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ON A RELATION BETWEEN TWO TERNARY CUBIC FORMS.

[From the *Philosophical Magazine*, vol. xx. 1860, pp. 512—514.]

THE cubic form

$$x^3 + y^3 + z^3 + 6lxyz$$

is in general linearly transformable into the form

$$(X + Y + Z)^3 + 27kXYZ;$$

in fact, writing

$$X = 2lx - y - z,$$
$$Y = 2ly - z - x,$$
$$Z = 2lz - x - y,$$

we have identically

$$(1 - 2l + 4l^2)(X + Y + Z)^3 + 24(l - 1)^3 XYZ = 8(2l + 1)^2 (l - 1)^3 (x^3 + y^3 + z^3 + 6lxyz);$$

and the value of k consequently is

$$k = \frac{8(l - 1)^3}{9(1 - 2l + 4l^2)}.$$

If, however, $l = 1$ or $l = -\frac{1}{2}$, the transformation fails. In the former case, viz. for $l = 1$, the equations for the linear transformation become

$$X = 2x - y - z,$$
$$Y = 2y - z - x,$$
$$Z = 2z - x - y,$$

which give $X+Y+Z=0$, so that X, Y, Z are no longer independent; and the formula of transformation becomes

$$(X+Y+Z)^3=0.$$

It may be noticed that the invariant S of the form

$$x^3+y^3+z^3+6lxyz$$

is $S=-l+l^4$, so that $l=1$ is one of the values which make S vanish. And the above transformation is not applicable to the cubic form $x^3+y^3+z^3+6xyz$, which is a form for which S vanishes. The transformation, however, holds good for $l=0$, which is another value which makes S vanish; or it does apply to the form $x^3+y^3+z^3$, for which S vanishes. The transformation, in fact, is

$$(X+Y+Z)^3+24XYZ=-8(x^3+y^3+z^3),$$

with the linear equations

$$X=-y-z,$$
$$Y=-z-x,$$
$$Z=-x-y.$$

The above two forms for which S vanishes, viz.

$$x^3+y^3+z^3+6xyz,$$
$$x^3+y^3+z^3,$$

are, notwithstanding, equivalent to each other, as appears by the identical equation

$$(x+y+z)^3+(x+\omega y+\omega^2 z)^3+(x+\omega^2 y+\omega z)^3=3(x^3+y^3+z^3+6xyz),$$

where ω is an imaginary cube root of unity. In the latter of the two cases of failure, viz. for $l=-\frac{1}{2}$, the equations for the linear transformations are

$$X=Y=Z=-x-y-z;$$

so that X, Y, Z are not only independent, but they are connected by *two* linear relations. And the formula of transformation becomes

$$(X+Y+Z)^3-27XYZ=0,$$

which is, in fact, true in virtue of the equations $X=Y=Z$.

The two forms of equation,

$$x^3+y^3+z^3+6lxyz=0$$
$$(x+y+z)^3+27kxyz=0,$$

represent each of them equally well a curve of the third order without a double point. In the first form the three real points of inflexion are given by

$$(x=0,\ y+z=0),\ (y=0,\ z+x=0),\ (z=0,\ x+y=0);$$

or what is the same thing, the points in question are the intersections of the lines $x=0$, $y=0$, $z=0$ with the line $x+y+z=0$; or we have $x+y+z=0$ for the equation of the line through the three points of inflexion; and the equations of the tangents at the points of inflexion are

$$2lx-y-z=0, \quad 2ly-z-x=0, \quad 2lz-x-y=0.$$

For the second form it is obvious that the points of inflexion are the intersections of the lines $x=0$, $y=0$, $z=0$ with the line $x+y+z=0$; and, moreover, that the lines $x=0$, $y=0$, $z=0$ are the tangents at the point of inflexion.

The first of the above-mentioned forms, however, cannot represent a curve with a double point. In fact the condition for its doing so would be $1+8l^3=0$; but when this condition is satisfied, the left-hand side breaks up into linear factors, and the equation represents, not a proper curve of the third order, but a system of three lines. The second form *can* represent a curve having a double point; viz. if $k=-1$, the curve will have a conjugate or isolated point at the point $x=y=z$. It is clear *à priori* that ($x=0$, $y=0$, $z=0$ being real lines) neither of the forms can represent a curve of the third order having a double point with two real branches through it, since in this case the curve has only one real point of inflexion.

I have elsewhere used the word "node" to denote a double point, and I take the opportunity of suggesting the employment of the words "crunode" (*crus*) and "acnode" (*acus*) to denote respectively a double point with two real branches through it, and a conjugate or isolated point.

2, *Stone Buildings, W.C., October* 19, 1860.