241.

ON POINSOT'S FOUR NEW REGULAR SOLIDS.

[From the *Philosophical Magazine*, vol. XVII. (1859), pp. 123—128.]

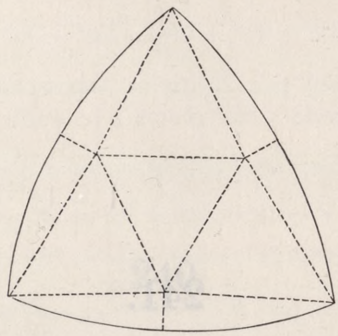
It is shown by Poinsot; in the "Mémoire sur les Polygones et les Polyèdres," *Jour. École Polyt.* vol. IV. pp. 16 to 48 (1810), that, besides the regular polyhedrons of ordinary geometry, there are (of course in an extended signification of the term) four new regular polyhedrons, viz. an icosahedron, which I will call the great icosahedron (No. 33 of the Memoir), and three dodecahedrons, which I will call the great dodecahedron (No. 37), the great stellated dodecahedron (No. 38), and the small stellated dodecahedron (No. 39). The nature of Poinsot's generalization will be best understood by conceiving, as he does, that the polyhedron is projected on a concentric sphere, so that the faces become spherical polygons. Then for the ordinary polyhedrons of geometry, the sum of the angles at a vertex $=4$ right angles; but it may, according to the more general notion, be $=e$ times 4 right angles. In like manner for the ordinary polyhedrons, the sides of a face subtend at the centre angles the sum of which is $=4$ right angles; but according to the more general notion, this sum may be (viz. if the polygons are stellated) $=e'$ times four right angles. And finally, the sum of the spherical polygons is ordinarily equal to the entire spherical surface; but according to the more general notion, it may be $=D$ times the spherical surface. (e is Poinsot's e; e' does not occur in Poinsot; and, for a reason which will appear, I have written D for Poinsot's E.)

The new polyhedra are constructed as follows:

1. *The great Icosahedron.*—Each face is made up of seven faces, or rather four faces and six half faces of the ordinary icosahedron, in the manner shown by fig. 1. There are, as in the ordinary icosahedron, five angles at each vertex; but these make up together, not four, but eight right angles, or $e=2$; but, as in the ordinary poly-

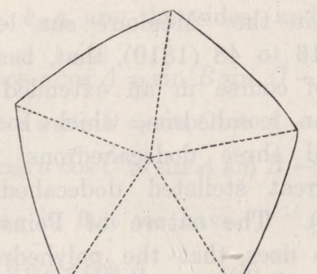
hedra, $e'=1$; and the sum of all the faces is obviously seven times the spherical surface, or $D=7$. (Also $E=7$.)

Fig. 1.



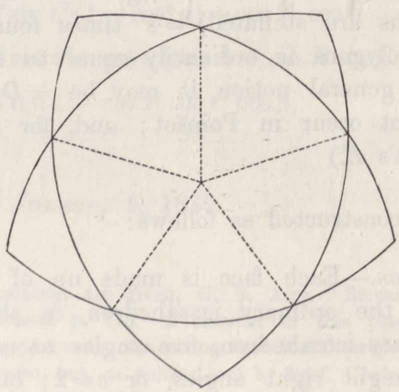
2. *The great Dodecahedron.*—Each face is made up of five faces of the ordinary icosahedron in the manner shown by the figure 2. There are five angles at each vertex, and these make up together eight right angles, or $e=2$; but, as in ordinary polyhedra, $e'=1$; and the sum of all the faces is obviously $12 \times \frac{5}{20}$, that is three times the spherical surface, or $D=3$. (Also $E=3$.)

Fig. 2.



3. *The great stellated Dodecahedron.*—Each face is formed by stellating a face of the great dodecahedron in the manner shown by fig. 3. There are, as in the ordinary

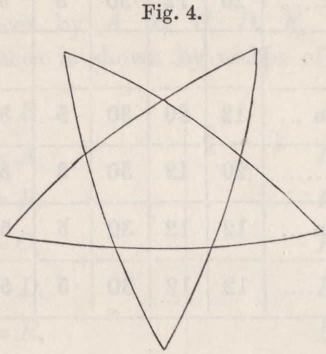
Fig. 3.



dodecahedron, three angles at each vertex, and the sum of these is simply four right angles, or $e=1$. On account of the stellation, $e'=2$. Each of the projecting parts of the face is equal $\frac{1}{3}$ of the face of the ordinary icosahedron; and if we reckon the area of the stellated pentagon to be that of the interior pentagon *plus* the projecting parts, the area of the face will be $5+\frac{5}{3}$, or $\frac{20}{3}$ of the face of the ordinary icosahedron; and the sum of the faces will be four times the spherical surface, and accordingly Poinsot writes $E=4$. If, however, what seems preferable, we reckon the area of the stellated pentagon as five times the triangle having for its vertex the centre of the face and standing upon a side (or what is the same thing, reckon the stellated pentagon as *twice* the interior pentagon *plus* the projecting parts), then the area of the face will be $10+\frac{5}{3}$ or $\frac{35}{3}$ of the face of the ordinary icosahedron, and the sum of the faces will be seven times the spherical surface, or $D=7$.

4. *The small stellated Dodecahedron.*—Each face is formed by stellating a face of the ordinary dodecahedron, as shown by fig. 4. There are five angles at each vertex; and the sum of these is four right angles, or $e=1$. On account of the stellation, $e'=2$. The area of each of the projecting parts is $\frac{1}{5}$ of the interior pentagon or face of the ordinary dodecahedron; and, according to the first mode of measurement, the

Fig. 4.



area of the stellated face is twice that of the face of the ordinary dodecahedron, and the sum of the faces is twice the spherical surface, and accordingly Poinsot writes $E=2$. But according to the second mode of measurement, the area of the stellated pentagon is three times that of the face of the ordinary dodecahedron, and the sum of the faces is three times the spherical surface, or we have $D=3$.

I form now the following Table, comprising as well the ordinary five figures as the new ones of Poinsot, and where we have

H, the number of faces.

S, the number of vertices.

A, the number of edges.

n, the number of sides to a face.

n', the number of sides (angles) at a vertex.

e, viz. the angles at a vertex make together e times four right angles.

e', viz. the angles which the sides of a face subtend at the centre of the face make together e' times four right angles.

E, viz. the faces make together E times the spherical surface, the area of a stellated face being reckoned (as by Poinsot), each portion being taken once only.

D, viz. the faces make together D times the spherical surface, the area of a stellated face being reckoned as the sum of the triangles having their vertices at the centre of the face and standing on the sides.

The Table is

Designation.	$H.$	$S.$	$A.$	$n.$	$n'.$	$e.$	$e'.$	$D.$		$E.$
Tetrahedron	4	4	6	3	3	1	1	1		1
Hexahedron	6	8	12	4	3	1	1	1		1
Octahedron	8	6	12	3	4	1	1	1		1
Dodecahedron	12	20	30	5	3	1	1	1		1
Icosahedron	20	12	30	3	5	1	1	1		1
Great stellated dodecahedron ...	12	20	30	5	3	1	2	7		4
Great icosahedron......	20	12	30	3	5	2	1	7		7
Small stellated dodecahedron ...	12	12	30	5	5	1	2	3		2
Great dodecahedron	12	12	30	5	5	2	1	3		3

where the figures which are polar reciprocals of each other are written in pairs: viz. as is well known, the tetrahedron is its own reciprocal, the hexahedron and octahedron are reciprocals, and the dodecahedron and icosahedron are reciprocals; moreover the great stellated dodecahedron and the great icosahedron are reciprocals, and the small stellated dodecahedron and the great dodecahedron are reciprocals. The number which I have called D is reciprocal to itself; this is not the case for Poinsot's E; and I have not been able to define E in such a manner as to enable me to form the definition of a reciprocal number E': this may be possible, but in the mean time it seems better to discard E altogether, and use instead of it the number D.

Euler's well-known relation applying to ordinary polyhedra is

$$S + H = A + 2.$$

Poinsot in his memoir has (by an extension of Legendre's demonstration of Euler's theorem) obtained the more general relation,

$$eS + H = A + 2E,$$

which, however, does not apply to the two stellated figures where e' is different from unity; the general form is

$$eS + e'H = A + 2D,$$

which applies to all the nine figures. This applies to all polyhedra, regular or not, which are such that e has the same value for each vertex, and e' the same value for each face. To prove it, we have only to further extend Legendre's demonstration. If for any face, stellated or not, the sum of the angles is s, and the number of sides n, then, according to the foregoing mode of reckoning, the area of the face (measured in right angles) is

$$s + 4e' - 2n.$$

Now the sum of all the faces is D times the spherical surface, $= 8D$. But the sum of the term s is equal to the sum of the angles about each vertex, $= 4eS$; the sum of the term $4e'$ is $= 4e'H$, the sum of the term $2n$ is four times the number of edges, $= 4A$. Hence $4eS + 4e'H - 4A = 8D$, or $eS + e'H = 2D$.

I remark that the small stellated dodecahedron and the great dodecahedron are descriptively the same figures, and that, if we represent the vertices by $a, b, c, d, e, f, g, h, i, j, p, q$, and the faces by $A, B, C, D, E, F, G, H, I, J, P, Q$, then the relations of the vertices and faces is shown by either of the following Tables:

$a\ b\ c\ d\ e = P,$	$A\ C\ E\ B\ D = p,$
$p\ b\ i\ h\ e = A,$	$P\ I\ E\ B\ H = a,$
$p\ e\ j\ i\ a = B,$	$P\ J\ A\ C\ I = b,$
$p\ d\ f\ j\ b = C,$	$P\ F\ B\ D\ J = c,$
$p\ e\ g\ f\ c = D,$	$P\ G\ C\ E\ F = d,$
$p\ a\ h\ g\ d = E,$	$P\ H\ D\ A\ G = e,$
$j\ c\ d\ g\ q = F,$	$J\ D\ Q\ C\ G = f,$
$f\ d\ e\ h\ q = G,$	$F\ E\ Q\ D\ H = g,$
$g\ e\ a\ i\ q = H,$	$G\ A\ Q\ E\ I = h,$
$h\ a\ b\ j\ q = I,$	$H\ B\ Q\ A\ J = i,$
$i\ b\ c\ f\ q = J,$	$I\ C\ Q\ B\ F = j,$
$f\ g\ h\ i\ j = Q,$	$F\ H\ J\ G\ I = q,$

where it is to be noticed that in either Table each non-consecutive duad of any pentad occurs once, and only once, as a non-consecutive duad of another pentad. The restriction that a non-consecutive duad of any multiplet is *not* to occur as a duad, consecutive or non-consecutive, of any other multiplet (see my note appended to Mr Kirkman's paper "On Autopolar Polyhedra," *Phil. Trans.* 1857, p. 183 [259]), applies only to ordinary polyhedra, and not to the class here considered.

2, *Stone Buildings, W.C., January* 13, 1859.