227.

ON THE THEORY OF RECIPROCAL SURFACES.

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THE present note is intended to be supplementary to Mr Salmon's memoir "On the Degree of a Surface reciprocal to a given one" (*Trans. R. Irish Acad.*, vol. XXI. pp. 461—488, 1857). I find that Mr Salmon's equations admit of a transformation which appears important in reference to the geometrical theory, and the object of the note is to present the system of equations under the new form.

Mr Salmon writes:

n, the order of the surface.

a, the order of the tangent cone drawn from any point to the surface.

δ, the number of the double edges of the cone.

κ, the number of its cuspidal edges.

b, the order of any double curve upon the surface.

k, the number of apparent double points of the double curve.

t, the number of triple points on the double curve.

c, the order of any cuspidal curve on the surface.

h, the number of apparent double points of the cuspidal curve.

β, the number of intersections of the double and cuspidal curves which are stationary points on the cuspidal curve.

γ, the number of intersections which are stationary points on the double curve.

i, the number of intersections which are not stationary points upon either curve.

ρ, the number of the points where the double curve is met by the curve of contact of the tangent cone.

σ, the number of the points where the cuspidal curve is met by the curve of contact.

And the accented letters denote the corresponding singularities of the reciprocal surface, or, if we choose that they should refer to the given surface, and its tangential or *class* singularities, then we have:

n', the class of the surface.

a', the class of the curve of intersection by any plane.

δ', the number of double tangents of the curve.

κ', the number of its cusps.

b', the class of the node-couple develope.

k', the number of apparent double planes of the node-couple develope.

t', the number of triple planes of the node-couple develope.

c', the class of the spinode develope.

h', the number of the apparent double planes of the spinode develope

β', the number of common tangent planes of the node-couple and spinode developes, stationary planes of the spinode develope.

γ', the number of common tangent planes, stationary planes of the node-couple develope.

i', the number of the common tangent planes which are not stationary planes of either develope.

ρ', the number of the common tangent planes of the node-couple develope, and the tangent cone.

σ', the number of the common tangent planes of the spinode develope, and the tangent cone.

The terminology made use of is that of my paper "On the Singularities of Surfaces" (*Cambridge and Dublin Mathematical Journal*, vol. VII., 1852), [106]. To explain it, I need only remark that the term node is used as synonymous with double point, and the term spinode as synonymous with cusp; a spinode plane is a tangent plane meeting the surface in a curve, having a spinode at the point of contact; and a node-couple plane is a double tangent plane, or plane meeting the surface in a curve having two nodes; the term develope is used instead of developable surface.

To collect all the formulæ, it is proper to write also:

r, the class of the cuspidal curve.

q, the class of the double curve.

r', the order of the spinode develope.

q', the order of the node-couple develope,

where q' is what Mr Salmon, who only uses it incidentally in referring to a result of Professor Schläfli's, calls (after him) A.

Mr Salmon obtains, between the twenty-eight quantities,

$$n,\ a,\ \delta,\ \kappa,\ b,\ k,\ t,\ c,\ h,\ \beta,\ \gamma,\ i,\ \rho,\ \sigma,$$

$$n',\ a',\ \delta',\ \kappa',\ b',\ k',\ t',\ c',\ h',\ \beta',\ \gamma',\ i',\ \rho',\ \sigma',$$

the twenty-one equations,

$$a = a',$$

$$a' = n(n-1) - 2b - 3c,$$

$$\kappa' = 3n(n-2) - 6b - 8c,$$

$$\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{9}{2}c(c-1),$$

$$a(n-2) = \kappa + \rho + 2\sigma,$$

$$b(n-2) = \rho + 2\beta + 3\gamma + 3t,$$

$$c(n-2) = 2\sigma + 4\beta + \gamma,$$

$$a(n-2)(n-3) = 2\delta + 2ab + 3ac - 4\rho - 9\sigma,$$

$$b(n-2)(n-3) = 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho,$$

$$c(n-2)(n-3) = 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma,$$

$$n' = n(n-1)^2 - n(7b+12c) + 4b^2 + 9c^2 + 8b + 15c - 8k - 18h + 18\beta + 12\gamma + 12i - 9t,$$

$$a = n'(n'-1) - 2b' - 3c',$$

$$\kappa = 3n'(n'-2) - 6b' - 8c',$$

$$*\delta = \tfrac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \tfrac{9}{2}c'(c'-1),$$

$$a'(n'-2) = \kappa' + \rho' + 2\sigma',$$

$$b'(n'-2) = \rho' + 2\beta' + 3\gamma' + 3t',$$

$$c'(n'-2) = 2\sigma' + 4\beta' + \gamma',$$

$$a'(n'-2)(n'-3) = 2\delta' + 2a'b' + 3a'c' - 4\rho' - 9\sigma',$$

$$b'(n'-2)(n'-3) = 4k' + a'b' + 3b'c' - 9\beta' - 6\gamma' - 3i' - 2\rho',$$

$$c'(n'-2)(n'-3) = 6h' + a'c' + 2b'c' - 6\beta' - 4\gamma' - 2i' - 3\sigma',$$

$$*n = n'(n'-1)^2 - n'(7b'+12c') + 4b'^2 + 9c'^2 + 8b' + 15c' - 8k' - 18h' + 18\beta' + 12\gamma' + 12i' - 9t';$$

to which may be joined

$$q = b^2 - b - 2k - 3\gamma - 6t,$$

$$r = c^2 - c - 2h - 3\beta,$$

$$q' = b'^2 - b' - 2k' - 3\gamma' - 6t',$$

$$r' = c'^2 - c' - 2h' - 3\beta',$$

Considering the twenty-one equations, and taking as data n, b, c, β, γ, h, k, then, by means of the several equations, other than the two equations marked (*), we may express in terms of the above data a, δ, κ, t, i, ρ, σ, n', a', δ', κ', b', c', ρ', σ', $2\beta'+3\gamma'+3t'$, $4\beta'+\gamma'$, $4k'-3i'$, $6h'-2i'$; the quantities which enter into the first of the marked equations are then all given in terms of the above data, and it is clear that the equation must be satisfied identically: the quantities which enter into the second of the marked equations are given in terms of the data and of t', i', and it is not clear, *a priori*, but that the equation might lead to a relation between the data and t', i'; it will, however, appear in the sequel that the equation must be satisfied identically, independently of any particular values of t', i'. Thus, Mr Salmon's theory does not determine the values of these two quantities, nor, consequently, the values of β', γ', h', k'; it does, however, determine the values of the combinations $4\beta'+\gamma'$, $8k'-18h'$. But the twenty-one equations between the twenty-eight quantities may be replaced by seventeen equations between the twenty quantities

$$n,\ a,\ \delta,\ \kappa,\ b,\ c,\ \rho,\ \sigma,\ 4\beta+\gamma,\ 8k-18h,$$
$$n',\ a',\ \delta',\ \kappa',\ b',\ c',\ \rho',\ \sigma',\ 4\beta'+\gamma',\ 8k'-18h';$$

this will clearly be the case if it is only shown that the equation which gives n' can by the other equations be transformed into one of the form in question; for a similar transformation will, of course, apply to the equation for n, and then we have only to reject the equation containing t, and to replace the two equations which contain i, by the equation given by the elimination of this quantity, and in like manner to reject the equation containing t', and to replace the two equations containing i', by the equation given by the elimination of this quantity, and the system will be reduced to the required form.

The reduction of the equation which gives n' is effected as follows; we have

$$(2b+3c)(n-2)(n-3)=8k+18h+a(2b+3c)+12bc-36\beta-24\gamma-12i-4\rho-9\sigma,$$
$$3b(n-2)=6\beta+9\gamma+9t+3\rho;$$

and thence

$$\begin{aligned}(2b+3c)&(n-2)(n-3)+3b(n-2)\\ &=a(2b+3c)+12bc+8k+18h-12i+9t-\rho-9\sigma-30\beta-15\gamma\\ &=a(2b+3c)+12bc+8k+18h-12i+9t-18\beta-12\gamma-\rho-9\sigma-3(4\beta+\gamma);\end{aligned}$$

and consequently

$$\begin{aligned}-8k&-18h+18\beta+12\gamma+12i-9t\\ &=\{a-(n-2)(n-3)\}(2b+3c)-3b(n-2)+12bc-\rho-9\sigma-3(4\beta+\gamma),\end{aligned}$$

which (observing that the left-hand side is precisely the combination of terms which enters into the equation for n') shows that the reduction is possible; to complete it, putting for a its value $n(n-1)-2b-3c$, we have

$$\begin{aligned}-8k&-18h+18\beta+12\gamma+12i-9t\\ &=b(5n-6)+c(12n-18)-4b^2-9c^2-\rho-9\sigma-3(4\beta+\gamma);\end{aligned}$$

and substituting this value in the equation for n', we obtain

$$n' = n(n-1)^2 - b(2n-2) - 3c - \rho - 9\sigma - 3(4\beta+\gamma).$$

Some of the other equations admit of simplification: the equation

$$a(n-2)(n-3) = 2\delta + a(2b+3c) - 4\rho - 9\sigma,$$

if we put for a its value $n(n-1) - 2b - 3c$, becomes

$$(4n - 6 - 2b - 3c)(n-2)(n-3) = 2\delta - 4\rho - 9\sigma,$$

and the prescribed combination

$$(2b - 3c)(n-2)(n-3) = 8k - 18h + a(2b-3c) - 4\rho + 9\sigma,$$

gives in like manner, putting for a its value,

$$(-n^2 + n + 4b)(n-2)(n-3) = (8k - 18h) - 4\rho + 9\sigma.$$

The system of seventeen equations then is:

$$\begin{aligned}
&a = a',\\
&a' = n(n-1) - 2b - 3c,\\
&\kappa' = 3n(n-2) - 6b - 8c,\\
&\delta' = \tfrac{1}{2}n(n-2)(n^2-9) - (n^2-n-6)(2b+3c) + 2b(b-1) + 6bc + \tfrac{9}{2}c(c-1),\\
&a(n-2) = \kappa + \rho + 2\sigma,\\
&c(n-2) = 2\sigma + (4\beta+\gamma),\\
&(4n-6-2b-3c)(n-2)(n-3) = 2\delta - 4\rho - 9\sigma,\\
&(-n^2+n+4b)(n-2)(n-3) = (8k-18h) - 4\rho + 9\sigma,\\
&n' = n(n-1)^2 - b(2n-2) - 3c - \rho - 9\sigma - 3(4\beta+\gamma),\\
&a = n'(n'-1) - 2b' - 3c',\\
&\kappa = 3n'(n'-2) - 6b' - 8c',\\
{}^*&\delta = \tfrac{1}{2}n'(n'-2)(n'^2-9) - (n'^2-n'-6)(2b'+3c') + 2b'(b'-1) + 6b'c' + \tfrac{9}{2}c'(c'-1),\\
&a'(n'-2) = \kappa' + \rho' + 2\sigma',\\
&c'(n'-2) = 2\sigma' + (4\beta'+\gamma'),\\
&(4n'-6-2b'-3c')(n'-2)(n'-3) = 2\delta - 4\rho' - 9\sigma',\\
&(-n'^2+n'+4b')(n'-2)(n'-3) = (8k'-18h') - 4\rho' + 9\sigma',\\
{}^*&n = n'(n'-1)^2 - b'(2n'-2) - 3c' - \rho' - 9\sigma' - 3(4\beta'+\gamma').
\end{aligned}$$

We may here take as data n, b, c, $4\beta+\gamma$, $8k-18h$, the equations exclusively of the two marked (*), then give a, δ, κ, ρ, σ, n', a', δ', κ', b', c', ρ', σ', $4\beta'+\gamma'$, $8k'-18h'$

and then, since all the quantities entering into the two excepted equations are expressed in terms of the data, these equations are satisfied identically, and it is easy to see that this proves what was before assumed, viz., that in the system of twenty-one equations, the second of the equations marked (*) is satisfied identically.

Several of the other quantities may be expressed without difficulty in terms of the data n, b, c, $4\beta+\gamma$, $8k-18h$: we in fact have (besides a, a', κ', δ', which are originally so expressed):

$$2\sigma = (n-2)\,c - (4\beta+\gamma),$$

$$8\rho = (16n-24)\,b - (15n-18)\,c - 2\,(4b^2-9c^2) + 2\,(8k-18h) - 9\,(4\beta+\gamma),$$

$$8\kappa = 8n\,(n-1)\,(n-2) - (32n-56)\,b - (17n-46)\,c \\ + 2\,(4b^2-9c^2) - 2\,(8k-18h) + 17\,(4\beta+\gamma),$$

$$2\delta = n\,(n-1)\,(n-2)\,(n-3) - (4n^2-20n+24)\,b - (6n^2-15n+18)\,c \\ + 12bc + 18c^2 + (8k-18h) - 9\,(4\beta+\gamma),$$

$$8n' = 8n\,(n-1)^2 - (32n-40)\,b - (21n-30)\,c + 2\,(4b^2-9c^2) - 2\,(8k-18h) + 21\,(4\beta+\gamma),$$

$$c' = 4n\,(n-1)\,(n-2) - (16n-28)\,b - (10n-26)\,c + (4b^2-9c^2) - (8k-18h) + 10\,(4\beta+\gamma),$$

but the expressions for the remaining quantities, viz., b', ρ', σ', $4\beta'+\gamma'$, $8k'-18h'$ would be very complicated. If we suppose that b, c, $4\beta+\gamma$, $8k-18h$, vanish, or, what is the same thing, attend only to the terms which contain n alone, we have:

$$2b' = n\,(n-1)\,(n-2)\,(n^3-n^2+n-12),$$

$$\rho' = n\,(n-2)\,(n^3-n^2+n-12),$$

$$\sigma' = 4n\,(n-2),$$

$$4\beta' + \gamma' = 4n^2\,(n-2)\,(n^3-3n^2+3n-3),$$

$$8k'-18h' = n\,(n-2)\,(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+403n^4 \\ -482n^3+348n^2-242n+60),$$

which agree with the values which Mr Salmon has obtained for β', γ', h', k' by means of the twenty-one equations, and the additional equations (peculiar to the case in question, of a surface of the degree n without singularities, and which are obtained by him from independent considerations), $i'=0$, and $\beta' = 2n\,(n-2)\,(11n-24)$.

The system of seventeen equations completely accounts for the reduction of the order of the given surface considered as the reciprocal of the reciprocal surface, but the omitted equations are important for other purposes. We may by means of them express i, t in terms of the data for the system of twenty-one equations, viz., n, b, c, β, γ, h, k; and, effecting this, and annexing the corresponding values of i', t', we have the supplementary system:

$$4i = (5n-6)\,c - 6c^2 + 12h - 5\gamma,$$

$$24t = -(8n-8)\,b + (15n-18)\,c + 2\,(4b^2-9c^2) - 16k + 36h + 20\beta - 15\gamma,$$

$$4i' = (5n'-6)\,c' - 6c'^2 + 12h' - 5\gamma',$$

$$24t' = -(8n'-8)\,b' + (15n'-18c') + 2\,(4b'^2-9c'^2) - 16k' + 36h' + 20\beta' - 15\gamma,$$

to which I annex also, without transformation, the four equations for q, r, q', r', viz.:

$$q = b^2 - b - 2k - 3\gamma - 6t,$$
$$r = c^2 - c - 2h - 3\beta,$$
$$q' = b'^2 - b' - 2k' - 3\gamma' - 6t',$$
$$r' = b'^2 - c' - 2h' - 3\beta,$$

the last two of which, neglecting singularities, give

$$q' = 4n(n-2)(n-3)(n^2+2n-4),$$
$$r' = 2n(n-2)(3n-4),$$

which are the values given by Mr Salmon. I remark, in conclusion, that there is considerable difficulty in the geometrical conception of the points i and the planes i', and the subject appears to require further examination. In the case of a surface of the order n without multiple lines, we have not only $i=0$ (which is a matter of course), but also $i'=0$. In my paper before referred to, I showed, or attempted to show, by geometrical reasoning, that the common tangent planes of the spinode develope and the node-couple develope are stationary planes of the one or the other of the two developes, that is, $i'=0$, and the reasoning seems correct as far as it goes, but it was not shown how the demonstration would (as it ought to do) fail in the case of a surface having a double or cuspidal curve. I showed also that in the case where the common tangent plane is a stationary plane of the spinode develope (that is for the planes β'), the spinode curve and the node-couple curve touch instead of simply intersecting; it would seem that the tangent plane at such point is to be counted once, and not twice, in reckoning the number β' of such tangent planes: the like remark applies, of course, also to the points of intersection β of the double and cuspidal curves.