \\ lm(n+q)p \\ lmn pq; \end{array}$$

and so on.

C. XII.

6. Observe that, if the two factors contain i numbers and j numbers respectively, $i >$ or $= j$, then in the product we have

$$\begin{aligned}
 & [i]^j \text{ terms each containing } i \text{ numbers,} \\
 & \frac{[j]^1}{[1]^1} [i]^{j-1} \quad \text{,,} \quad \text{,,} \quad i+1 \quad \text{,,} \\
 & \frac{[j]^2}{[2]^2} [i]^{j-2} \quad \text{,,} \quad \text{,,} \quad i+2 \quad \text{,,} \\
 & \vdots \\
 & \frac{[j]^j}{[j]^j} [i]^0, = 1, \text{ term containing } i+j \quad \text{,,}
 \end{aligned}$$

so that the whole number of terms in the product is

$$\{i, j\} = [i]^j + \frac{j}{1} [i]^{j-1} + \dots + 1.$$

We may, if we please, take the smaller number first, and we then have

$$\{j, i\} = \{i, j\};$$

the $\{j, i\}$ series in fact begins with zero terms, but following these we have terms which are identical each to each with those of the $\{i, j\}$ series. Thus

$$\begin{aligned}
 \{5, 3\} &= [5]^3 + 3[5]^2 + 3[5]^1 + [5]^0 = 60 + 3 \cdot 20 + 3 \cdot 5 + 1, = 136, \\
 \{3, 5\} &= [3]^5 + 5[3]^4 + 10[3]^3 + 10[3]^2 + 5[3]^1 + [3]^0 = 0 + 0 + 10 \cdot 6 + 10 \cdot 6 + 5 \cdot 3 + 1, = 136
 \end{aligned}$$

and it is easy to see that the general theorem can be verified in like manner.

In particular, we have

Putting $i = 1$, these give

$$\begin{aligned}
 \{i, 1\} &= [i]^1 + 1 &= i + 1 & , \{1, 1\} = 2, \\
 \{i, 2\} &= [i]^2 + 2[i]^1 + 1 &= i^2 + i + 1 & , \{1, 2\} = 3, \\
 \{i, 3\} &= [i]^3 + 3[i]^2 + 3[i]^1 + 1 &= i^3 + 2i + 1 & , \{1, 3\} = 4, \\
 \{i, 4\} &= [i]^4 + 4[i]^3 + 6[i]^2 + 4[i]^1 + 1 &= i^4 - 2i^3 + 5i^2 + 1 & , \{1, 4\} = 5, \\
 \{i, 5\} &= [i]^5 + 5[i]^4 + 10[i]^3 + 10[i]^2 + 5[i]^1 + 1 &= i^5 - 5i^4 + 15i^3 - 15i^2 + 9i + 1, & \{1, 5\} = 6,
 \end{aligned}$$

which agree with

$$\{i, 1\} = i + 1.$$

Hence also the values of

{1, 1},	are 2,
{1, 2}, {2, 2},	3, 7,
{1, 3}, {2, 3}, {3, 3},	4, 13, 34,
{1, 4}, {2, 4}, {3, 4}, {4, 4},	5, 21, 73, 209,
{1, 5}, {2, 5}, {3, 5}, {4, 5}, {5, 5},	6, 31, 136, 501, 1546.

7. In forming a product $lmn \dots pqr \dots$, we may have equalities among the numbers l, m, n, \dots of the first symbol, and also between the numbers p, q, r, \dots of the second symbol: moreover (whether there are or are not any such equalities), we may have equalities presenting themselves between the numbers $l+p, m+q, \dots$ of any symbol $(l+p)(m+q) \dots$ on the right-hand side of the equation: and the process must be performed so as to take account of all these equalities. The actual process is best shown by an example: say we require the product 3332.322 , which is of the deg. weight 6.18.

3332		322	÷ 12		
322		6	6552	2	1
32 2		12	6543	1	1
22 3		6	5553	6	3
32	2	12	65322	2	2
3 2	2	6	64332	2	1
2 3	2	6	55332	4	2
22	3	6	55332	4	2
2 2	3	6	54333	6	3
3	22	3	633222	12	3
3	22	1	533322	12	1
2	32	6	533322	12	6
2	32	2	433332	24	4
	322	1	3333222	144	12
		73			

8. Observe first that $\{4, 3\} = 24 + 36 + 12 + 1, = 73$. In placing all or any of the numbers of the 322 under those of the 3332, we do this in all the really distinct ways, inserting a numerical coefficient for the frequency of each way. Thus when the whole of the 322 is thus placed, the 3 may be under a 3, and the two 2's may then be under 33 or under 32: or else the 3 may be under a 2, and the two 2's must then be under 33: there are thus three ways, and these have the frequencies 6, 12, 6 respectively. For as to the first way, the 3 may be under any one of the three 3's, and for each such position of the 3, the 22 can be placed in either of two orders under the other two 3's; the frequency is thus $3 \cdot 2, = 6$. And similarly the other two frequencies are 12 and 6; the sum $6 + 12 + 6, = 24$, is, the first term of $\{4, 3\}$. In like manner, when two of the numbers of the 322 are placed under the 3332, there are five ways having the frequencies 12, 6, 6, 6, 6 respectively, the sum of these frequencies is 36, which is the second term of $\{4, 3\}$. And so when one number of the 322 is placed under the 3332, there are four ways having the frequencies 3, 1, 6, 2 respectively: the sum of these frequencies is 12, which is the

third term of {4, 3}. Finally, when no number of the 322 is placed under the 3332, there is a single way, having the frequency 1, which is the last term of {4, 3}. These agreements are a useful verification for the frequencies: the sum of all the frequencies is of course = 73, the value of {4, 3}.

9. We next form the column of symmetric functions by adding to each line the 3332: to avoid accidental errors of addition, it is proper to verify for each of these that the weight is the sum, = 18, of the weights 11 and 7 of the two factors respectively. It is to be observed that the same symmetric function may present itself more than once: thus we have the functions 55332, and 533322, each of them twice. We then form a column of multiplicities: in 6552, the two 5's give the multiplicity 2; in 5553, the three 5's give 6: in 633222, the two 3's and the three 2's give 2. 6, = 12, and so on. There is in like manner for the factor 3332 a multiplicity 6, and for the factor 322 a multiplicity 2; and these combined together give 6.2, = 12, viz. this is the heading ÷ 12 of the column. And then forming the products 6.2, 12.1, 6.6, &c., of the corresponding frequencies and multiplicities and dividing in each case by 12, we have the right-hand column of numerical coefficients.

10. The result is to be read

$$\begin{aligned} 3332.322 &= 1.6552 \\ &+ 1.6543 \\ &+ 3.5553 \\ &+ \&c., \end{aligned}$$

the coefficients 2, 2 and 1, 6 of the repeated terms being of course united together, so that we have

$$\begin{aligned} &+ 4.55332 \\ &+ 7.533322. \end{aligned}$$

With a little practice, the operation is performed without difficulty and with very small risk of error.

11. We may apply the process to obtain analytical formulæ for certain forms of products. Consider, for instance, the product $2^\alpha \cdot 2^\beta$, where $\alpha >$ or $= \beta$: and in the coefficients write for shortness α , instead of $[\alpha]^\alpha$ or $\Pi\alpha$, to denote $1.2.3 \dots \alpha$. We have

$$\frac{2^\alpha}{2^A} \left| \frac{2^\beta}{2^{\beta-A}} \right. \frac{\alpha}{A \cdot \alpha - A} \frac{\beta}{\beta - A} \cdot 4^A 2^{\alpha+\beta-2A} \cdot A \cdot \alpha + \beta - 2A \left. \right| \frac{\alpha + \beta - 2A}{\alpha - A \cdot \beta - A},$$

viz. the formula is

$$2^\alpha \cdot 2^\beta = \sum \frac{\alpha + \beta - 2A}{\alpha - A \cdot \beta - A} \cdot 4^A 2^{\alpha+\beta-2A},$$

where A has any integer value from β to 0; the first term is $= 1 \cdot 4^\beta 2^{\alpha-\beta}$, and the last term is $\frac{\alpha + \beta}{\alpha \cdot \beta} 2^{\alpha+\beta}$. The weight of any term is $4A + 2(\alpha + \beta - 2A)$, $= 2\alpha + 2\beta$, as it should be.

In explanation as to the frequency, observe that out of the β 2's we take any A 2's and place them in any order under any A of the α 2's. The number of combinations taken is $\frac{\beta}{A \cdot \beta - A}$, which (in A orders) gives $\frac{\beta}{\beta - A}$ sets to be placed under $\frac{\alpha}{A \cdot \alpha - A}$ sets out of the α 2's: we thus have the foregoing coefficient of frequency $\frac{\alpha}{A \cdot \alpha - A} \cdot \frac{\beta}{\beta - A}$, where as mentioned above α , &c. are written to denote $[\alpha]^a$, &c.

12. We may in like manner find a formula for the product $3^a 2^\beta \cdot 3^\gamma 2^\delta$. Taking $\gamma = x + y + z$, any partition of γ , and $\delta = p + q + r$, any partition of δ , and writing also $A = x$, $B = y + p$, $C = q$, we have

$$\frac{3^a 2^\beta \cdot 3^\gamma 2^\delta}{3^{x+2p} 3^{y+2q}} \mid \frac{3^{\gamma} 2^{\delta}}{3^{2r}} M \cdot 6^A 5^{B+4C} 3^{\alpha+\gamma-2A-B} 2^{\beta+\delta-B-2C} \cdot N \mid \div \alpha \cdot \beta \cdot \gamma \cdot \delta$$

where for the frequency, M , we have

$$\begin{aligned} \text{No. of terms } 3^x &= \frac{\gamma}{x \cdot \gamma - x}; & 3^y &= \frac{\gamma - x}{y \cdot z}, \\ \text{,, ,, } 2^p &= \frac{\delta}{p \cdot \delta - p}; & 2^q &= \frac{\delta - p}{q \cdot r}, \end{aligned}$$

to be placed under

$$\begin{aligned} \text{sets of terms } 3^a &= \frac{\alpha}{x + p \cdot \alpha - x - p}; & 2^\beta &= \frac{\beta}{y + q \cdot \beta - y - q}, \\ \text{in orders,} &= x + p; & &= y + q, \end{aligned}$$

whence, multiplying, we have for the frequency

$$M = \frac{\alpha \cdot \beta \cdot \gamma \cdot \delta}{x \cdot y \cdot z \cdot p \cdot q \cdot r \cdot \alpha - x - p \cdot \beta - y - q};$$

and for the multiplicity, N , we have

$$\begin{aligned} N &= A \cdot B \cdot C \cdot \alpha + \gamma - 2A - B \cdot \beta + \delta - B - 2C, \\ &= x \cdot B \cdot q \cdot \alpha + \gamma - 2A - B \cdot \beta + \delta - B - 2C. \end{aligned}$$

13. The coefficient is thus

$$\begin{aligned} \frac{M \cdot N}{\alpha \cdot \beta \cdot \gamma \cdot \delta} &= \frac{B \cdot \alpha + \gamma - 2A - B \cdot \beta + \delta - B - 2C}{y \cdot z \cdot p \cdot r \cdot \alpha - x - p \cdot \beta - y - q}, \\ &= \frac{B \cdot \alpha + \gamma - 2A - B \cdot \beta + \delta - B - 2C}{y \cdot p \cdot \gamma - A - y \cdot \alpha - A - p \cdot \beta - C - y \cdot \delta - C - p}, \end{aligned}$$

or, putting also

$$\begin{aligned} D &= \alpha + \gamma - 2A - B, \\ E &= \beta + \delta - B - 2C, \end{aligned}$$

whence

$$6A + 5B + 4C + 3D + 2E = 3(\alpha + \gamma) + 2(\beta + \delta),$$

the formula finally is

$$3^\alpha 2^\beta \cdot 3^\gamma 2^\delta = \Sigma \Lambda \cdot 6^A 5^B 4^C 3^D 2^E,$$

where

$$\Lambda = \frac{B \cdot D \cdot E}{y \cdot p \cdot \gamma - A - y \cdot \alpha - A - p \cdot \beta - C - y \cdot \delta - C - p},$$

or, as this may also be written,

$$= \frac{B}{y \cdot B - y} \cdot \frac{D}{z \cdot D - z} \cdot \frac{E}{r \cdot E - r},$$

so that the coefficient Λ is in fact the product of three binomial coefficients: it must, however, be recollected that the same term $6^A 5^B 4^C 3^D 2^E$ may occur more than once, with different coefficients Λ , so that in the final result, when these terms are united together, the numerical coefficients are not each of them of the form in question.

14. The limits of the summation are conveniently defined by means of the diagram

A	p	$\alpha - A - p$	α	$D = \alpha + \gamma - 2A - B,$
y	C	$\beta - C - y$	β	$E = \beta + \delta - B - 2C,$
z	r			<i>ut suprâ,</i>
γ	δ			

viz. here the sums of the first and second lines are α and β respectively, and the sums of the first and second columns are γ and δ respectively; we have $B = y + p$, a partition of B ; but the values of y, p must be such as not to render negative any one of the four terms $z, r, \alpha - A - p, \beta - C - y$ of the diagram: for if any one of these numbers were negative, the corresponding factorial in the denominator of Λ would be infinite and we should have $\Lambda = 0$. For any given term $6^A 5^B 4^C 3^D 2^E$ of the proper weight $3(\alpha + \gamma) + 2(\beta + \delta)$, there may be no suitable values of y, p , and the coefficient is then $= 0$: there may be a single pair of values, and the coefficient is then (as remarked above) = a product of three binomial coefficients: or there may be more than a single pair of values, and the entire coefficient of the term has not in this case a like simple form.

15. To exhibit the working of the formula, I apply it to the recalculation of the foregoing product $3332 \cdot 322$ ($\alpha, \beta, \gamma, \delta = 3, 1, 1, 2$). Properly the whole series of symbols 666, 6642, 6633, &c., of the symmetric functions of the weight 18 should be written down, and the coefficient be calculated for each of them: but I write down only those which have coefficients not $= 0$: for each of these, I take *all* the partitions

$B = y + p$, several of these giving, as will be seen, zero values, and the others giving the values already obtained for the coefficients of the several terms.

		1			2			3			1	2	3						
	A	B	C	D	E	B	y	p	D	$\gamma - A - y$	$\alpha - A - p$	E	$\beta - C - y$	$\delta - C - p$					
6552	1	2	0	0	1	2	1	1	0	-1	1	1	0	1	1	0	0	1	1.6552
6543	1	1	1	1	0	1	0	1	1	-1	2	0	-1	1	1	0	0	1	1.6543
6532 ²	1	1	0	1	2	1	1	0	1	-1	2	2	0	2	1	0	1	1	2.6532 ²
64332	1	0	1	2	1	0	0	0	2	0	2	1	0	1	1	1	1	1	1.64332
6332 ³	1	0	0	2	3	0	0	0	2	0	2	3	1	2	1	1	3	3	6332 ³
5553	0	3	0	1	0	3	2	1	1	-2	3	0	-1	1	3	0	0	3	3.5553
55332	0	2	0	2	1	2	1	1	2	-1	3	2	0	1	2	1	1	2	4.55332
54333	0	1	1	3	0	1	0	1	3	0	3	0	-1	1	1	1	0	1	3.54333
53332 ²	0	1	0	3	2	1	1	0	3	0	3	2	0	1	1	1	1	1	7.53332 ²
433332	0	0	1	4	1	0	0	0	4	1	3	1	0	1	1	4	1	4	4.433332
33332 ³	0	0	0	4	3	0	0	0	4	1	3	3	1	2	1	4	3	12	33332 ³

It is quite possible that abbreviations and verifications might be introduced, but the process as it stands seems to be at once less expeditious and less safe than the one first made use of.

16. Particular cases of the general formula are

$$2^\beta \cdot 2^\delta = \Sigma \Lambda 4^C 2^E; \quad E = \beta + \delta - 2C, \text{ and thence } 4C + 2E = 2(\beta + \delta),$$

$$\Lambda = \frac{E}{\beta - C \cdot \delta - C},$$

which agrees with a result before obtained. Again,

$$3^\alpha 2^\beta \cdot 2^\delta = \Sigma \Lambda \cdot 5^B 4^C 3^D 2^E; \quad D = \alpha - B,$$

$$E = \beta + \delta - B - 2C,$$

and thence

$$5B + 4C + 3D + 2E = 3\alpha + 2\beta + 2\delta,$$

$$\Lambda = \frac{E}{\beta - C \cdot \delta - B - C}.$$

17. We may, in like manner with the formula for $3^\alpha 2^\beta \cdot 3^\gamma 2^\delta$, obtain the following:

$$4^\theta 3^\alpha 2^\beta \cdot 2^\delta = \Sigma \Lambda \cdot 6^A 5^B 4^C 3^D 2^E,$$

where

$$D = \alpha - B,$$

$$E = 2\theta + \beta + \delta - 3A - B - 2C,$$

and thence

$$6A + 5B + 4C + 3D + 2E = 4\theta + 3\alpha + 2\beta + 2\delta,$$

and the value of the coefficient Λ is

$$\Lambda = \frac{C \cdot E}{\theta - A \cdot C - \theta + A \cdot \theta + \beta - A - C \cdot \theta + \delta - 2A - B - C},$$

viz. Λ is the product of two binomial coefficients: and since here a given term $6^A 5^B 4^C 3^D 2^E$ occurs once only, each numerical coefficient is actually of this form.

In the particular case $\theta = 0$, we must have $A = 0$ (this appears *à posteriori* from the denominator factor $-A$, a factorial which is infinite for any positive value of A); and we thus obtain the first given formula for $3^\alpha 2^\beta \cdot 2^\delta$.

Capitation and Decapitation. Art. Nos. 18 to 21.

18. I explain the converse processes of Decapitation and Capitation. In any symmetric function, for instance 6552 of the degree 6, the whole coefficient of α^6 is 552, this symbol referring in the first instance to the series of remaining roots $\beta, \gamma, \delta, \dots$; but as the series of roots is unlimited, we may ultimately replace these by $\alpha, \beta, \gamma, \dots$, and so use 552 in its original sense. Similarly, in 6652, the whole coefficient

of α^6 is = 652, the only difference being that, while in the former case the degree is reduced to 5, in the present case it remains = 6. In every case we decapitate the symbol by striking out the highest number—in the case of two or more equal numbers, one only of these being struck out. Observe that by decapitation we always diminish the weight, but we do or do not diminish the degree. In a product such as 3332.322 we obtain in like manner the whole coefficient of α^6 by the decapitation of each factor, viz. the coefficient is = 332.22: and in any equation such as that obtained above for the product 3332.322, the whole coefficient of the highest power of α must be = 0, viz. we can by decapitation obtain a new equation of lower weight: thus from the equation in question of weight 18, we obtain the new equation of weight 12,

$$\begin{aligned} 332.22 &= 1.552 \\ &+ 1.543 \\ &+ 2.5322 \\ &+ 1.4332 \\ &+ 3.33222, \end{aligned}$$

where observe that the terms of a degree lower than 6 in the original equation give no term in the new equation. The new equation of the deg. weight 5.12 might of course be obtained independently in like manner with the original equation.

19. We capitate a symbol by prefixing to it a number which is not less than the highest number contained in it: thus 552 may be capitated into 5552, 6552, &c.: and so a product 332.22 may be capitated into 3332.222, 3332.322, &c.; moreover a single symbol may be capitated into a product, 552 into 5552.4: in fact, the capitation may be any operation such that by decapitation we reproduce the original symbol. The increase of weight may be any number not less than the degree of the original symbol: but it is usually taken to be a given number: thus for any symbol of a degree not exceeding 6, we may capitate so as to increase the weight by 6. The capitation does or does not increase the degree.

20. An identical equation may be capitated in a variety of ways, but instead of an identity we obtain only a congruence, that is, an equation which requires to be completed by the adjunction to it of proper terms of lower degrees. Thus from the above equation of the deg. weight 5.12 we may obtain

$$\begin{aligned} 3332.322 &\equiv 1.6552 \\ &+ 1.6543 \\ &+ 2.65322 \\ &+ 1.64332 \\ &+ 3.633222. \end{aligned}$$

Imagine here all the terms brought to the same side so that the form is $\Omega \equiv 0$: Ω is a function not containing α^6 , for the whole coefficient of α^6 therein is precisely

that function which by the equation of the deg. weight 5.12 is expressed to be $=0$: and hence Ω , *quâ* symmetric function cannot contain β^6, γ^6, \dots ; viz. Ω is a symmetric function of the degree 5 at most: the congruence $\Omega \equiv 0$ thus means, $\Omega =$ a properly determined function of the degree 5 at most. Obviously by development of the term 3332.322, that is, by substituting for this term its value as given by the equation of the deg. weight 6.18, the function of the degree 5 at most would be found to be $=$ the sum of those terms in the expression of 3332.322, which are of a degree inferior to 6: and the congruence $\Omega \equiv 0$ thus completed would be nothing else than the equation of the deg. weight 6.18.

21. We might have capitated in a different manner: for instance

$$\begin{aligned} 4332.222 &= 1.6552 \\ &+ 1.6543 \\ &+ 2.65322 \\ &+ 1.44332.2 \\ &+ 3.33222.3, \end{aligned}$$

there being here three products requiring to be developed: on replacing them by their values, there would be found for Ω a value which would be a determinate function of a degree less than 6: and putting $\Omega =$ this value, the congruence $\Omega \equiv 0$ would be completed into an equation.

Seminvariants of a given Degree; Perpetuants, &c. Art. Nos. 22 to 38.

22. We consider now seminvariants according to their degrees; in particular, those of the degrees 2, 3, 4, 5 and 6, or say quadric, cubic, quartic, quintic and sextic seminvariants: the forms of these are $2^{\alpha+1}, 3^{\alpha+1}2^{\beta}, 4^{\alpha+1}3^{\beta}2^{\gamma}, 5^{\alpha+1}4^{\beta}3^{\gamma}2^{\delta}, 6^{\alpha+1}5^{\beta}4^{\gamma}3^{\delta}2^{\epsilon}$, where each exponent $\alpha, \beta, \gamma, \delta, \epsilon$ is a positive integer not excluding zero: the exponent of the highest number is in each case written $\alpha+1$, and has thus the value 1 at least, for otherwise the form would not be of the proper degree. The several weights are $2(\alpha+1), 3(\alpha+1)+2\beta, 4(\alpha+1)+3\beta+2\gamma, \&c.$, or, what is the same thing, we have for the several degrees respectively

$$\begin{aligned} w-2 &= 2\alpha, \\ w-3 &= 3\alpha+2\beta, \\ w-4 &= 4\alpha+3\beta+2\gamma, \\ w-5 &= 5\alpha+4\beta+3\gamma+2\delta, \\ w-6 &= 6\alpha+5\beta+4\gamma+3\delta+2\epsilon, \end{aligned}$$

and we have for a given degree and weight as many seminvariants as there are systems of exponents satisfying the corresponding equation.

23. These numbers are at once expressible by means of a series of Generating Functions (G. F.), viz. writing for shortness **2**, **3**, **4**, **5**, **6** to denote $1 - x^2$, $1 - x^3$, $1 - x^4$, $1 - x^5$, $1 - x^6$, the G. F.'s are

$$\frac{x^2}{2}, \frac{x^3}{2.3}, \frac{x^4}{2.3.4}, \frac{x^5}{2.3.4.5}, \frac{x^6}{2.3.4.5.6}.$$

In fact, the number of seminvariants of a given weight is = coefficient of x^w in the corresponding G. F.; for the quadric seminvariants (or those of deg. weight $2.w$) in $\frac{x^2}{2}$; for the cubic seminvariants (or those of deg. weight $3.w$) in $\frac{x^3}{2.3}$; and so for the others.

24. A seminvariant of a given degree may be a sum of products (of that degree) of seminvariants of lower degrees, and of seminvariants of lower degrees: and it is in this case said to be reducible: a seminvariant which is not reducible is said to be irreducible, or otherwise to be a perpetuant. This notion of a perpetuant is due to Sylvester, see his Memoir "On Subinvariants, i.e. Semi-Invariants to Binary Quantics of an Unlimited Order," *American Journal of Mathematics*, vol. v. (1882—83), pp. 78—137 (§ 4 Perpetuants, pp. 105—118). In speaking of the number of perpetuants of a given deg. weight, we assume throughout that these are independent perpetuants, not connected by any linear relation.

25. Since the seminvariants used for the reduction of a given reducible seminvariant can themselves be expressed in terms of perpetuants, we may say more definitely that a seminvariant of a given degree, which is a sum of products (of that degree) of perpetuants of lower degrees, and of perpetuants of lower degrees, is reducible. The words "of that degree" are essential to the definition: a seminvariant may be expressible as a sum of products (of a higher degree) of perpetuants of lower degrees, and of perpetuants of lower degrees, and it is not on this account reducible: a seminvariant so expressible is said to be a "syzygant"; but as to this, see No. 49.

26. Every quadric or cubic seminvariant is obviously a perpetuant: the quadric and cubic perpetuants have thus the before-mentioned G. F.'s $\frac{x^2}{2}$ and $\frac{x^3}{2.3}$ respectively.

27. A reducible quartic seminvariant can only be a sum of products (2.2) of two quadric perpetuants, and of quadric perpetuants, and it is clear that no quartic seminvariant the symbol of which contains a **3** is thus expressible. If the symbol does not contain a **3**, viz. when the form is $4^{a+1} 2^r$, the seminvariant is reducible: we have for instance

$$4 = 2.2 - 2.22,$$

$$42 = 22.2 - 3.222, \text{ \&c.}$$

To show that this is so in general, observe that any symmetric function $2^{a+1} 1^r$, *quâ* symmetric function can be expressed as a rational and integral function of the

degree $2(\alpha + 1) + \gamma$, of the coefficients 1, 1^2 , 1^3 , &c.: instead of the roots considering their squares, we have thence an expression for the quartic seminvariant $4^{\alpha+1} 2^\gamma$ in terms of the quadric perpetuants 2, 2^2 , 2^3 , &c., and such expression will be of the same degree $4(\alpha + 1) + 2\gamma$, as the quartic seminvariant.

It thus appears, as regards the quartic seminvariants, that whenever the symbol contains a 3, and in this case only, the seminvariant is a perpetuant: or, what is the same thing, the form of a quartic perpetuant is $4^{\alpha+1} 3^{\beta+1} 2^\gamma$: for the weight w , the number is equal to that of the sets of values α, β, γ , such that $w - 7 = 4\alpha + 3\beta + 2\gamma$: or, what is the same thing, the G. F. of the quartic perpetuants is $= \frac{x^7}{2.3.4}$.

28. Sylvester, in the memoir referred to, obtained this result in a different manner: the quartic seminvariants of a given weight are the quartic perpetuants of that weight and also the products (of that weight) of two quadric perpetuants, the same or different: say (4) is the G. F. for the perpetuants, and (2, 2) for the products: then the G. F. for the quartic seminvariants being as already mentioned $\frac{x^4}{2.3.4}$, we have his equation

$$(4) + (2, 2) = \frac{x^4}{2.3.4}.$$

He deduces (2, 2) from the G. F. $= \frac{x^2}{2}$ of the quadric perpetuants and thence obtains

$$(4), = \frac{x^7}{2.3.4}, \text{ as above.}$$

29. Write for a moment ϕx to represent the G. F. $= \frac{x^2}{2}$ of the quadric perpetuants, and A, B, C, \dots to represent these quadric perpetuants: we have, in an algorithm which will be readily understood,

$$\phi x = (A + B + C \dots),$$

$$(\phi x)^2 = (A + B + C \dots)^2, = A^2 + 2AB + \dots,$$

$$\phi x^2 = A^2 + \&c.,$$

and thence

$$\frac{1}{2} \{(\phi x)^2 + \phi x^2\} = A^2 + AB + \&c.;$$

viz. the G. F. (2, 2), is

$$\frac{1}{2} \{(\phi x)^2 + \phi x^2\}, = \frac{1}{2} \left(\frac{x^4}{2.2} + \frac{x^4}{4} \right), = \frac{1}{2} \left(\frac{x^4(1+x^2)}{2.4} + \frac{x^4(1-x^2)}{2.4} \right), = \frac{x^4}{2.4},$$

and we thence have

$$(4) + \frac{x^4}{2.4} = \frac{x^4}{2.3.4},$$

that is, (4) $= \frac{x^7}{2.3.4}$, the same result as was found above by independent considerations.

30. Sylvester established in like manner (but without the terms S which will be presently explained) the equations

$$(5) + (3, 2) = \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + S_5,$$

$$(6) + (4, 2) + (3, 3) + (2, 2, 2) = \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + S_6,$$

viz. here (5) is the G. F. for the quintic perpetuants, (2, 3) that for the products of the quadric and the cubic perpetuants: and similarly (6) is the G. F. for the sextic perpetuants, (4, 2) that for the products of the quadric and the quartic perpetuants, (3, 3) for the products of two cubic perpetuants, the same or different: and (2, 2, 2) for the products of three quadric perpetuants, the same or different. We have at once $(3, 2) = (3) \cdot (2)$; $(4, 2) = (4) \cdot (2)$; $(3, 3)$ is found by the same process as was used for finding (2, 2), substituting only $\frac{x^3}{3}$ for $\frac{x^2}{2}$; and (2, 2, 2) is found by a like process, viz. the G. F. for $A^3 + A^2B + ABC$ is $\frac{1}{6} \{(\phi x)^3 + 3\phi x \cdot \phi x^2 + 2\phi x^3\}$, where $\phi x = \frac{x^2}{2}$ as before, viz. this is $\frac{1}{6} \left\{ \frac{x^6}{2 \cdot 2 \cdot 2} + \frac{3x^6}{2 \cdot 4} + \frac{2x^6}{6} \right\}$: reducing to the common denominator 2.4.6, the numerator is

$$= \frac{1}{6} x^6 \{ (1 + x^2)(1 + x^2 + x^4) + 3(1 - x^6) + 2(1 - x^2)(1 - x^4) \},$$

viz. this is $= x^6$. The several functions thus are

$$(3, 2) = \frac{x^5}{2 \cdot 2 \cdot 3}, \quad (4, 2) = \frac{x^6}{2 \cdot 2 \cdot 3 \cdot 4},$$

$$(3, 3) = \frac{x^6}{3 \cdot 6}, \quad (2, 2, 2) = \frac{x^6}{2 \cdot 4 \cdot 6}.$$

31. Mr Hammond, in regard to the equation for the quintic perpetuants, made the very important observation—see his paper “On the Solution of the Differential Equation of Sources,” *American Journal of Mathematics*, t. v. (1882), pp. 218—227—that the products (3, 2) of a cubic perpetuant and a quadric perpetuant are not independent: we have between them syzygies such as $32 \cdot 2 - 3 \cdot 22 \equiv 0$ (viz. the difference of the two products contains no term of the degree 5; the actual value is $= 43 + 322$: Hammond’s equation (12), p. 222), hence the necessity in the equation of a term S_5 referring to these syzygies; and he moreover obtained the expression, $S_5 = \frac{x^7}{2 \cdot 4}$ of the G. F. for these syzygies.

The equation gives

$$(5) = \frac{-x^7 + x^{10} + x^{12}}{2 \cdot 3 \cdot 4 \cdot 5} + S_5,$$

where of course the first term is the value of (5) which would be given by the equation without the term S_5 ; and substituting herein for S_5 the foregoing value, viz.

$$S_5 = \frac{x^7}{2 \cdot 4}, = \frac{x^7(1 - x^3)(1 - x^5)}{2 \cdot 3 \cdot 4 \cdot 5},$$

we find

$$(5) = \frac{x^{15}}{2 \cdot 3 \cdot 4 \cdot 5},$$

which is the correct value of the G. F. for the quintic perpetuants: the lowest quintic perpetuant is thus of the weight 15.

32. The equation for the sextic perpetuants gives

$$(6) = \frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + S_6,$$

which is an equation connecting (6) and S_6 , the G. F.'s for the sextic perpetuants, and the sextic syzygies respectively. I have, in the investigation of the value of S_6 , met with a difficulty which I have not been able to overcome: but I find that

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18} + \omega(x)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6},$$

where $\omega(x)$ is possibly the monomial function x^{31} , but this result (which Capt. MacMahon believes to be true) is not yet completely established; it is a function containing no term lower than x^{31} . We have therefore

$$(6) = \frac{\omega(x)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6},$$

and there is, it would appear, no sextic perpetuant of a weight lower than 31.

33. But before entering on the investigation, it is proper to further develop the theory of the quintic perpetuants. We have quintic seminvariants: 5 for weight 5; 52 for weight 7; 53 for weight 8; 54, 522 for weight 9; 55, 532 for weight 10; 542, 533, 5222 for weight 11; and so on. These are reduced by means of the products 3.2 of a cubic perpetuant and a quadric perpetuant; viz. for any given weight, we have all the products $3^a 2^b$ of that weight. Thus for weight 5 there is only the product 3.2, and this in fact serves to reduce the seminvariant 5: we have $3 \cdot 2 = 5 + 32$, and therefore $5 = 3 \cdot 2 - 32$.

For the weight 7; there is only the seminvariant 52, and there are the two products 32.2 and 3.22: either of these would serve for the reduction: we have

$$\begin{aligned} 32 \cdot 2 &= 52, & 3 \cdot 22 &= 52, \\ &+ 43, & &+ 322, \\ &+ 2 \cdot 322, & & \end{aligned}$$

and these two equations imply the before-mentioned syzygy, $32 \cdot 2 - 3 \cdot 22 \equiv 0$, in virtue of which the two reductions become equivalent; or say there remains a single equation serving for the reduction: the most simple form is $52 = 3 \cdot 22 - 322$.

Weight 8: there is only the seminvariant 53, and the product 33.2: this gives the reduction.

Weight 9: seminvariants 54, 522: products $32^2 \cdot 2$, $32 \cdot 2^2$, $3 \cdot 2^3$; there is between the first and last of these a syzygy; and this being satisfied, there remain two equations for the reductions.

Weight 10: seminvariants 55, 532: products 332.2, 33.22: these give the reductions.

Weight 11: seminvariants 533; 542, 5222: products 333.2; 32³.2, 32².2², 32.2³, 32⁴.

34. Observe that the seminvariants, and in like manner the products, form two classes, according as the symbols contain three odd numbers or a single odd number: these correspond separately to each other, for the development of any product will contain only seminvariants having each of them as many odd numbers as there are odd numbers in the product. Hence for the weight 11 just referred to, 333.2 serves for the reduction of 533; 32³.2, 32².2², 32.2³, 3.2⁴ are connected the first and fourth of them by a syzygy, and the second and third by a syzygy; and there remain two equations serving for the reduction of the two seminvariants 542, 5222.

It is easy to show in this manner that there is no quintic perpetuant for any weight under 15; and that there is a single quintic perpetuant for the weight 15.

35. Generally the syzygies exist only for an odd weight $w = 2\beta + 3$ between the products $32^{\beta-1}.2, 32^{\beta-2}.2^2, \dots, 32.2^{\beta-1}, 3.2^{\beta}$; viz. there is a syzygy between the first and last terms: a syzygy between the second and last but one terms; and so on. The existence of these syzygies at once appears from the principle of decapitation: decapitating $32^{\beta-1}.2 - 3.2^{\beta}$, we have $2^{\beta-1} - 2^{\beta-1}$, which is identically = 0, hence the function contains no term of the degree 5, that is, $32^{\beta-1}.2 - 3.2^{\beta} \equiv 0$; and similarly for the other pairs of terms.

There are β terms: hence in the case β even, we have $\frac{1}{2}\beta$ pairs of terms and therefore $\frac{1}{2}\beta$ syzygies: in the case β odd, there is a middle term, not connected by a syzygy with any other term, and the number of syzygies is thus $= \frac{1}{2}(\beta - 1)$: writing for β its value $\frac{1}{2}(w - 3)$, the number is $= \frac{1}{4}(w - 3)$, and $\frac{1}{4}(w - 5)$ in the two cases respectively: and it thus appears that the G. F. is $= \frac{w^7}{2.4}$ as already mentioned.

36. In the case w an odd number, we have seminvariants, and in like manner products, containing respectively one odd number, three odd numbers, five odd numbers, and so on: thus $w = 15$, we have

Seminvariants.	Products.	
555	3332 ² .2	3 equations,
5532	3332.2 ²	
5433	333.2 ³	
5332 ²		
5442	32 ⁵ .2	$\frac{1}{2} 6, = 3$ equations;
542 ³	32 ⁴ .2 ²	
52 ⁵	32 ³ .2 ³	
	32 ² .2 ⁴	
	32.2 ⁵	
	3.2 ⁶	

hence the seminvariants 555 , 5532 , 5433 , 5332^2 with three odd numbers are not reducible, but they can be linearly expressed in terms of the 3 like products $3332^2 \cdot 2$, $3332 \cdot 2^2$, $333 \cdot 2^3$, and of $4-3$, = 1 arbitrary quantity (observe, however, that this must not be the seminvariant 5332^2 , for this is in fact reducible): the seminvariants 5442 , 542^2 , 52^5 , with one odd number, are reducible. And the like as regards any other odd value of w .

37. In the case w an even number, we have seminvariants, and in like manner products, containing respectively two odd numbers, four odd numbers, six odd numbers, and so on. Thus $w=16$, we have

Seminvariants.	Products.	
5533	$33332 \cdot 2$	2 equations,
53332	$3333 \cdot 2^2$	
5542	$332^4 \cdot 2$	5 equations;
552^3	$332^3 \cdot 2^2$	
5443	$332^2 \cdot 2^3$	
5432^2	$332 \cdot 2^4$	
532^4	$33 \cdot 2^5$	

and thus the seminvariants with four odd numbers, and those with two odd numbers, are each set reducible.

38. I give in the case $w=19$ the following results: the expression for the G. F. shows that there are 2 quintic perpetuants: viz. two forms X , Y such that every quintic seminvariant of the weight 19 is expressible as a linear function of these, of products (3.2) of a cubic and a quadric perpetuant, and of forms of a degree inferior to 5, that is, quartic, cubic and quadric perpetuants. Attending only to the terms in X , Y , the actual values are:

$$\begin{aligned}
 5554 &= X, \\
 5552^2 &= Y, \\
 55432 &= -X, \\
 55333 &= 0, \\
 5532^3 &= -Y, \\
 54442 &= 0, \\
 54433 &= X, \\
 5442^3 &= 0, \\
 54332^2 &= Y, \\
 542^5 &= 0, \\
 533332 &= 0, \\
 5332^4 &= 0, \\
 52^7 &= 0,
 \end{aligned}$$

viz. of the 13 quintic seminvariants of the weight in question there are 7, which as not containing either X or Y are each of them reducible; while the remaining 6 can only be expressed as linear functions of X and Y . It would be allowable to select $5554(=X)$ and $5552^2(=Y)$ as the two representative perpetuants, but there is no particular advantage in this.

Sextic Perpetuants and Sextic Syzygies: Syzygants. Art. Nos. 39 to 51.

39. Returning now to the sextic seminvariants, these are weight 6; 6: weight 8; 62: weight 9; 63: weight 10; 64, 62²: and so on. And they are reducible by means of the products (4, 2), (3, 3) and (2, 2, 2), that is, of a quartic perpetuant and a quadric perpetuant, of two cubic perpetuants, and of three quadric perpetuants: this last form of product existing only in the case of an even weight.

40. For weight 6, we have seminvariant 6, and the two products 3.3 and 2.2.2; this implies a syzygy $3.3 - 2.2.2 \equiv 0$; and there then remains a single equation for the reduction of the seminvariant. The formulæ are

$$\begin{aligned} 3.3 &= 6 & 2.2.2 &= 6 \\ &+ 2.33, & &+ 3.22.2 \\ & & &- 3.222, \end{aligned}$$

so that the complete syzygy, and the most simple reduction, are

$$\begin{aligned} &3.3 & 6 &= & 3.3 \\ &- 2.2.2 & & & - 2.33, \\ &- 2.33 & & & \\ &+ 3.22.2 & & & \\ &- 3.222 = 0, & & & \end{aligned}$$

and we might in this way verify that, for the successive weights 8, 9, 10, &c., there are no sextic perpetuants; and find for these weights respectively, the number of the sextic syzygies. But such direct investigation becomes soon impracticable.

41. I endeavour to determine the number of sextic syzygies for the weight w ; and for this purpose I establish the following relation:

$$(S_6) = ((0)) + ((2))' + ((3))' + ((2, 2))' + ((3, 2))' + (S_6)' + (S_6)' - ((5))' + ((\theta))',$$

where (S_6) is the number of sextic syzygies for the weight w , or, what is the same thing, it is the coefficient of x^w in the function S_6 , which is the G. F. for these syzygies: $((0))$ has the value 1 for $w=6$, and the value 0 in all other cases. The accented symbols refer to the weight $w-6$; $(S_6)'$ is thus the number of sextic syzygies for this weight: and for the same weight $w-6$, $((2))'$ denotes the number of quadric perpetuants, or coefficient of x^{w-6} in the function (2) which is the G. F. for these perpetuants: $((3))'$ the number of cubic perpetuants, $((2, 2))'$ the number of

products of two quadric perpetuants, the same or different, $((3, 2))'$ the number of products of a cubic perpetuant and a quadric perpetuant, $(S_5)'$ the number of quintic syzygies, $((5))'$ the number of quintic perpetuants, and $((\theta))'$ a term of unascertained form which will be explained further on. Transposing the term $(S_5)'$ to the left-hand side, and passing to the generating functions, we have

$$(1 - x^6) S_6 = (0) + (2)' + (3)' + (2, 2)' + (3, 2)' + S_5' - (5)' + (\theta)',$$

or, as this may be written,

$$= x^6 \{1 + (2) + (3) + (2, 2) + (3, 2) + (S_5) - (5) + (\theta)\},$$

where (2), (3), &c., have the values already obtained for these G. F.'s respectively:

viz. writing $x^6(\theta) = \frac{\omega x}{2.3.4.5}$, the equation is

$$6. S_6 = x^6 + \frac{x^8}{2} + \frac{x^9}{3} + \frac{x^4}{2.4} + \frac{x^{11}}{2.2.3} + \frac{x^{13}}{2.4} - \frac{x^{21}}{2.3.4.5} + \frac{\omega(x)}{2.3.4.5}.$$

42. Reducing on the right-hand side the known terms to the common denominator 2.3.4.5, the numerator is

x^6	$.1 - x^2$	$.1 - x^3$	$.1 - x^4$	$.1 - x^5 = x^6 - x^8 - x^9 - x^{10}$	$+ x^{12} + 2x^{13} + x^{14}$	$- x^{16} - x^{17} - x^{18} + x^{20}$
$+ x^8$	$.1 - x^3$	$.1 - x^4$	$.1 - x^5$	$+ x^8$	$- x^{11} - x^{12} - x^{13}$	$+ x^{15} + x^{16} + x^{17} - x^{20}$
$+ x^9$	$.1 - x^4$	$.1 - x^5$		$+ x^9$	$- x^{13} - x^{14}$	$+ x^{18}$
$+ x^{10}$	$.1 - x^3$	$.1 - x^5$		$+ x^{10}$	$- x^{13} - x^{15}$	$+ x^{18}$
$+ x^{11}$	$.1 + x^2$	$.1 - x^5$		$+ x^{11}$	$+ x^{13} - x^{16}$	$- x^{18}$
$+ x^{13}$	$.1 - x^3$	$.1 - x^5$			$+ x^{13} - x^{16}$	$- x^{18} + x^{21}$
$- x^{21}$						$- x^{21}$
				x^5	$+ x^{13}$	$- 2x^{16} - x^{18}$

whence, dividing by 6, we have the before-mentioned formula

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18} + \omega(x)}{2.3.4.5.6}.$$

43. We have to prove the formula for (S_6) : this symbol denotes, for the weight w , the number of syzygies between the products (4, 2), (3, 3) and (2, 2, 2): we have to consider separately the cases w odd, and w even. First, if w be odd, there are no products (2, 2, 2) and the only forms are (4, 2) and (3, 3).

I consider a particular value of w , say $w = 15$. The whole series of seminvariants weight 15 is

663	555	4443	33333
654	5532	4432 ²	3332 ³
652 ²	5442	43332	32 ⁵ ;
6432	5433	432 ⁴ ,	
6333	542 ³		
632 ³ ,	5332 ²		
	52 ⁵ ,		

and from the quartic and cubic forms we obtain the forms of the products (4, 2) and (3, 3), viz. these are

4432 . 2	3333 . 3
443 . 2 ²	333 . 33
4333 . 2	332 ³ . 3
432 ³ . 2	332 ² . 32
432 ² . 2 ²	332 . 32 ²
432 . 2 ³	33 . 32 ³ ;
43 . 2 ⁴ ,	

and (S_6) will denote the number of syzygies between these products. Now from any such syzygy, we obtain by decapitation, it may be an identity, but if not an identity, then a syzygy, of the weight $15 - 6 = 9$: and from these lower identities or syzygies we can pass back to the syzygies of the weight 15. To show how this is, I decapitate the several products, thus obtaining the forms

432	333	; viz. the distinct	432	; and of these	333	occur each of
43 . 2	33 . 3	forms are	43 . 2		32 ³	them twice.
333	32 ³		333		32 ² . 2	
32 ³	32 ² . 2	33 . 3	32 ³		32 . 2 ²	
32 ² . 2	32 . 2 ²		32 ² . 2		3 . 2 ³	
32 . 2 ²	3 . 2 ³		32 . 2 ²			
3 . 2 ³			3 . 2 ³			

The forms occurring each twice are 333, 32³, viz. these are the forms (3), or cubic perpetuants of the weight 9: and 32² . 2, 32 . 2², 3 . 2³, viz. these are the forms (3, 2) or products of a cubic perpetuant and a quadric perpetuant for the weight 9; and any form thus occurring twice gives a syzygy of the weight 15: thus 333, we capitate it with 4 . 2 or with 3 . 3, and so obtain the syzygy $4333 . 2 - 3333 . 3 \equiv 0$; and in like manner for each of the other forms 32³, 32² . 2, 32 . 2², 3 . 2³. And so for any other odd weight: (S_6) contains the terms ((3))' and ((3, 2))', and for an odd value of w we may assume that S_6 contains also the terms ((2))' and ((2, 2))': for these, it is clear, vanish for any odd value of w .

44. When w is even, it appears by a similar investigation that (S_6) contains the terms ((2))', and ((2, 2))', which in this case do not vanish, and also the before-mentioned terms ((3))' and ((3, 2))': so that whether w be even or odd, (S_6) contains the terms ((2))', ((3))', ((2, 2))', ((3, 2))'.

In the particular case $w=6$, there is the sextic syzygy $3 . 3 - 2 . 2 . 2 \equiv 0$, obtained by capitation from the identity $1 - 1 = 0$; and by reason hereof, we introduce into the formula the term ((0)), = 1 for $w=6$, and = 0 in every other case.

In what immediately follows, I revert to the instance $w=19$, but this now represents indifferently an odd or an even value of w , there being no distinction between the two cases.

obtained, will serve as an example: 54 is not a quintic perpetuant, but ignoring this, it is by the syzygy in question expressible in the form

$$\begin{aligned} 54 &= 43.2 \\ &- 33.3 \\ &- 432 \\ &- 3.333, \end{aligned}$$

viz. as a Sextic Syzygant, inasmuch as on the right-hand side we have terms 43.2 and 33.3, of the degree 6, which exceeds the degree 5 of the seminvariant 54 in question. Referring back to the definition of reduction, No. 25, observe that this is *not* a reduction of the seminvariant 54. It may be remarked that for the weight 19 we have 15 sextic syzygies: the number of quintic perpetuants is = 3: so that while it is conceivable that the 15 equations might be such that they would fail to determine the 3 perpetuants, it is *primâ facie* very unlikely that this should be so. I have in fact ascertained that the equations are sufficient for the determination; that is, that (weight 19) each of the three quintic perpetuants is a sextic syzygant. So in the case $w=23$, the number of the sextic syzygies is = 28, and that of the quintic perpetuants is = 5; here also the 28 equations are sufficient to determine the 5 perpetuants, viz. (weight 23) each of the 5 quintic perpetuants is a sextic syzygant.

50. Supposing that for any given weight $w-6$, each of the quintic perpetuants is a sextic syzygant: this implies that the number of sextic syzygies $(S_6)'$ is at least equal to the number $((5))'$ of quintic perpetuants (for each expression of a quintic perpetuant as a sextic syzygant is in fact a sextic syzygy): and not only so, but it further implies that the number of the sextic syzygies, which do not contain a quintic perpetuant, is precisely equal $(S_6)' - ((5))'$: for if besides the equations which serve to express the perpetuants as syzygants, we have any other sextic syzygy, then either this does not contain a quintic perpetuant, or it can (by substituting therein for every quintic perpetuant its value as a sextic syzygant) be reduced to a syzygy which does not contain any quintic perpetuant.

51. In the general case, we have $(S_6)'$ sextic syzygies of the weight $w-6$, and $((5))'$ quintic perpetuants of this weight: but it may happen that certain of the quintic perpetuants do not enter into any of the sextic syzygies; and those which enter, may do so in definite combinations: by elimination of these combinations of perpetuants we obtain (it may be) α sextic syzygies not containing any quintic perpetuant; and the remaining $(S_6)' - \alpha$ equations will then serve to express each of them a quintic perpetuant, or combination of quintic perpetuants, as a sextic syzygant. The number α is at most = $((5))'$, or taking it to be = $((5))' - ((\theta))'$, the number of sextic syzygies not containing any quintic perpetuant will be = $(S_6)' - ((5))' + ((\theta))'$, that is, the number of sextic syzygies not containing any quintic perpetuant will be equal to the whole number $(S_6)'$ of sextic syzygies diminished by some number $((5))' - ((\theta))'$, which is less than or at most equal to the whole number $((5))'$ of quintic perpetuants of the weight in question $w-6$. But as already mentioned, I have not been able to obtain the expression of the function (θ) , = $\frac{\omega(x)}{2.3.4.5}$, which is the G. F. of the number $((\theta))'$.

Cambridge, England, 17th March, 1884.

and it can be capitated, for instance, into

$$\begin{aligned}
 & 443.2^2 \\
 & - 333.33 \\
 & - 432.2^3 \\
 & + 2.43.2^4 \\
 & + 3.4333.2 \\
 & + 432^3.2 \equiv 0,
 \end{aligned}$$

the form of capitation being (for the reason mentioned above) quite immaterial. Observe that in every case where the sextic syzygy contains in the first instance any quintic seminvariants, it is assumed that each of these is expressed in terms of quintic perpetuants, as shown in No. 38; and this being done, the sextic syzygy exhibits itself as a syzygy containing, or else not containing, a quintic perpetuant or perpetuants.

47. The conclusion is that from any sextic syzygy of the weight $w-6$, which does not contain a quintic perpetuant, we can obtain by capitation a sextic syzygy of the weight w . The number of sextic syzygies of the weight $w-6$ is $(S_6)'$, and the number of quintic perpetuants of the same weight is $((5))'$: the former of these is (for not too large values of w) the greater; and at first sight it would appear that we can, by elimination of the quintic perpetuants, obtain from the $(S_6)'$ syzygies, $(S_6)' - ((5))'$ syzygies which do not contain a quintic perpetuant: if this was always the case, we should have in (S_6) the term $(S_6)' - ((5))'$, completing the series of terms, and the formula would be

$$(S_6) = ((0)) + ((2))' + ((3))' + ((2, 2))' + ((3, 2))' + (S_5)' + (S_6)' - ((5))',$$

leading to

$$S_6 = \frac{x^6 + x^{13} - 2x^{16} - x^{18}}{2.3.4.5.6}.$$

48. But this result is on the face of it wrong, for as remarked by Sylvester in the memoir referred to, from the mere fact that the sum $1+1-2-1$ of the numerator coefficients is negative, it follows that the coefficients of the development ultimately become negative; and the actual calculation showing when this happens is given by him. And it is further to be noticed that not only the formula cannot be correct beyond the point at which the coefficients become negative, but it cannot be correct beyond the point for which $(S_6)' - ((5))'$ becomes negative: the sextic syzygies of the weight $w-6$ may add nothing to, but they cannot take anything away from, the number of the sextic syzygies of the weight w .

49. If for a moment we further consider these syzygies of the weight $w-6$; so long as the number of these is greater than the number of quintic perpetuants of the same weight, we can by means of them presumably express each of the quintic perpetuants in terms of sextic products, viz. in the language of Capt. MacMahon, express each quintic perpetuant as a "Sextic Syzygant." The syzygy of the weight 9, above

In the new tables we have a property in regard to the sums of the numbers in a *line*: viz. except for the last line of each table, where there is only a single number +1 or -1, this sum is always = 0. I have given in the several tables on the right-hand of each line, the sums for the positive and the negative coefficients separately: thus V (b), line 1, the number ± 375 means that these sums are + 375 and - 375 respectively, the sum of all the coefficients being of course = 0. The property is an important verification as well of the original tables (b) as of the new tables derived from them; and I had the pleasure of thus ascertaining that there was not a single inaccuracy in the original tables (b).

The symbols in the left-hand outside column of each table denote symmetric functions of the roots $\alpha, \beta, \gamma, \dots$; $5 = \Sigma \alpha^5$, $41 = \Sigma \alpha^4 \beta$, &c.: and the tables are read according to the lines: thus in table V (b),

$$5 (= \Sigma \alpha^5) = \frac{1}{120} (5f + 25be + 50cd - 100b^2d - 150bc^2 + 300b^3c - 120b^5),$$

$$41 (= \Sigma \alpha^4 \beta) = \frac{1}{120} (5f - 5be - 50cd + 20b^2d + 90bc^2 - 60b^3c), \text{ \&c.}$$

I (b)

=	b	- 1
1	- 1	

II (b)

	÷ 2	
	c	b ²
2	- 2	+ 2
1 ²	+ 1	+ 1

III (b)

	÷ 6		
	d	bc	b ³
3	- 3	+ 9	- 6
21	+ 3	- 3	± 3
1 ³	- 1	- 1	

IV (b)

	÷ 24				
	e	bd	c ²	b ² c	b ⁴
4	- 4	+ 16	+ 12	- 48	+ 24
31	+ 4	- 4	- 12	+ 12	± 16
2 ²	+ 2	- 8	+ 6	± 8	
21 ²	- 4	+ 4	± 4		
1 ⁴	+ 1	+ 1			

V (b)

	÷ 120						
	f	be	cd	b ² d	bc ²	b ³ c	b ⁵
5	- 5	+ 25	+ 50	- 100	- 150	+ 300	- 120
41	+ 5	- 5	- 50	+ 20	+ 90	- 60	± 115
32	+ 5	- 25	+ 10	+ 40	- 30	± 55	
31 ²	- 5	+ 5	+ 20	- 20	± 25		
2 ² 1	- 5	+ 15	- 10	± 15			
21 ³	+ 5	- 5	± 5				
15	- 1	- 1					