255.

ON A PROBLEM OF DOUBLE PARTITIONS.

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If $a+b+c+\ldots=m$, $\alpha+\beta+\gamma+\ldots=\mu$ (the quantities being all positive integer numbers, not excluding zero), then $(a, \alpha)+(b, \beta)+(c, \gamma)+\ldots$ is considered as a partition of (m, μ): and the partible quantity (m, μ), and parts (a, α), (b, β), &c. being each of them composed of two elements, such partition is said to belong to the theory of *Double Partitions*. The subject (so far as I am aware) has hardly been considered except by Professor Sylvester, and it is greatly to be regretted that only an outline of his valuable researches has been published: the present paper contains the demonstration of a theorem, due to him, by which (subject to certain restrictions) the question of Double Partitions is made to depend upon the ordinary theory of Single Partitions.

Let the question be proposed, "In how many ways can (m, μ) be made up of the given parts (a, α), (b, β), (c, γ), &c." under the following conditions (which are, it will be seen, necessary in the demonstration of the theorem constituting the solution), viz.

$$\frac{a}{\alpha},\ \frac{b}{\beta},\ \frac{c}{\gamma},\ \text{\&c. are unequal fractions, each in its least terms,}$$

and

$$\alpha,\ \beta,\ \gamma,\ \text{\&c., are each less than } \mu+2.$$

The number of partitions is

$$= \text{coeff. } x^m y^\mu \text{ in } \frac{1}{(1-x^a y^\alpha)(1-x^b y^\beta)(1-x^c y^\gamma)\ldots}$$

the fraction being developed in ascending powers of x, y.

Considering the fraction as a function of y, it may be expressed as a sum of partial fractions in the form

$$\frac{A(x, y)}{1-x^a y^\alpha}+\frac{B(x, y)}{1-x^b y^\beta}+\frac{C(x, y)}{1-x^c y^\gamma}+\ldots,$$

where

$A(x, y)$ is rational in x, rational and integral of degree $\alpha-1$ in y,

$B(x, y)$ " " " $\beta-1$ "

$C(x, y)$ " " " $\gamma-1$ "

&c.

To find $A(x, y)$ we have, when $y=x^{-\frac{a}{\alpha}}$,

$$A(x, y)=\frac{1}{(1-x^b y^\beta)(1-x^c y^\gamma)\ldots};$$

or what is the same thing, we have

$$A(x, x^{-\frac{a}{\alpha}})=\frac{1}{(1-x^{b-\frac{a\beta}{\alpha}})(1-x^{c-\frac{a\gamma}{\alpha}})\ldots}.$$

This in fact determines $A(x, y)$; for the right-hand side of the equation may be reduced to the form

$$\frac{A_0+A_1 x^{-\frac{a}{\alpha}}\ldots+A_{\alpha-1}x^{-\frac{(\alpha-1)a}{\alpha}}}{(1-x^{ab-\alpha\beta})(1-x^{ac-\alpha\gamma})\ldots},$$

where A_0, $A_1\ldots A_{\alpha-1}$ are rational functions of x: to do this, it is only necessary (taking ω an imaginary α-th root of unity) to multiply the numerator and denominator by

$$\Pi(1-\omega x^{b-\frac{a\beta}{\alpha}})\,\Pi(1-\omega x^{c-\frac{a\gamma}{\alpha}})\ldots,$$

where Π denotes the product of the factors corresponding to the $\alpha-1$ values of ω; the denominator is thus converted into

$$(1-x^{\alpha(b-\frac{a\beta}{\alpha})})(1-x^{\alpha(c-\frac{a\gamma}{\alpha})})\ldots,$$

which is of the form in question; and the numerator becomes a rational function of x and $x^{-\frac{a}{\alpha}}$, integral as regards $x^{-\frac{a}{\alpha}}$, and therefore at once expressible in the form in question. And the equation, viz.

$$A(x, x^{-\frac{a}{\alpha}})=\frac{A_0+A_1 x^{-\frac{a}{\alpha}}\ldots+A_{\alpha-1}x^{-\frac{(\alpha-1)a}{\alpha}}}{(1-x^{ab-\alpha\beta})(1-x^{ac-\alpha\gamma})\ldots},$$

remains true if instead of $x^{-\frac{a}{\alpha}}$ we write $\omega x^{-\frac{a}{\alpha}}$; in fact, instead of writing in the first instance $y=x^{-\frac{a}{\alpha}}$, it would have been allowable to write $y=\omega x^{-\frac{a}{\alpha}}$, ω being any α-th root, real or imaginary, of unity. Hence recollecting that $A(x, y)$ is a rational and integral function of the degree $\alpha - 1$ in y, the equation

$$A(x, y) = \frac{A_0 + A_1 y \ldots + A_{\alpha-1} y^{\alpha-1}}{(1 - x^{\alpha b - a\beta})(1 - x^{\alpha c - a\gamma})\ldots},$$

which is true for the α values $\omega x^{-\frac{a}{\alpha}}$ of y, must be true identically; or this equation gives the value of $A(x, y)$. And the values of $B(x, y)$, $C(x, y)$, &c. are of course of the like form.

Now consider the term

$$\frac{A(x, y)}{1 - x^a y^\alpha},$$

where $A(x, y)$ is a rational and integral function of the degree $\alpha - 1$ in y, and $\frac{a}{\alpha}$ is by hypothesis a fraction in its least terms. The coefficient therein of $x^m y^\mu$ (the fraction being developed in ascending powers of x, y) is

$$= \text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1 - x^{\frac{a}{\alpha}} y}$$

(the fraction being developed in ascending powers of x, y). In fact the two fractions only differ by a wholly *irrational* function of x, as is at once obvious by developing $\frac{1}{1 - x^{\frac{a}{\alpha}} y}$ in ascending powers of y. We have, separating the integral part,

$$\frac{A(x, y)}{1 - x^{\frac{a}{\alpha}} y} = U + \frac{A(x, x^{-\frac{a}{\alpha}})}{1 - x^{\frac{a}{\alpha}} y},$$

where U is a rational and integral function of the degree $\alpha - 2$ in y. But α being by hypothesis $< \mu + 2$, or what is the same thing, $\alpha - 2 < \mu$, U does not contain any term of the form $x^m y^\mu$, and therefore

$$\text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1 - x^{\frac{a}{\alpha}} y}$$

$$= \quad \text{do.} \quad \text{in } \frac{A(x, x^{-\frac{a}{\alpha}})}{1 - x^{\frac{a}{\alpha}} y};$$

and this last is

$$= \text{coeff. } x^m \text{ in } x^{\frac{\mu a}{\alpha}} A(x, x^{-\frac{a}{\alpha}}),$$

$$= \text{coeff. } x^{m - \frac{\mu a}{\alpha}} \text{ in } A(x, x^{-\frac{a}{\alpha}}),$$

$$= \text{coeff. } x^{\alpha m - a\mu} \text{ in } A(x^\alpha, x^{-a});$$

and from the foregoing equation

$$A\left(x, x^{-\frac{\alpha}{a}}\right) = \frac{1}{\left(1 - x^{b - \frac{a\beta}{a}}\right)\left(1 - x^{c - \frac{a\gamma}{a}}\right)\ldots},$$

this is

$$= \text{coeff. } x^{\alpha m - a\mu} \text{ in } \frac{1}{(1 - x^{\alpha b - a\beta})(1 - x^{\alpha c - a\gamma})\ldots}.$$

The last-mentioned expression is thus the value of

$$\text{coeff. } x^m y^\mu \text{ in } \frac{A(x, y)}{1 - x^a y^\alpha};$$

and hence, *Theorem*,

$$\text{coeff. } x^m y^\mu \text{ in } \frac{1}{(1 - x^a y^\alpha)(1 - x^b y^\beta)(1 - x^c y^\gamma)\ldots}$$

$$= \text{coeff. } x^{\alpha m - a\mu} \text{ in } \frac{1}{(1 - x^{\alpha b - a\beta})(1 - x^{\alpha c - a\gamma})\ldots}$$

$$+ \text{coeff. } x^{\beta m - b\mu} \text{ in } \frac{1}{(1 - x^{\beta a - b\alpha})(1 - x^{\beta c - b\gamma})\ldots}$$

$$+ \text{coeff. } x^{\gamma m - c\mu} \text{ in } \frac{1}{(1 - x^{\gamma a - c\alpha})(1 - x^{\gamma b - c\beta})\ldots}$$

$$+ \&\text{c.},$$

the fraction on the left-hand side being expanded in ascending powers of x, y, and those on the right-hand side being expanded in ascending powers of x, and the data satisfying the above-mentioned conditions. The number of partitions of (m, μ) is thus found to be equal to the expression on the right-hand side. It is to be noticed that on the right-hand side, when any of the indices $\alpha m - a\mu$, $\beta m - b\mu, \ldots$ is negative, the corresponding coefficient vanishes; and that when the index of the power of x in any factor of a denominator is negative, e.g. if $\alpha b - a\beta = -p$, then (in order to develope in ascending powers of x) we must in the place of $\frac{1}{1 - x^{\alpha b - a\beta}} = \frac{1}{1 - x^{-p}}$ write $\frac{x^p}{x^p - 1}$, $= -\frac{x^p}{1 - x^p}$, and develope in the form $-(x^p + x^{2p} + x^{3p} + \ldots)$. The right-hand side is thus seen to be the sum of a series of positive or negative numbers, each of which *taken positively* denotes the number of the single partitions of a given partible number into given parts.

If, using a term of Professor Sylvester's, we say that

$$\text{coeff. } x^m \text{ in } \frac{1}{(1 - x^a)(1 - x^b)\ldots}$$

(where $m, a, b, c \ldots$ are positive or negative integers, and the fraction is developed in ascending powers of x) is

$$= \text{Denumerant of } m \text{ in respect to the elements } (a, b, c, \ldots), \text{ say}$$

$$= \text{Denumerant } (m\,;\ a,\ b,\ c\ldots),$$

then when $m, a, b, c\ldots$ are positive, but not otherwise, Denumerant $(m\,;\ a,\ b,\ c\ldots)$ denotes the number of ways in which m can be made up of the parts $a, b, c\ldots$; and the foregoing result shows that the number of ways in which (m, μ) can be made up of the parts (a, α), (b, β), (c, γ), &c. is equal to the sum

$$\begin{aligned} &\text{Denumerant } (\alpha m - a\mu\,;\ \alpha b - a\beta,\ \ \alpha c - a\gamma, \ldots) \\ +\ &\text{Denumerant } (\beta m - b\mu\,;\ \beta a - b\alpha,\ \ \beta c - b\gamma, \ldots) \\ +\ &\text{Denumerant } (\gamma m - c\mu\,;\ \gamma a - c\alpha,\ \ \gamma b - c\beta, \ldots) \\ +\ &\text{\&c.} \end{aligned}$$

But, as appears from what precedes, a denumerant may be equal to zero, or may denote a number of partitions taken negatively; and it is not allowable, in the place, e.g., of the first denumerant, to write *simpliciter*, number of partitions of $\alpha m - a\mu$ in respect of $\alpha b - a\beta$, $\alpha c - a\gamma$, &c. The notion of a Denumerant is, in fact, an important generalization of the notion of a number of partitions.

2, *Stone Buildings, W.C., October* 4, 1860.