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ON CERTAIN DEVELOPABLE SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. VI. (1864), pp. 108—126.[1]]

If $U=0$ be the equation of a developable surface, or say a developable, then the Hessian HU vanishes, not identically, but only by virtue of the equation $U=0$ of the surface; that is, HU contains U as a factor, or we may write $HU=U.PU$; the function PU, which for the developable replaces as it were the Hessian HU, is termed the *Prohessian;* and (since if r be the order of U the order of HU is $4r-8$) we have $3r-8$ for the order of the Prohessian. If $r=4$, the order of the Prohessian is also 4, and in fact, as is known, the Prohessian is in this case $=U$. The Prohessian is considered, but not in much detail, in Dr Salmon's *Geometry of Three Dimensions*, (1862), pp. 338 and 426 [Ed. 4 (1882), p. 408]: the theorem given in the latter place is almost all that is known on the subject. I call to mind that the tangent plane along a generating line of the developable meets the developable in this line taken 2 times, and in a curve of the order $r-2$; the line touches the curve at the point of contact, or say the ineunt, on the edge of regression, and besides meets it in $r-4$ points. The ineunt taken 3 times, and the $r-4$ points form a linear system of the order $r-1$, and the Hessian of this system (considered as a curve of one dimension, or binary quantic) is a linear system of $2r-6$ points; viz. it is composed of the ineunt taken 4 times, and of $2r-10$ other points. This being so, the theorem is that the generating line meets the Prohessian in the ineunt taken 6 times, in the $r-4$ points, and in the $2r-10$ points $(6+r-4+2r-10=3r-8)$; it is assumed that $r=5$ at least.

The developables which first present themselves are those which are the envelopes of a plane

$$(a,\ b,\ \ldots \between t,\ 1)^n=0,$$

[1] Presented to the Royal Society and read 27 Nov., 1862, but withdrawn by permission of the Council.

where t is an arbitrary parameter, and the coefficients $(a, b, \ldots)$ are linear functions of the coordinates; the equation of the developable is

$$\text{Disct. } (a, b, \ldots \between t, 1)^n = 0,$$

the discriminant being taken in regard to the parameter t. Such developable is in general of the order $2n-2$, but if the second coefficient b is $=0$, or, more generally, if it is a mere numerical multiple of a, then a will divide out from the equation, and we have a developable of the order $2n-3$: the like property of course exists in regard to the last but one, and the last, of the coefficients of the function. We thus obtain developables of the orders 4, 5, and 6, sufficiently simple to allow of the actual calculation of their Prohessians, and the chief object of the present Memoir is to exhibit these Prohessians; but the Memoir contains some other researches in relation to the developables in question.

Quartic Developable, Nos. 1 *to* 6.

1. I consider first the developable of the fourth order

$$U = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2,$$

derived from the cubic function $(a, b, c, d \between t, 1)^3$, and which is in fact the general quartic developable.

2. Taking (a, b, c, d) as coordinates and omitting common numerical factors, the first derived functions are

$$\begin{aligned} &ad^2 - 3bcd + 2c^3, \\ -&3acd + 6b^2d - 3bc^2, \\ -&3abd + 6ac^2 - 3b^2c, \\ &a^2d - 3abc + 2b^3, \end{aligned}$$

(quantities which, if $(X, Y, Z, W \between t, 1)^3$ denote the cubicovariant of $(a, b, c, d \between t, 1)^3$, are equal to $(-W, 3Z, -3Y, X)$ respectively). And the second derived functions are

$$\begin{array}{cccc} d^2, & -3cd, & -3bd+6c^2, & 2ad-3bc, \\ -3cd, & 12bd-3c^2, & -3ad-6bc, & -3ac+6b^2, \\ -3bd+6c^2, & -3ad-6bc, & 12ac-3b^2, & -3ab, \\ 2ad-3bc, & -3ac+6b^2, & -3ab, & a^2. \end{array}$$

3. Representing these by

$$\begin{array}{cccc} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & N, \\ L, & M, & N, & P, \end{array}$$

and expressing the determinant in the partially developed form

$$
\begin{aligned}
= \; & (AM-LH)(FN-CM) \\
& +(AN-LG)(FM-BN) \\
& +(AP-L^2\;)(BC\;-F^2\;) \\
& +(HN-GM)^2 \\
& +(HP-LM)(FG-CH) \\
& +(GP-LN)(HF-BG),
\end{aligned}
$$

and proceeding to the calculation, we find

$AM-LH$ $=3\times$	$FN-CM$ $=9\times$	$AN-LG$ $=3\times$	$FM-BN$ $=9\times$	$AP-L^2$ $=3\times$	$BC-F^2$ $=9\times$
$acd^2 + 1$	$a^2bd + 1$	$abd^2 + 1$	$a^2cd + 1$	$a^2d^2 - 1$	$a^2d^2 - 1$
$b^2d^2 + 2$	$a^2c^2 + 4$	$ac^2d - 4$	$ab^2d + 2$	$abcd + 4$	$abcd + 12$
$bc^2d - 3$	$ab^2c - 7$	$b^2cd - 3$	$abc^2 + 1$	$b^2c^2 - 3$	$ac^3 - 4$
	$b^4 + 2$	$bc^3 + 6$	$b^3c - 4$		$b^3d - 4$
					$b^2c^2 - 3$

$HN-GM$ $=18\times$	$HP-LM$ $=3\times$	$FG-CH$ $=9\times$	$GP-LN$ $=3\times$	$HF-BG$ $=9\times$
$ac^3 + 1$	$a^2cd + 1$	$abd^2 + 1$	$a^2bd + 1$	$acd^2 + 1$
$b^3d + 1$	$ab^2d - 4$	$ac^2d + 2$	$a^2c^2 + 2$	$b^2d^2 + 4$
$b^2c^2 - 2$	$abc^2 - 3$	$b^2cd + 1$	$ab^2c - 3$	$bc^2d - 7$
	$b^3c + 6$	$bc^3 - 4$		$c^4 + 2$

4. Hence, forming the six parts and collecting, we find

$27\times$

a^4d^4	$+1$			$+1$			
a^3bcd^3	-12	$+1$	$+1$	-16		$+1$	$+1$
$a^3c^3d^2$	$+8$	$+4$	-4	$+4$		$+2$	$+2$
$a^2b^3d^3$	$+8$	$+2$	$+2$	$+4$		-4	$+4$
$a^2b^2c^2d^2$	$+30$	-2	-10	$+54$		-10	-2
a^2bc^4d	-48	-12	$+2$	-16		-10	-12
a^2c^6	$+16$				$+12$		$+4$
ab^4cd^2	-48	-12	-10	-16		$+2$	-12
ab^3c^3d	$+68$	$+21$	$+25$	-48	$+24$	$+25$	$+21$
ab^2c^5	-24		$+6$	$+12$	-48	$+12$	-6
b^6d^2	$+16$	$+4$			$+12$		
b^5c^2d	-24	-6	$+12$	$+12$	-48	$+6$	
b^4c^4	$+9$		-24	$+9$	$+48$	-24	

where the first column is the Hessian. This is in fact $= U^2$, and hence the Prohessian is

$$PU = U = \begin{array}{|ll|} a^2d^2 & + 1 \\ abcd & - 6 \\ ad^3 & + 4 \\ b^3c & + 4 \\ b^2c^2 & \ \ 3 \end{array}$$

5. To complete the theory it is proper to calculate the inverse coefficients

$$\begin{array}{llll} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L}, \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M}, \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N}, \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P}. \end{array}$$

We have for example

$$\mathfrak{P} = ABC - AF^2 - BG^2 - CH^2 + 2FGH,$$

which is found to be

$$9 \times$$

$$= \begin{array}{|l|l|} a^2d^4 & - 1 \\ abcd^3 & + 6 \\ ac^3d^2 & - 4 \\ b^3d^3 & - 16 \\ b^2c^2d^2 & + 39 \\ bc^4d & - 36 \\ c^6 & + 12 \end{array}$$

which, omitting the factor 9, is

$$= 3\,(ad^2 - 3bcd + 2c^3)^2 - 4d^2\,(a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2),$$

that is $= 3W^2 - 4d^2U$; and calculating in like manner the other coefficients, the system is found to be

$$\begin{array}{llll} 3X^2 - 4a^2U, & 3XY - 4abU, & 3XZ + (2ac - 6b^2)\,U, & 3XW + (5ad - 9bc)\,U, \\ 3YX - 4abU, & 3Y^2 - 4acU, & 3YZ - (1ad + 3bc)\,U, & 3YW + (2bd - 6c^2)\,U, \\ 3ZX + (2ac - 6b^2)\,U, & 3ZY - (1ad + 3bc)\,U, & 3Z^2 - 4bdU, & 3ZW - 4cd\,U, \\ 3WX + (5ad - 9bc)\,U, & 3WY + (2bd - 6c^2)\,U, & 3WZ - 4cdU, & 3W^2 - 4d^2U. \end{array}$$

6. Let $(\lambda, \mu, \nu, \rho)$ be any arbitrary multipliers, and write

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = \left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P} \end{array}\right)\!(\lambda, \mu, \nu, \rho),$$

then if $\theta = 3(\lambda X + \mu Y + \nu Z + \rho W)$, we have

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = (\theta X + \alpha U,\ \theta Y + \beta U,\ \theta Z + \gamma U,\ \theta W + \delta U).$$

The function $(X, Y, Z, W \between t, 1)^3$ is the cubicovariant of $(a, b, c, d \between t, 1)^3$ and if for a moment these functions are represented by v, u respectively, and if we also write $U = a^2d^2 - \&c. = \tilde{U}(a, b, c, d)$, then

$$(a + \theta X,\ b + \theta Y,\ c + \theta Z,\ d + \theta W \between t, 1)^3 = u + \theta v,$$

and thence

$$\begin{aligned}\tilde{U}(a + \theta X,\ b + \theta Y,\ c + \theta Z,\ d + \theta W) &= \text{Disct.}\,(u + \theta v),\\ &= (1 - \theta^2 U)^2\, U,\end{aligned}$$

by a formula given in my "Fifth Memoir on Quantics," *Phil. Trans.*, t. CXLVIII. (1858), see p. 442 [156]; the function on the left-hand side thus contains U as a factor, and it at once follows that the function

$$\tilde{U}(a + \mathfrak{X},\ b + \mathfrak{Y},\ c + \mathfrak{Z},\ d + \mathfrak{W});$$

viz., the function obtained from U by writing therein $(a + \mathfrak{X},\ b + \mathfrak{Y},\ c + \mathfrak{Z},\ d + \mathfrak{W})$ in the place of (a, b, c, d) respectively, contains U as a factor, and therefore vanishes if $U = 0$; that is $a + \mathfrak{X}$, $b + \mathfrak{Y}$, $c + \mathfrak{Z}$, $d + \mathfrak{W}$, are the coordinates of a point on the surface $U = 0$; they are in fact the coordinates of a point on the generating line through (a, b, c, d); this is a theorem which applies to any developable whatever, as appears by the following considerations.

Remarks on the General Theory of Developables, Nos. 7 to 9.

7. In general for any surface whatever, taking a point on the surface, the successive polars of this point (the last of them being the tangent plane) all touch at this point; and not only so, but the tangents to the two branches of the curve in which the surface itself (or any of its polars down to the quadric polar) is intersected by the ultimate polar or tangent plane, are respectively coincident. Suppose that for any point on the surface, the quadric polar becomes a cone: the vertex of this cone is *not* the point itself; hence the tangent plane at the point touches the cone along a generating line; that is the tangents to the curve of intersection with the surface, or with any of its polars, coincide with the generating line of the cone—and the curve of intersection of the tangent plane with the surface, or any of its polars, at the point of contact (instead of, as in general, a node) has a cusp. In particular the curve of intersection with the surface has at the point of contact a cusp. The condition that the quadric polar may be a cone is $HU = 0$, and when this differential equation is satisfied in virtue of the equation $U = 0$ (that is, when we have identically $HU = U.PU$), the surface is a developable. Now all that is proved in the first instance by the equation $HU = 0$ is that *every point* of the surface has the above mentioned property; viz., that the tangent plane at the point cuts the surface in a curve having a cusp at the point in question.

8. What really happens in the case of a developable is more than this; viz. the curve of intersection is made up of the generating line taken twice, and of a curve of an order less by 2 than the order of the surface. Let (x, y, z, w) be the coordinates of the point on the developable, $U=0$ the equation of the developable, $(A, B, C, P, F, G, H, L, M, N)$ the second derived functions of U, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{P}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N})$ the inverse system, K the determinant formed with the second derived functions, so that we have $K=HU=0$. The coordinates of the vertex of the cone are given by the equations

$$\begin{aligned}\alpha : \beta : \gamma : \delta &= \mathfrak{A} : \mathfrak{H} : \mathfrak{G} : \mathfrak{L}\\ &= \mathfrak{H} : \mathfrak{B} : \mathfrak{F} : \mathfrak{M},\\ &= \mathfrak{G} : \mathfrak{F} : \mathfrak{C} : \mathfrak{N},\\ &= \mathfrak{L} : \mathfrak{M} : \mathfrak{N} : \mathfrak{P};\end{aligned}$$

these several sets of ratios being equivalent to each other in virtue of the equation $K=0$. Hence $(\lambda, \mu, \nu, \rho)$ being arbitrary multipliers, if we write

$$(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}) = \left(\begin{array}{cccc}\mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{C}\\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M}\\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N}\\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{P}\end{array}\right)(\lambda, \mu, \nu, \rho);$$

the coordinates of the vertex of the cone will be as $\mathfrak{X} : \mathfrak{Y} : \mathfrak{Z} : \mathfrak{W}$, and hence observing that the absolute magnitudes of these quantities are arbitrary, $x+\mathfrak{X} : y+\mathfrak{Y} : z+\mathfrak{Z} : w+\mathfrak{W}$ will represent the coordinates of any point on the line joining the point (x, y, z, w) with the vertex of the cone, that is, the generating line through the point (x, y, z, w); which is the theorem in question, the coordinates being in the present investigation denoted by (x, y, z, w) instead of the (a, b, c, d) of the example.

9. Reverting to the developable $U = a^2d^2 - \&c. = 0$, the results previously obtained show that the coordinates of the vertex of the cone which is the quadric polar of the point (a, b, c, d) are as $X : Y : Z : W$ (these quantities denoting as above the coefficients of the cubicovariant), and thence also that the coordinates of any point on the generating line will be as $a+\theta X : b+\theta Y : c+\theta Z : d+\theta W$, where θ is arbitrary.

Special Quintic Developable, Nos. 10 *to* 25.

10. We have, secondly, the developable of the fifth order

$$U = a^3e^2 + 6a^2c^2e - 24ab^2ce + 9ac^4 + 16b^4e - 8b^2c^3 = 0,$$

derived from the quartic function $(a, 2b, 3c, 0, -27e \between t, 1)^4$, or, what is the same thing, $at^4 + 8bt^3 + 18ct^2 - 27e = 0$, where it will be observed that, as well in the quartic function as in the equation of the developable, the sum of the numerical coefficients is $=$ zero; it was on this account that the foregoing form of the quartic function was selected in preference to the form $(a, b, c, 0, e \between t, 1)^4$. The last mentioned form has for its discriminant

$$(ae+3c^2)^3 - 27\,(ace - b^2e - c^3)^2, \quad = e\,(a^3e^2 - 18a^2c^2e + 54ab^2ce + 81ac^4 - 27b^4e - 54b^2c^3),$$

and writing therein in place of (a, b, c, e), the values $(a, 2b, 3c, -27e)$, the second factor divided by 729 gives the foregoing expression for U, belonging to the form

$$(a,\ 2b,\ 3c,\ 0,\ -27e\between t,\ 1)^4.$$

11. Taking (a, b, c, e) as coordinates, and omitting common numerical factors, the first derived functions of U are

$$\begin{aligned}&3a^2e^2+12ac^2e-24b^2ce+\ 9c^4\ ,\\ &-48abce+64b^3e\ -16bc^3\ ,\\ &12a^2ce-24ab^2e+36ac^3\ -24b^2c^2,\\ &2a^3e\ +\ 6a^2c^2-24ab^2c+16b^4\ ,\end{aligned}$$

and the second derived functions are

$$\begin{array}{llll} 3\,(ae^2+2c^2e)\ , & -24bce\ , & 6\,(2ace-2b^2e+3c^3), & 3\,(a^2e+2ac^2-4b^2c), \\ -24bce\ , & 8\,(-3ace+12b^2e-c^3), & 24\,(-abe-bc^2)\ , & 8\,(-3abc+4b^3)\ , \\ 6\,(2ace-2b^2e+3c^3), & 24\,(-abe-bc^2)\ , & 6\,(a^2e+9ac^2-4b^2c), & 6\,(a^2c-2ab^2)\ , \\ 3\,(a^2e+2ac^2-4b^2c), & 8\,(-3abc+4b^3)\ , & 6\,(a^2c-2ab^2)\ , & a^2\ . \end{array}$$

12. Representing these by

$$\begin{array}{cccc} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & N, \\ L, & M, & N, & P, \end{array}$$

and expressing the determinant in the partially developed form

$$\begin{aligned}&P\,(ABC-AF^2-BG^2-CH^2+2FGH)\\ &\quad-L^2(BC-F^2)-M^2(CA-G^2)-N^2(AB-H^2)\\ &\quad-2\,MN(GH-AF)-2NL\,(HF-BG)-2LM\,(FG-CH),\end{aligned}$$

then, proceeding to the calculation, we have

$BC-F^2$ $=48\times$	$CA-G^2$ $=18\times$	$AB-H^2$ $=24\times$	$GH-AF$ $=72\times$	$HF-BG$ $=48\times$	$FG-CH$ $=144\times$	$ABC-$&c. $=288\times$
$a^3ce^2\ -\ 3$	$a^3e^2\ +\ 1$	$a^2ce^3\ -\ 3$	$a^2be^3\ +\ 1$	$a^2ce^2\ +\ 6$	$a^2bce^2\ -\ 1$	$a^4ce^4\ -\ 3$
$a^2c^3e\ -\ 28$	$a^2c^2e^2\ +\ 3$	$ab^2e^3\ +\ 12$	$abc^2e^2\ -\ 1$	$ab^2ce^2\ -\ 18$	$ab^3c^2\ +\ 2$	$a^3c^3e^3\ -\ 10$
$ab^2c^2e\ +\ 96$	$ab^2ce^2\ +\ 12$	$ac^3e^2\ -\ 7$	$b^3ce^2\ +\ 4$	$ac^4e\ +\ 11$	$abc^3e\ +\ 4$	$a^2b^2c^2e^3+\ 24$
$ac^5\ -\ 9$	$ac^4e\ -\ 6$	$c^5e\ -\ 2$	$bc^4e\ -\ 4$	$b^4e^2\ +\ 24$	$b^3c^2e\ -\ 2$	$a^2c^5e^2\ +\ 15$
$b^4ce\ -\ 48$	$b^4e^2\ -\ 8$			$b^2c^3e\ -\ 26$	$bc^5\ -\ 3$	$ab^4ce^3\ +\ 72$
$b^2c^4\ -\ 8$	$b^2c^3e\ +\ 16$			$c^6\ +\ 3$		$ab^2c^4e^2-168$
	$c^6\ -\ 18$					$ac^7e\ +\ 60$
						$b^6e^3\ -\ 96$
						$b^4c^3e^2\ -200$
						$b^3c^6e\ -112$
						$c^9\ +\ 18$

$L^2 = 9 \times$	$M^2 = 64 \times$	$N^2 = 96 \times$	$2MN = 96 \times$	$2NL = 36 \times$	$2LM = 48 \times$
$a^4e^2 + 1$	$a^2b^2c^2 + 9$	$a^4c^2 + 1$	$a^3bc^2 - 3$	$a^4ce + 1$	$a^3bce - 3$
$a^3c^2e + 4$	$ab^4c - 24$	$a^3b^2c - 4$	$a^2b^3c + 10$	$a^3b^2e - 2$	$a^2b^3e + 4$
$a^2b^2ce - 8$	$b^6 + 16$	$a^2b^4 + 4$	$ab^5 - 8$	$a^3c^3 + 2$	$a^2bc^3 - 6$
$a^2c^4 + 4$				$a^2b^2c^2 - 8$	$ab^3c^2 + 20$
$ab^2c^3 - 16$				$ab^4c + 8$	$b^5c - 16$
$b^4c^2 + 16$					

13. It is now easy to form the seven component terms of the determinant, and thence the determinant itself; each of the component terms divides by 576, and omitting this factor, the sum of the seven terms divides by 2; the result is

a^7ce^4	$+ 3$	$- 3$	$+ 9$					
$a^6c^3e^2$	$+ 28$	$- 10$	$+ 120$		$+ 18$		$- 72$	
$a^5b^2c^2e^3$	$- 96$	$+ 24$	$- 360$	$- 72$	$- 144$	$+ 144$	$+ 360$	$- 144$
$a^5c^5e^2$	$+ 90$	$+ 15$	$+ 399$		$+ 42$		$- 276$	
$a^4b^4ce^3$	$+ 24$	$+ 72$	$+ 144$	$+ 192$	$+ 360$	$- 480$	$- 720$	$+ 480$
$a^4b^2c^4e^2$	$- 384$	$- 168$	$- 1944$	$- 216$	$- 168$	$- 144$	$+ 1584$	$+ 288$
a^4c^7e	$+ 108$	$+ 60$	$+ 444$		$+ 12$		$- 300$	
$a^3b^6e^3$	$+ 32$	$- 96$		$- 128$	$- 288$	$+ 384$	$+ 576$	$- 384$
$a^3b^4c^3e^2$	$+ 568$	$+ 200$	$+ 3024$	$- 288$	$- 168$	$+ 1056$	$+ 3504$	$+ 480$
$a^3b^2c^6e$	$- 224$	$- 112$	$- 2616$	$+ 432$	$- 48$	$- 576$	$+ 1752$	$+ 720$
a^3c^9	$+ 27$	$+ 18$	$+ 108$				$- 72$	
$a^2b^6c^2e^2$	$+ 384$		$- 1152$	$+ 2496$		$- 2304$	$+ 4032$	$- 2304$
$a^2b^4c^5e$	$- 696$		$+ 6336$	$- 2304$	$+ 48$	$+ 1920$	$- 3552$	$- 3840$
$a^2b^2c^8$	$+ 192$		$- 336$	$+ 1296$			$+ 288$	$- 864$
ab^8ce^2	$- 1152$			$- 3072$		$+ 1536$	$- 2304$	$+ 1536$
ab^6c^4e	$+ 1440$		$- 6912$	$+ 3840$		$- 1536$	$+ 2496$	$+ 4992$
ab^4c^7	$- 408$		$+ 48$	$- 3456$			$- 288$	$+ 2880$
$b^{10}e^2$	$+ 512$			$+ 1024$				
b^8c^3e	$- 640$		$+ 2304$	$- 2048$				$- 1536$
b^6c^6	$+ 192$		$+ 384$	$+ 2304$				$- 2304$

where the first column is the Hessian.

14. This divides as it should do by

$U =$		
	a^3e^2	$+ 1$
	a^2c^2e	$+ 6$
	ab^2ce	$- 24$
	ac^4	$+ 9$
	b^4e	$+ 16$
	b^2c^3	$- 8$

and the quotient, which is the Prohessian, is found to be

$PU =$		
	a^4ce^2	$+ 3$
	a^3c^3e	$+ 10$
	$a^2b^2c^2e$	$- 24$
	a^2c^5	$+ 3$
	ab^4ce	$- 24$
	ab^2c^4	$+ 24$
	b^6e	$+ 32$
	b^4c^3	$- 24$

15. But before discussing the Prohessian, I will further consider the developable itself. Regarding it as derived from the equation $(a, 2b, 3c, 0, -27e\between t, 1)^4 = 0$, we have

$$eU = (27)^3 \{(ae-c^2)^3 + (-3ace + 4b^2e - c^3)^2\} = 0,$$

and observing that

$$-3ace + 4b^2e - c^3 = c(ae-c^2) - 4e(ac-b^2),$$

it appears that the equations of the cuspidal curve or edge of regression of the developable are $ae - c^2 = 0$, $ac - b^2 = 0$ (so that the cuspidal curve is a curve of the fourth order, the intersection of two quadric surfaces, or say a quadri-quadric curve). This is perhaps better seen by writing the equation of the developable in the form

$$U = a(ae-c^2)^2 - 8c(ae-c^2)(ac-b^2) + 16e(ac-b^2)^2 = 0,$$

or what is the same thing

$$U = (a, c, e\between ae-c^2, 4ac-4b^2)^2 = 0,$$

where the discriminant of the quadric function is $= ae - c^2$, which vanishes for the curve $(ae - c^2 = 0, ac - b^2 = 0)$.

16. Another form of the equation is

$$U = a(ae+3c^2)^2 - 8b^2(3ace - 2b^2e + c^3) = 0,$$

which shows that the conic $ae + 3c^2 = 0$, $b = 0$, is a nodal curve on the developable.

And, again, another form is

$$U = c^3(9ac - 8b^2) + e(a^3e + 6a^2c^2 - 24ab^2c + 16b^4) = 0,$$

which shows that the conic $9ac - 8b^2 = 0$, $e = 0$, is a simple line on the developable.

17. In my paper "On the Developable Surfaces which arise from Two Surfaces of the Second Order," *Camb. and Dub. Math. Jour.*, t. v. (1850), pp. 46—57, [84], I considered first the developable having for its edge of regression the intersection of two quadric surfaces; in the general case the developable is of the order 8; but if the two surfaces have an ordinary contact it is of the order 6; and if they have a singular contact (as there explained) it is of the order 5. And in the last mentioned case, if the equations of the two quadric surfaces are taken to be $x^2 - 2wz = 0$, $y^2 - 2zx = 0$, then the equation of the developable was found to be

$$4z^3w^2 + 12z^2x^2w + 9zx^4 - 24zxy^2w - 4x^3y^2 + 8y^4w = 0,$$

which putting therein $z = a$, $y = 2b$, $x = 2c$, $w = 2e$, becomes

$$a^3e^2 + 6a^2c^2e - 24ab^2ce + 9ac^4 + 16b^4e - 8b^2c^3 = 0,$$

which is the before mentioned developable $U = 0$; the two equations $x^2-2wz=0$, $y^2-2zx=0$ become by the same substitution $ae - c^2 = 0$, $ac - b^2 = 0$. We have in fact already seen that the developable $U = 0$ has this curve for its edge of regression.

18. But in the paper just referred to, it is also shown that considering the developable which is the envelope of the common tangent planes of two quadric surfaces; in the general case the developable is of the order 8, but if the two surfaces have an ordinary contact it is of the order 6, and if they have a singular contact it is of the order 5.

In the last mentioned case the surfaces may without loss of generality be reduced to conics, and their equations may be taken to be $(y^2-2zx=0,\ w=0)$ and $(x^2-2zw=0,\ y=0)$ [1], and this being so the equation of the developable is

$$32z^3w^2-32z^2x^2w+72zxy^2w+8zx^4-27y^4w-4x^3y^2=0.$$

This is really a developable of the same kind with the first mentioned developable of the order 5; for writing $x=12c,\ y=8b,\ z=3a,\ w=-8e$, the equation becomes

$$a^3e^2+6a^2c^2e-24ab^2ce+9ac^4+16b^4e-8b^2c^3=0,$$

which is the before mentioned developable $U=0$. The equations of the two conics become $(9ac-8b^2=0,\ e=0)$ and $(ae+3c^2=0,\ b=0)$, and the developable is thus the envelope of the common tangent planes of these two conics. It has been seen that the first conic is a simple line, but the second conic a nodal line, on the developable.

19. Recapitulating, the developable of the fifth order $U=0$, which is the envelope of the plane $(a,\ 2b,\ 3c,\ 0,\ -27e)(t,\ 1)^4=0$ is the locus of the tangents of the quadri-quadric curve $(ae-c^2=0,\ ac-b^2=0)$, and it is also the envelope of the common tangent planes of the conics $(9ac-8b^2=0,\ e=0)$ and $(ae+3c^2=0,\ b=0)$.

20. Returning now to the Prohessian, its equation may be written in the form

$$PU=(3a^2c,\ -3b^2c,\ ace+2b^2e \between ae-c^2,\ 4ac-4b^2)^2=0,$$

and the discriminant of the quadric function is

$$3a^2ce(ac+2b^2)-9b^4c^2,$$

which is

$$=3c\{(a^2e+3b^2c)(ac-b^2)+3ab^2(ae-c^2)\},$$

and recollecting that the equations of the cuspidal curve or edge of regression of the developable are $ae-c^2=0,\ ac-b^2=0$, it thus appears that the curve in question is also a cuspidal curve on the Prohessian.

21. Consider for a moment the surface

$$(3a^2c,\ -3b^2c\theta,\ ace+2\theta b^2e \between ae-c^2,\ 4ac-4b^2)^2=0,$$

[1] These are in fact the conics made use of, p. 57 in the paper above referred to [and p. 495 in the reprint], but the equations are by mistake given as $(x^2-2yz=0,\ w=0)$, $(y^2-2zw=0,\ x=0)$, that is, x and y are interchanged.

where θ is an arbitrary parameter; this is a surface having for a nodal curve the cuspidal curve of the developable; but if the discriminant of the quadric function vanishes, that is if

$$3a^2ce\,(ac + 2b^2\theta) - 9b^4c^2\theta^2 = 0,$$

for $ae - c^2 = 0$, $ac - b^2 = 0$, then the curve in question will be a cuspidal curve on the surface. But the last mentioned equation is

$$3c\,[(a^2e + 3b^2c\theta^2)(ac - b^2) + ab^2\{(1 + 2\theta)\,ae - 3\theta^2c^2\}] = 0,$$

which for $ae - c^2 = 0$, $ac - b^2 = 0$, becomes $1 + 2\theta - 3\theta^2 = 0$, that is $\theta = 1$, which gives the Prohessian, or $\theta = -\frac{1}{3}$.

22. For the latter value the surface is

$$(9a^2c,\ 3b^2c,\ 3ace - 2b^2e \between ae - c^2,\ 4ac - 4b^2)^2 = 0,$$

or, expanding, the surface ($\theta = -\frac{1}{3}$) is

a^4ce^2	$+\ 9$	$=0,$
a^3c^2e	$+\ 30$	
$a^2b^2c^2e$	$-\ 104$	
a^2c^5	$+\ 9$	
ab^4ce	$+\ 88$	
ab^2c^4	$-\ 24$	
b^6e	$-\ 36$	
b^4c^3	$-\ 24$	

the before mentioned discriminant being

$$= c\,\{(3a^2e - b^2c)(ac - b^2) + ab^2\,(ae - c^2)\};$$

but I have not further examined the geometrical signification of this surface, or inquired into its relation to the Prohessian.

23. The equation of the Prohessian may be written

$$PU = (ac - b^2)\,\{16e\,(ac - b^2)^2 + 3a\,(ae - c^2)^2\} + b^2U = 0,$$

or what is the same thing

$$PU = (ac - b^2)\,\{a\,(ae + 3c^2)(3ae + c^2) - 32ab^2ce + 16b^4e\} + b^2U = 0,$$

the latter of which shows that the conic ($ae + 3c^2 = 0$, $b = 0$), which is the nodal line of the developable, is a simple line on the Prohessian.

24. Consider the curve of intersection of the developable and the Prohessian; this is of the order 5×7, $= 35$. We have $ac - b^2 = 0$, $U = 0$, or else

$$16e\,(ac - b^2)^2 + 3a\,(ae - c^2)^2 = 0,\ U = 0.$$

Consider for a moment the second system, this is

$$3a\,(ae - c^2)^2 + \qquad\qquad 16e\,(ac - b^2)^2 = 0,$$

$$3a\,(ae - c^2)^2 - 24c\,(ae - c^2)(ac - b^2) + 48e\,(ac - b^2)^2 = 0,$$

which give

$$(ac-b^2)\{3c(ae-c^2)-4e(ac-b^2)\}=0,\ U=0,$$

and these are equivalent to

$$(ac-b^2=0,\ U=0) \text{ and } \{3c(ae-c^2)-4e(ac-b^2)=0,\ U=0\},$$

so that the entire intersection is made up of $(ac-b^2=0,\ U=0)$ twice, and of $\{4e(ac-b^2)-3c(ae-c^2)=0,\ U=0\}$ once.

25. The first part is at once seen to give

the cuspidal curve	$(ac-b^2=0;\ ae-c^2=0)$	4 times, order	16
the line	$(a=0,\ b=0)$	4 " "	4
			20

The second part gives

$$(ae-c^2)\{4c(ac-b^2)+\ a(ae-c^2)\}=0,$$

$$\{4e(ac-b^2)-3c(ae-c^2)\}=0,$$

this consists of 1° the part $ae-c^2=0,\ e(ac-b^2)=0$, viz.

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	once, order	4
the line	$(c=0,\ e=0)$	twice, "	2
			6

and 2° the part

$$4c(ac-b^2)+a(ae-c^2)=0,$$

$$4e(ac-b^2)-3c(ae-c^2)=0,$$

which contains

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	once, order	4,

and by writing the two equations in the form

$$c(ae+3c^2)-4b^2e=0,$$

$$a(ae+3c^2)-4b^2c=0,$$

it is clear that it contains also

the nodal curve	$(ae+3c^2,\ b=0)$	twice, order	4
and the line	$(c=0,\ e=0)$	once, "	1
			9

whence the complete intersection of the developable and the Prohessian is made up as follows, viz.

the cuspidal curve	$(ac-b^2=0,\ ae-c^2=0)$	6 times, order	24
the nodal curve	$(ae+3c^2=0,\ b=0)$	2 times, "	4
the line	$(a=0,\ b=0)$	4 times, "	4
the line	$(a=0,\ e=0)$	3 times, "	3
			35

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26. Lastly, we have the developable

$$U = a^3e^3 - 12a^2bde^2 - 27a^2d^4 - 6ab^2d^2e - 27b^4e^2 - 64b^3d^3 = 0,$$

derived from the quartic function $(a, b, 0, d, e \between t, 1)^4$; the discriminant is in fact $(ae - 4bd)^3 - 27(-ad^2 - b^2e)^2$, which is equal to the foregoing value of U.

27. Taking a, b, d, e as coordinates, then omitting common numerical factors, the first derived functions are

$$a^2e^3 - 8abde^2 - 18ad^4 - 2b^2d^2e,$$
$$-4a^2de^2 - 4abd^2e - 36b^3e^2 - 64b^2d^3,$$
$$-4a^2be^2 - 36a^2d^3 - 4ab^2de - 64b^3d^2,$$
$$a^3e^2 - 8a^2bde - 2ab^2d^2 - 18b^4e,$$

and the second derived functions (changing, for greater convenience, the signs) are

$$\begin{array}{llll} 2(-ae^3 + 4bde^2 + 9d^4), & 4(2ad^2 + bd^2e), & 4(2abe^2 + 18ad^3 + b^2de), & -3a^2e^2 + 16abde + 2b^2d^2, \\ 4(2ade^2 + bd^2e), & 4(ad^2e + 27b^2e^2 + 32bd^3), & 4(a^2e^2 + 2abde + 48b^2d^2), & 4(2a^2de + abd^2 + 18b^3e), \\ 4(2abe^2 + 18ad^3 + b^2de), & 4(a^2e^2 + 2abde + 48b^2d^2), & 4(27a^2d^2 + ab^2e + 32b^3d), & 4(2a^2be + ab^2d), \\ -3a^2e^2 + 16abde + 2b^2d^2, & 4(2a^2de + abd^2 + 18b^3e), & 4(2a^2be + ab^2d), & 2(-a^3e + 4a^2bd + 9b^4). \end{array}$$

28. Representing these by

$$\begin{array}{cccc} A, & H, & G, & L, \\ H, & B, & F, & M, \\ G, & F, & C, & M, \\ L, & M, & N, & P, \end{array}$$

and employing for the determinant the same partially developed form as in the first example, then proceeding to the calculation, we find

$AM - LH = 4 \times$	$FN - CM = 16 \times$	$AN - LG = 4 \times$	$FM - BN = 16 \times$	$AP - L^2 = 1 \times$	$BC - F^2 = 16 \times$	$HN - GM = 288 \times$
$a^3de^4 + 2$	$a^4be^3 + 2$	$a^3be^4 + 2$	$a^4de^3 + 2$	$a^4e^4 - 5$	$a^4e^4 - 1$	$a^3d^4e - 2$
$a^2bd^2e^3 - 15$	$a^4d^3e - 54$	$a^3d^3e^2 + 54$	$a^3bd^2e^2 + 3$	$a^3bde^3 + 64$	$a^3bde^3 - 4$	$a^2bd^5 - 1$
$a^2d^5e + 36$	$a^3b^2de^2 + 3$	$a^2b^2de^3 - 15$	$a^2b^3e^3 - 36$	$a^3d^4e - 36$	$a^3d^4e + 27$	$ab^4e^3 - 2$
$ab^3e^4 - 36$	$a^3bd^4 - 27$	$a^2bd^4e - 252$	$a^2b^2d^3e + 33$	$a^2b^2d^2e^2 - 180$	$a^2b^2d^2e^2 + 630$	$ab^3d^3e - 18$
$ab^2d^3e^2 - 12$	$a^2b^3d^2e - 453$	$ab^3d^2e^2 - 12$	$ab^4de^2 + 9$	$a^2bd^5 + 144$	$a^2bd^5 + 864$	$b^5de^2 - 1$
$abd^6 + 18$	$ab^5e^2 - 18$	$ab^2d^5 - 18$	$ab^3d^4 + 16$	$ab^4e^3 - 36$	$ab^4e^3 + 27$	
$b^4de^3 + 144$	$ab^4d^3 + 16$	$b^4d^3e - 2$	$b^5d^2e + 864$	$ab^3d^3e - 64$	$ab^3d^3e - 128$	
$b^3d^4e + 322$	$b^6de - 576$			$b^5de^2 + 144$	$b^5de^2 + 864$	
				$b^4d^4 + 320$	$b^4d^4 - 1280$	
$+459$	-1107	-243	$+891$	$+351$	-999	-24

$HP-LM$ $=4\times$		$FG-CH$ $=16\times$		$GP-LN$ $=4\times$		$HF-BG$ $=16\times$	
a^4de^3	+ 2	a^3be^4	+ 2	a^4be^3	+ 2	a^3de^4	+ 2
$a^3bd^2e^2$	− 15	$a^3d^3e^2$	− 36	a^4d^3e	− 36	$a^2bd^2e^3$	+ 3
$a^2b^3e^3$	+ 54	$a^2b^2de^3$	+ 3	$a^3b^2de^2$	− 15	a^2d^5e	− 18
$a^2b^2d^3e$	− 12	a^2bd^4e	+ 9	a^3bd^4	+ 144	ab^3e^4	− 54
ab^4de^2	− 252	$ab^3d^2e^2$	+ 33	$a^2b^3d^2e$	− 12	$ab^2d^3e^2$	− 453
ab^3d^4	− 2	ab^2d^5	+ 864	ab^5e^2	+ 36	abd^6	− 576
b^5d^2e	− 18	b^4d^3e	+ 16	ab^4d^3	+ 322	b^4de^3	− 27
				b^6de	+ 18	b^3d^4e	+ 16
	− 243		+ 891		+ 459		− 1107

where, by way of verification, I have annexed to each column the sum of the coefficients.

29. Forming from these the seven component terms of the determinant, and collecting, the result divides by 5, and we have

a^8e^8	+ 1			+ 5			
a^7bde^7	+ 4	+ 16	+ 16	− 44		+ 16	+ 16
$a^7d^4e^5$	− 135	− 432	+ 432	− 99		− 288	− 288
$a^6b^2d^2e^6$	− 722	− 96	− 96	− 3226		− 96	− 96
$a^6bd^5e^4$	+ 432	+ 3312	− 1368	− 2592		+ 2232	+ 576
$a^6d^8e^2$	+ 2916	− 7776		− 972	+ 20736		+ 2592
$a^5b^4e^7$	− 135	− 288	− 288	− 99		+ 432	− 432
$a^5b^3d^3e^5$	+ 6784	+ 3876	− 7788	+ 41744		− 7788	+ 3876
$a^5b^2d^6e^3$	+ 21492	+ 4788	+ 3960	+ 27180		+ 8100	+ 63432
a^5bd^9e	+ 11664	− 7776		− 27216	+ 20736		+ 72576
$a^4b^5de^6$	+ 432	+ 576	+ 2232	− 2592		− 1368	+ 3312
$a^4b^4d^4e^4$	+ 352	+ 2524	+ 36220	− 117200	+ 41472	+ 36220	+ 2524
$a^4b^3d^7e^2$	− 26352	− 133272	− 30024	− 61920	+ 373248	− 51984	− 227808
$a^4b^2d^{10}$	− 40824	− 1944		+ 124416	+ 5184		− 331776
$a^3b^6d^2e^5$	+ 21492	+ 63432	+ 8100	+ 27180		+ 3960	+ 4788
$a^3b^4d^5e^3$	+ 50944	+ 6504	+ 177576	− 154912	+ 41472	+ 177576	+ 6504
$a^3b^4d^8e$	+ 11232	− 65088	− 18504	− 19008	+ 186624	− 41544	+ 13680
$a^2b^8e^6$	+ 2916	+ 2592		− 972	+ 20736		− 7776
$a^2b^7d^3e^4$	− 26352	− 227808	− 51984	− 61920	+ 373248	− 30024	− 133272
$a^2b^6d^6e^2$	− 140624	− 668472	− 872592	+ 689024	+ 1689984	− 872592	− 668472
$a^2b^5d^9$	− 131328	+ 1152	− 1152	+ 92160		− 6912	− 741888
ab^9de^5	+ 11664	+ 72576		− 27216	+ 20736		− 7776
$ab^8d^4e^3$	+ 11232	+ 13680	− 41544	− 19008	+ 186624	− 18504	− 65088
ab^7d^7e	− 25088	− 20864	− 62336	+ 40960		− 62336	− 20864
$b^{10}d^2e^4$	− 40824	− 331776		+ 124416	+ 5184		− 1944
$b^9d^5e^2$	− 131328	− 741888	− 6912	+ 92160		− 1152	+ 1152
b^8d^8	− 81920			− 409600			
	+ 492075	− 2032452	− 866052	+ 350649	+ 2985984	− 866052	− 2032452

where the first column is the Hessian.

30. This divides, as it should do, by

$U =$

a^3e^3	$+$	1
a^2bde^2	$-$	12
a^2d^4	$-$	27
ab^2d^2e	$-$	6
b^4e^2	$-$	27
b^3d^3	$-$	64
		-135

and the other factor, which is the Prohessian, is

$PU =$

a^5e^5	$+$	1
a^4bde^4	$+$	16
$a^4d^4e^2$	$-$	108
$a^3b^2d^2e^3$	$-$	524
a^3bd^5e	$-$	432
$a^2b^4e^4$	$-$	108
$a^2b^3d^3e^2$	$+$	656
$a^2b^2d^6$	$+$	1512
ab^5de^3	$-$	432
ab^4d^4e	$+$	272
$b^6d^2e^2$	$+$	1512
b^5d^5	$+$	1280
		$+3645.$

31. To simplify this, I first collect the six terms

$$\begin{aligned} &-108\,a^2e^2\ \ (a^2d^4+b^4e^2) \\ &-432\,abde\,(a^2d^4+b^4e^2) \\ &+1512\,b^2d^2\ \ (a^2d^4+b^4e^2), \end{aligned}$$

and then putting $a^2d^4+b^4e^2=(ad^2+b^2e)^2-2ab^2d^2e$, we have the terms

$$\begin{aligned} &+\ \ 216\,a^3b^2d^2e^3 \\ &+\ \ 864\,a^2b^3d^3e^2 \\ &-3024\,ab^4d^4e\,, \end{aligned}$$

which combined with the remaining terms give

$$\begin{aligned} &+\quad\ 1\,a^5e^5 \\ &+\quad 16\,a^4bde^4 \\ &-\ \ 308\,a^3b^2d^2e^3 \\ &+1520\,a^2b^3d^3e^2 \\ &-2752\,ab^4d^4e \\ &+1280\,b^5d^5\quad ; \end{aligned}$$

which is found to be divisible by $(ae-4bd)^3$: and we thus obtain for PU the form

$$PU = -108\,(a^2e^2+4\,abde-14b^2d^2)\,(ad^2+b^2e)^2+(a^2e^2+28\,abde-20\,b^2d^2)\,(ae-4\,bd)^3,$$

which puts in evidence that the cuspidal line $(ae-4bd=0,\ ad^2+b^2e=0)$ of the developable is also a cuspidal line of the Prohessian.

32. Writing the equations of the developable and the Prohessian under the forms

$$A^3-27\,B^2=0,$$

$$LA^3-108\,MB^2=0,$$

and substituting in the second equation $A^3=27\,B^2$, it becomes $B^2(L-4M)=0$, that is the intersection is made up of $A^3=0$, $B^2=0$, which is the cuspidal curve taken six times (order 36), and of the curve $A^3-27\,B^2=0$, $L-4M=0$ (order 24). But substituting for L, M their values, the equation $L-4M=0$ becomes

$$a^2e^2-4abde-12b^2d^2=0,$$

that is

$$(ae+2bd)\,(ae-6bd)=0,$$

so that the last mentioned curve is composed of the intersections of the developable by the two quadric surfaces

$$ae+2bd=0,\quad ae-6bd=0.$$

33. Now combining with the equation of the developable the equation $ae+2bd=0$, and observing that in consequence of the last mentioned equation we have

$$(ae-4bd)^3=(-6bd)^3=-216\,b^3d^3=+108\,ab^2d^2e,$$

the equation of the developable gives $(ad^2-b^2e)^2=0$, or we have (taken twice) the curve $ae+2bd=0$, $ad^2-b^2e=0$, which is a curve of the sixth order made up of the lines $(a=0,\ b=0)$, $(d=0,\ e=0)$, and of a quartic curve (an excubo-quartic[1]) the nodal line on the developable. If in like manner with the equation of the developable we combine the equation $ae-6\,bd=0$, then from this equation we have

$$(ae-4bd)^3=(2bd)^3=8b^3d^3=\tfrac{4}{3}ab^2d^2e,$$

and the equation of the developable then gives

$$(ad^2+b^2e^2)-\tfrac{4}{81}\,ab^2d^2e=0\,;$$

that is,

$$ad^2+\theta b^2e=0,\quad ad^2+\frac{1}{\theta}\,b^2e=0,\quad \text{if}\ \ \theta+\frac{1}{\theta}=2-\tfrac{4}{81}=\tfrac{158}{81}.$$

[1] A quartic curve which is the complete intersection of two quadric surfaces is termed a quadriquadric; a quartic curve of the kind which is not such complete intersection but can only be represented by means of a cubic surface is termed an excubo-quartic.

The curve $ae-6bd=0$, $ad^2+\theta b^2e=0$ is made up of the lines $(a=0,\ b=0)$, $(d=0,\ e=0)$, and of an excubo-quartic, and the curve $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2e=0$ is made up of the same two lines and of an excubo-quartic.

34. Hence we see that the intersection of the developable and the Prohessian which is of the order $(6+10=)$ 60 is made up as follows, viz.,

cuspidal curve $ae-4bd=0$, $ad^2+b^2e=0$,	taken 6 times,		$6\times6=36$
line $(a=0,\ b=0)$	„ 4	„	$1\times4=\ 4$
line $(d=0,\ e=0)$	„ 4	„	$1\times4=\ 4$
nodal curve (excubo-quartic) $ae+2bd=0$, $ad^2-b^2e=0$	„ 2	„	$4\times2=\ 8$
excubo-quartic $ae-6bd=0$, $ad^2+\theta b^2e=0$	„ 1	„	$4\times1=\ 4$
excubo-quartic $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2e=0$	„ 1	„	$4\times1=\ 4$
			60

35. It is to be added that a generating line of the developable meets the Prohessian in the ineunt on the cuspidal edge taken 6 times, in a point of the nodal line taken 2 times, viz. the $r-4$ points (r being here $=6$) of the general theorem, in a point of the excubo-quartic $ae-6bd=0$, $ad^2+\theta b^2e=0$, and in a point of the excubo-quartic $ae-6bd=0$, $ad^2+\frac{1}{\theta}b^2=0$, (these being the $2r-10$ points of the general theorem); we have thus $(6+2+2=)$ 10 points of intersection of the generating line with the Prohessian.