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NOTE ON THE THEORY OF CUBIC SURFACES.

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THE equation

$$AX^3 + BY^3 + 6CRST = 0,$$

where $X + Y + R + S + T = 0$, represents a cubic surface of a special form, viz. each of the planes $R = 0$, $S = 0$, $T = 0$ is a triple tangent plane meeting the surface in three lines *which pass through a point* [1]; and, moreover, the three planes $AX^3 + BY^3 = 0$ are triple tangent planes intersecting in a line. It is worth noticing that the equation of the surface may also be written

$$ax^3 + by^3 + c(u^3 + v^3 + w^3) = 0,$$

where $x + y + u + v + w = 0$. In fact, the coordinates satisfying the foregoing linear equations respectively, we have to show that the equation

$$AX^3 + BY^3 + 6CRST = ax^3 + by^3 + c(u^3 + v^3 + w^3)$$

may be identically satisfied. We have

$$\begin{aligned} &ax^3 + by^3 + c\ (u^3 + v^3 + w^3) \\ &= ax^3 + by^3 + c\,[(u\ + v + w)^3 - 3\ (v + w)(w + u)(u + v)] \\ &= ax^3 + by^3 - c\ (x\ + y)^3 \qquad - 3c\,(v + w)(w + u)(u + v), \end{aligned}$$

[1] The tangent plane of a surface intersects the surface in a curve having at the point of contact a double point, and in like manner a triple tangent plane intersects the surface in a curve with three double points, viz. each point of contact is a double point; there is not in general any triple tangent plane such that the three points of contact come together, or (what is the same thing) there is not in general any tangent plane intersecting the surface in a curve having at the point of contact a triple point. A surface may, however, have the kind of singularity just referred to, viz. a tangent plane intersecting the surface in a curve having at the point of contact a triple point; such tangent plane may be termed a 'tritom' tangent plane, and its point of contact a 'tritom' point: for a cubic surface the intersection by a tritom tangent plane is of course a system of three lines meeting in the tritom point. The tritom singularity is sibi-reciprocal; it is, I think, a singularity which should be considered in the theory of reciprocal surfaces.

which is to be

$$= AX^3 + BY^3 + 6CRST;$$

and we may find X, Y, R, S, T, linear functions of x, y, u, v, w, so as to satisfy these equations, and so that in virtue of

$$x + y + u + v + w = 0,$$

we shall have also $X + Y + R + S + T = 0$. For, assuming

$$\begin{aligned} AX^3 + BY^3 &= ax^3 + by^3 - c(x+y)^3, \\ X + Y &= x + y, \\ R &= \tfrac{1}{2}(v + w), \quad C = -4c, \\ S &= \tfrac{1}{2}(w + u), \\ T &= \tfrac{1}{2}(u + v), \end{aligned}$$

we have identically

$$\begin{aligned} AX^3 + BY^3 + 6CRST &= ax^3 + by^3 - c(x+y)^3 - 3c(v+w)(w+u)(u+v), \\ X + Y + R + S + T &= x + y + u + v + w; \end{aligned}$$

and thus it only remains to show that we can find X, Y linear functions of x, y, such that

$$\begin{aligned} AX^3 + BY^3 &= ax^3 + by^3 - c(x+y)^3, \\ X + Y &= x + y. \end{aligned}$$

This is always possible; in fact if

$$U = ax^3 + by^3 - c(x+y)^3,$$

then taking Φ for the cubicovariant, and $\square$ for the discriminant of U, we have $\frac{1}{2}(\Phi + \sqrt{\square}U)$, $\frac{1}{2}(\Phi - \sqrt{\square}U)$ each a perfect cube, say

$$\tfrac{1}{2}(\Phi + \sqrt{\square}U) = (\lambda x + \mu y)^3,$$

$$\tfrac{1}{2}(\Phi - \sqrt{\square}U) = (\nu x + \rho y)^3,$$

and we then have

$$U = \frac{1}{\sqrt{\square}}\{(\lambda x + \mu y)^3 - (\nu x + \rho y)^3\} = AX^3 + BY^3,$$

which is satisfied by

$$X = l(\lambda x + \mu y),$$

$$Y = m(\nu x + \rho y),$$

if

$$Al^3 = \frac{1}{\sqrt{\square}}, \quad Bm^3 = -\frac{1}{\sqrt{\square}}.$$

The equation $X+Y=x+y$ then gives

$$l\lambda + m\nu = 1,$$

$$l\mu + m\rho = 1,$$

which give the values of l and m, and thence the values of A and B; and collecting all the equations, we have

$$\begin{aligned}
X &= \frac{\rho-\nu}{\lambda\rho-\mu\nu}(\lambda x+\mu y), & A &= \frac{1}{\sqrt{\Box}}\left(\frac{\lambda\rho-\mu\nu}{\rho-\nu}\right)^3, \\
Y &= -\frac{\mu-\lambda}{\lambda\rho-\mu\nu}(\nu x+\rho y), & B &= \frac{1}{\sqrt{\Box}}\left(\frac{\lambda\rho-\mu\nu}{\rho-\nu}\right)^3, \\
R &= \tfrac{1}{2}(v+w), & C &= -4c, \\
S &= \tfrac{1}{2}(w+u), \\
T &= \tfrac{1}{2}(u+v),
\end{aligned}$$

where

$$\lambda x + \mu y = \{\tfrac{1}{2}(\Phi + \sqrt{\Box}\,U)\}^{\frac{1}{3}},$$

$$\mu x + \rho y = \{\tfrac{1}{2}(\Phi - \sqrt{\Box}\,U)\}^{\frac{1}{3}}$$

$\left(\Phi, \Box\right.$ being respectively the cubicovariant and the discriminant of $\left.U = ax^3 + by^3 - c(x+y)^3\right)$, for the formulæ of the transformation

$$AX^3 + BY^3 + 6CRST = ax^3 + by^3 + c(u^3+v^3+w^3).$$

$$X+Y+R+S+T = x+y+u+v+w.$$

The equation $ax^3 + by^3 + c(u^3+v^3+w^3) = 0$, where

$$x+y+u+v+w=0,$$

presents over the other form the advantage that it is included as a particular case under the equation $ax^3+by^3+cu^3+dv^3+ew^3=0$ (where $x+y+u+v+w=0$) employed by Dr Salmon as the canonical form of equation for the general cubic surface.

5, *Downing Terrace, Cambridge, April* 29, 1864.