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ON AN ANALYTICAL THEOREM CONNECTED WITH THE DISTRIBUTION OF ELECTRICITY ON SPHERICAL SURFACES.

SECOND PART.

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THE theorem is certainly true; but its existence gives rise to a difficulty to which I shall advert in the sequel. I propose, in the first instance, to give a demonstration which starts from the expression for fx given by Plana's equation (115), instead of the deduced equation which was the basis of my former proof. It will be proper to explain the origin and meaning of the formulæ. We have two conducting spherical surfaces, radii 1 and b, in contact with each other (so that the distance between the centres is $1+b$): and then, if x is the distance from the centre of the sphere, radius 1, of an exterior point, and $\mu\,(=\cos\theta)$ the cosine of the inclination of this distance to the line from the centre to the centre of the other sphere, the potential $\phi(\mu,\ x)$ of the sphere, radius 1, at the point whose coordinates are $(x,\ \mu)$ is deduced from the potential fx of a point in the axis; that is, if

$$fx = A_0 \quad + A_1 x \quad + A_2 x^2 + \&c.,$$

then

$$\phi(\mu,\ x) = A_0 P_0 + A_1 P_1 x + A_2 P_2 x^2 + \&c.,$$

where P_0, P_1, P_2, &c. are Legendre's functions, viz. the functions of μ which are the coefficients of the successive powers of x in the development of $(1-2\mu x + x^2)^{-\frac{1}{2}}$ in ascending powers of x. And the electrical thickness y at any point of the surface of the sphere, radius 1, is given by the formula

$$y = x\frac{d\phi(\mu,\ x)}{dx} + 2\phi(\mu,\ x)$$

where, after the differentiation, $x=1$.

The problem consequently depends on the determination of the potential fx for a point on the axis; and this is determined by the functional equation

$$fx-\frac{b}{1+2b-(1+b)\,x}\,f\left(\frac{1+b-x}{1+2b-(1+b)\,x}\right)=h-\frac{bh}{1+b-x}$$

(Plana's equation (G), in which I have written for β, γ, H their values, and substituted also for g its value $=h$). The solution of this equation is (equation (H), writing therein $g=h$)

$$fx=\frac{P}{1-x}+bh\Sigma_{n_0}^{\infty}\frac{1}{b+n\,(1+b)-n\,(1+b)\,x}-bh\Sigma_{n_0}^{\infty}\frac{1}{(n+1)\,(1+b)-\big(1+n\,(1+b)\big)\,x},$$

where P is an arbitrary constant *quoad* the functional equation, viz. it is any function whatever which has the property of remaining unaltered when x is changed into $\dfrac{1+b-x}{1+2b-(1+b)\,x}$. Poisson, and Plana after him, arrive at the conclusion that in the physical problem $P=0$. It appears to me that there is ground for holding that this is only true *sub modo*, and that $\dfrac{P}{(1-x)^2}$ for $x=1$ (which, if P were retained, would be a term occurring in the expression for the thickness at the point of contact) is not of necessity zero. But the term, if it exists, can be replaced at the conclusion; and I write therefore

$$fx=bh\Sigma_{n_0}^{\infty}\frac{1}{b+n\,(1+b)-n\,(1+b)\,x}-bh\Sigma_{n_0}^{\infty}\frac{1}{(n+1)\,(1+b)-\big(1+n\,(1+b)\big)\,x}.$$

According to the process by which the solution of the functional equation was obtained, this is the true form of the solution; for although the series are non-convergent, and the two sums are in fact each of them infinite, there is nothing to show a relation between the number of terms which must be taken in each series. However, nothing immediately turns upon this, as the expression is only used for obtaining an expression for fx in the form of a definite integral, viz., equation (36),

$$fx=bh\int_0^1\frac{dt\,t^{b-1}\,(1-t^{1-x})}{1-t^{(1+b)\,(1-x)}};$$

or, equation (39),

$$fx=\frac{bh}{(1+b)\,(1-x)}\int_0^1\frac{dt\,(t^{-\frac{1}{1+b}}-1)\,t^{\frac{bx}{(1+b)(1-x)}}}{1-t};$$

the latter of which gives (equation (115), in which I have written for a its value $\dfrac{b.}{1+b}$)

$$fx=\frac{hb}{(1+b)\,(1-x)}\left\{Z'\left(\frac{1+b-x}{(1+b)\,(1-x)}\right)-Z'\left(\frac{b}{1+b-x}\right)\right\},$$

where $Z'(p)$ is Legendre's function $\dfrac{d}{dp}\log\Gamma p$, which is developable in the form

$$Z'p=\log p-\frac{1}{2p}-\frac{B_1}{2p^2}+\frac{B_3}{4p^4}-\frac{B_5}{6p^6}+\text{\&c.},$$

where B_1, B_3, &c. are Bernoulli's numbers.

This is the starting-point of the present investigation; and attending to the equations

$$\frac{\log(1+\Delta)}{\Delta}\, 0^1 \quad = -\tfrac{1}{2},$$

$$\frac{\log(1+\Delta)}{\Delta}\, 0^{2x-1} = 0,\ (x>1),$$

$$\frac{\log(1+\Delta)}{\Delta}\, 0^{2x} \quad = B_{2x-1},$$

we see that the development of $Z'p$ becomes

$$Z'p = \log p + \frac{\log(1+\Delta)}{\Delta}\left(\frac{0}{p} - \frac{0^2}{2p^2} + \frac{0^3}{2p^3} - \&c.\right) = \log p + \frac{\log(1+\Delta)}{\Delta}\log\left(1+\frac{0}{p}\right),$$

which, observing that

$$\left(\frac{\log(1+\Delta)}{\Delta} - 1\right)\log p = 0,$$

can be expressed under the more simple form

$$Z'p = \frac{\log(1+\Delta)}{\Delta}\log(p+0).$$

We deduce hence

$$fx = \frac{hb}{(1-x)(1+b)}\ \frac{\log(1+\Delta)}{\Delta}\left\{\log\left(\frac{1+b-x}{(1-x)(1+b)}+0\right) - \log\left(\frac{b}{(1-x)(1+b)}+0\right)\right\};$$

or what is the same thing,

$$fx = \frac{hb}{(1-x)(1+b)}\ \frac{\log(1+\Delta)}{\Delta}\{\log(1+b-x+(1-x)(1+b)\,0) - \log(b+(1-x)(1+b)\,0)\},$$

which may be converted into

$$fx = \frac{hb}{1+b}\ \frac{\log(1+\Delta)}{\Delta}\int_0^1 \frac{dt}{b+t(1-x)}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}\, 0\left\{\int_0^1 \frac{dt}{1-x+b+(1-x)(1+b)\,t0} - \int_0^1 \frac{dt}{b+(1-x)(1+b)\,t0}\right\};$$

or what is the same thing,

$$fx = \frac{hb}{1+b}\ \frac{\log(1+\Delta)}{\Delta}\int_0^1 \frac{dt}{b+t-tx}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}\, 0\left\{\int_0^1 \frac{dt}{(1+b)(1+t0)-x\,(1+(1+b)\,t0)} - \int_0^1 \frac{dt}{b+(1+b)\,t0-x(1+b)\,t0}\right\},$$

the object of the transformation being to express fx so that x may only enter under the form $\frac{1}{A-Bx}$. The factor $\frac{\log(1+\Delta)}{\Delta}$ which multiplies the first of the three definite integrals, might be reduced to unity, but it is more convenient not to make this change.

Now if a fraction $\frac{1}{A-Bx}$ be operated upon by expanding in ascending powers of x, and multiplying the successive terms of the development by P_0, P_1, P_2, &c., it is converted into

$$\frac{1}{(A^2-2AB\mu x+B^2x^2)^{\frac{1}{2}}}.$$

Hence from the foregoing expression for fx we pass at once to the expression for $\phi(\mu, x)$; that is, we have

$$\phi(\mu, x) = \frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}0\int_0^1\frac{dt}{(A^2-2AB\mu x+B^2x^2)^{\frac{1}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{dt}{(A'^2-2A'B'\mu x+B'^2x^2)^{\frac{1}{2}}}-\int_0^1\frac{dt}{(A''^2-2A''B''\mu x+B''^2x^2)^{\frac{1}{2}}}\right\};$$

where for shortness,

$$A=b+t,\quad A'=(1+b)(1+t0),\quad A''=b+(1+b)t0,$$
$$B=\quad t,\quad B'=1+(1+b)t0,\quad B''=\quad (1+b)t0;$$

and it may be remarked that

$$A'=1+b+B'',\quad B'=1+B'',\quad A''=b+B''.$$

We thence obtain

$$x\frac{d\phi(\mu, x)}{dx}+2\phi(\mu, x)=\frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}\int_0^1\frac{(A^2-B^2x^2)\,dt}{(A^2-2AB\mu x+B^2x^2)^{\frac{3}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{(A'^2-B'^2x^2)\,dt}{(A'^2-2A'B'\mu x+B'^2x^2)^{\frac{3}{2}}}-\int_0^1\frac{(A''^2-B''^2x^2)\,dt}{(A''^2-2A''B''\mu x+B''^2x^2)^{\frac{3}{2}}}\right\},$$

and writing $x=1$,

$$y=\frac{hb}{1+b}\frac{\log(1+\Delta)}{\Delta}\int_0^1\frac{(A^2-B^2)\,dt}{(A^2-2AB\mu+B^2)^{\frac{3}{2}}}$$

$$+\frac{hb\log(1+\Delta)}{\Delta}0\left\{\int_0^1\frac{(A'^2-B'^2)\,dt}{(A'^2-2A'B'\mu+B'^2)^{\frac{3}{2}}}-\int_0^1\frac{(A''^2-B'')\,dt}{(A''^2-2A''B''\mu+B''^2)^{\frac{3}{2}}}\right\},$$

the integrals in the foregoing expression are of the form

$$\int_0^1\frac{(G+Ht)\,dt}{(L+2Mt+Nt^2)^{\frac{3}{2}}};$$

the value of the indefinite integral is

$$\frac{1}{LN-M^2}\,\frac{(NG-MH)\,t+MG-LH}{(L+2Nt+Mt^2)^{\frac{1}{2}}},$$

from which the value of the definite integral can be at once found. It is easy, by means of the values to be presently given, to verify that, in each of the three definite integrals, $NG-MH=0$; and the expression for the definite integral is therefore

$$\frac{MG-LH}{LN-M^2}\left\{\frac{1}{(L+2M+N)^{\frac{1}{2}}}-\frac{1}{L^{\frac{1}{2}}}\right\}.$$

In the first integral we have

$$G=b^2,\quad L=b^2,$$
$$H=2b,\quad M=b\,(1-\mu),$$
$$N=2\,(1-\mu),$$

whence

$$LN-M^2\quad=b^2\,(1-\mu)\,(1+\mu),\qquad MG-LH=-\,b^3\,(1+\mu),$$
$$L+2M+N=b^2+2\,(1-\mu)\,(1+b),\qquad L=b^2\,;$$

and the integral is

$$\frac{-b}{1-\mu}\left\{\frac{1}{\sqrt{b^2+2\,(1-\mu)\,(1+b)}}-\frac{1}{b}\right\}.$$

For the second integral we have

$$G=b^2+2b,\qquad L=b^2+2\,(1-\mu)\,(1+b),$$
$$H=2b\,(1+b)\,0,\quad M=(1-\mu)\,(2+b)\,(1+b)\,0,$$
$$N=2\,(1-\mu)\,(1+b)^2\,0^2;$$

and thence

$$LN-M^2=(1-\mu)\,(1+\mu)\,b^2\,(1+b)^2\,0^2,\qquad MG-LH=-\,(1+\mu)\,b^3\,(1+b)\,0,$$
$$L+2M+N=b^2+2\,(1-\mu)\,(1+b)\,\{(1+0)^2+b\,(0+0^2)\};\quad L=b^2+2\,(1-\mu)\,(1+b)\,;$$

and the value of the integral is

$$-\frac{b}{(1+b)\,(1-\mu)\,0}\left\{\frac{1}{\sqrt{b^2+2\,(1-\mu)\,(1+b)\,((1+0)^2+b\,(0+0^2))}}-\frac{1}{\sqrt{b^2+2\,(1-\mu)\,(1+b)}}\right\}.$$

For the third integral,

$$G=b^2,\qquad L=b^2,$$
$$H=2b\,(1+b)\,0,\quad M=\ (1-\mu)\,b\,(1+b)\ 0,$$
$$N=2\,(1-\mu)\ \ (1+b)^2\,0^2,$$

and thence

$$LN-M^2=(1-\mu)\,(1+\mu)\,b^2\,(1+b)^2\,0^2,\qquad MG-LH=-\,(1+\mu)\,b^3\,(1+b)\,0,$$
$$L+2M+N=b^2+2\,(1-\mu)\,(1+b)\ \ (0^2+b\,(0+0^2)),\qquad L=b^2\,;$$

and the value of the integral is

$$\frac{-b}{(1+b)(1-\mu)\,0}\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)(0^2+b(0+0^2))}}-\frac{1}{b}\right\}.$$

Hence the expression for y is

$$y=\frac{-hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\left\{\frac{1}{\sqrt{b^2+2(1+\mu)(1+b)}}-\frac{1}{b}\right\},$$

$$-\frac{hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times$$

$$\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)((1+0)^2+b(0+0^2))}}-\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)}}\right\},$$

$$+\frac{hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times$$

$$\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)(0^2+b(0+0^2))}}-\frac{1}{b}\right\};$$

the top line is destroyed by the second terms of the other two lines, and we have

$$y=\frac{-hb^2}{(1-\mu)(1+b)}\frac{\log(1+\Delta)}{\Delta}\times$$

$$\left\{\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)((1+0)^2+b(0+0^2))}}-\frac{1}{\sqrt{b^2+2(1-\mu)(1+b)(0^2+b(0+0^2))}}\right\}.$$

This expression admits of expansion in positive integer powers of $1-\mu$; and when so expanded the result ought, according to Plana's theorem, to be identically equal to zero. And I proceed to show that this is in fact the case. The coefficient of $(1-\mu)^{m-1}$ is to a factor *près* of the form

$$\frac{\log(1+\Delta)}{\Delta}\{((1+0)^2-b(0+0^2))^m-(0^2+b(0+0^2))^m\},$$

which is the sum of a series of terms each of the form

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{2m-2n}-0^{2m-2n}\}(0+0^2)^n;$$

this is equal to

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{2m-n}\,0^n-0^{2m-n}(1+0)^n\},$$

which is of the form

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha\,0^\beta-(1+0)^\beta\,0^\alpha\},$$

where $\alpha+\beta=2m$ is even, or what is the same thing, $\alpha-\beta$ is even; and, as remarked in the first part of the present paper, such expression is in fact equal to zero. The demonstration, which is very simple, will be given in a note; but assuming for the moment the truth of the proposition, the coefficient of $(1-\mu)^{m-1}$ is the sum of a finite number of evanescent terms, and it is therefore identically equal to zero.

I consider this demonstration as identical *in principle* with that given by Plana; the same function is, by two processes, different indeed from each other, but which cannot but lead to the same result, developed in an infinite series of positive integer powers of $1-\mu$; and it is shown that the coefficient of each power of $1-\mu$ is equal to zero. But the difficulty I find is that the investigation *proves too much*, viz. it appears to prove that y is actually equal to zero. There are undoubtedly functions such as the function $e^{-\frac{1}{x^2}}$ (noticed by Cauchy and Sir W. R. Hamilton), which *in a sense* have the property in question, viz. that if we attempt to develope them in positive integer powers of x, the coefficients are found to be all of them zero; and it would appear that y is, in regard to $1-\mu$, a function of this nature. But it cannot be asserted *simpliciter* that $e^{-\frac{1}{x^2}}$ and its differential coefficients do in fact vanish for $x=0$; they only vanish for $x=0$ considered as the limit of an indefinitely small *real* positive or negative quantity. (This is quite consistent with a remarkable theorem of Cauchy's, by which it appears *à priori* that $e^{-\frac{1}{x^2}}$ cannot be expanded in positive integer powers of x, because it is discontinuous for the modulus zero.) And if, instead of a direct application of Maclaurin's theorem, we first expand $e^{-\frac{1}{x^2}}$, say in positive powers of $1-x$, and then develope the several terms in powers of x, we obtain for the coefficient of x^0, or any other power of x, an infinite series, which I apprehend is not convergent, and which can only be equal to zero in the same conventional sense in which $e^{-\frac{1}{x^2}}$ is equal to zero for $x=0$. This appears to be something very different from finding for the coefficient of x^0, or of any other power of x, an expression composed of a finite number of finite terms the sum whereof is identically equal to zero.

Plana has given for the calculation of y when μ is nearly equal to 1, an expression (equation (127)) which is deduced from the same development of $Z'p$ which is here made use of; but it appears to me that this expression is, for the following reason, open to objection. The expression referred to contains explicitly positive and integer powers of μ, and also powers of the radical $\sqrt{b^2+2(1-\mu)(1+b)}$: it would be, for anything that appears to the contrary, allowable to develope as well the positive and integer powers of μ as also the powers of the radical in question, in a series of positive and integer powers of $1-\mu$; but if this were done, we should obtain as a mere transformation of Plana's expression (127), an expression for y developed in a series of positive integer powers of $1-\mu$; and for consistency with the before-mentioned result, the coefficients of the different powers of $1-\mu$ must be each equal to zero. But if this be so, it does not appear how the original expression (127) can be anything else than zero. The difficulty is, I think, a real one; and I do not see how it is to be got over: it seems to render necessary a more careful study of the effect of the multiplication of the successive terms of the development of a function fx by Legendre's functions P_0, P_1, P_2, &c., so as to pass from fx to the function of two variables $\phi(\mu, x)$, as well generally as when this transformation is performed upon the as yet imperfectly studied transcendental function Z'.

I remark that the original expression for fx is of the form

$$fx = hb\Sigma_{n_0}^{\infty} \frac{1}{p - qx} - hb\Sigma_{n_0}^{\infty} \frac{1}{p' - q'x};$$

and this gives (Plana's equation (131))

$$y = hb\Sigma_{n_0}^{\infty} \frac{p^2 - q^2}{(p^2 - 2pq\mu + q^2)^{\frac{3}{2}}} - hb\Sigma_{n_0}^{\infty} \frac{p'^2 - q'^2}{(p'^2 - 2p'q'\mu + q'^2)^{\frac{3}{2}}},$$

the values of p, q, p', q' being

$$p = b + n(1 + b), \quad p' = (n+1)(1+b),$$
$$q = \quad n(1+b), \quad q' = 1 + n(1+b);$$

so that

$$p - q = b = p' - q', \text{ and } p' + q' = 2 + b + 2n(1+b) = p + q + 2.$$

Hence, putting $\mu = 1$, we find

$$y = hb\Sigma_{n_0}^{\infty}\left(\frac{p+q}{b^2} - \frac{p'+q'}{b^2}\right) = -\frac{2h}{b^2}\Sigma_{n_0}^{\infty} 1 = -\infty,$$

which is inconsistent with the expression $y = 0$, deduced from the definite integral. If, however, it is assumed that fx contains the term $\frac{P}{1-x}$, then the corresponding term of y will be

$$\frac{P(1-x^2)}{(1 - 2\mu x + x^2)^{\frac{3}{2}}},$$

which, when $\mu = 1$, becomes $\frac{P(1+x)}{(1-x)^2}$; and if P be put equal to zero, then it is conceivable that, for $x = 1$, $\frac{P}{1-x}$ may be equal to zero, but $\frac{P(1+x)}{(1-x)^2}$, or what will be the same thing, $\frac{2P}{(1-x)^2}$ may be finite or even infinite. This is perhaps the explanation of the apparent contradiction.

Note on the demonstration of the Theorem

$$\frac{\log(1+\Delta)}{\Delta}\{0^\alpha(1+0)^\beta - 0^\beta(1+0)^\alpha\} = 0, \quad \alpha-\beta \text{ even.}$$

Consider the function

$$\frac{e^t(t+z)}{e^{t+z}-1} = \phi(t, z),$$

which, it is clear, admits of expansion in positive integer powers of t and z. Changing the signs of t, z, we have

$$\frac{e^{-t}(-t-z)}{e^{-t-z}-1} = \phi(-t, -z),$$

or, what is the same thing,

$$\frac{e^z(t+z)}{e^{t+z}-1} = \phi(-t, -z),$$

and thence

$$\frac{(e^t-e^z)(t+z)}{e^{t+z}-1} = \phi(t, z) - \phi(-t, -z);$$

so that the development in positive integer powers of t, z, of the function on the right-hand side does not contain any term $t^\alpha z^\beta$ for which $\alpha-\beta$ is even. Writing the function under the form

$$\frac{e^t(t+z)}{e^{t+z}-1} - \frac{e^z(t+z)}{e^{t+z}-1},$$

and considering the two parts separately, then by Herschel's theorem extended to two variables, the coefficient of $t^\alpha z^\beta$ in the first term is

$$\frac{(1+\Delta_1)\log\{(1+\Delta_1)(1+\Delta_2)\}}{(1+\Delta_1)(1+\Delta_2)-1} 0_1{}^\alpha 0_2{}^\beta,$$

which is equal to

$$\frac{\log\{(1+\Delta_1)(1+\Delta_2)\}}{(1+\Delta_1)(1+\Delta_2)-1} (1+0_1)^\alpha 0_2{}^\beta,$$

or, what is the same thing,

$$\frac{\log(1+\Delta)}{\Delta}(1+0)^\alpha 0^\beta;$$

and forming in like manner the expression for the coefficient of $t^\alpha z^\beta$ in the second term, this is

$$\frac{\log(1+\Delta)}{\Delta}0^\alpha(1+0)^\beta;$$

the difference of the two expressions therefore vanishes when $\alpha-\beta$ is even, which is the above-mentioned theorem. It would be easy to obtain a variety of similar theorems.

2, *Stone Buildings, W.C., June* 29, 1859.