333.

NOTE ON THE NODAL CURVE OF THE DEVELOPABLE DERIVED FROM THE QUARTIC EQUATION $(a, b, c, d, e \between t, 1)^4 = 0$.

[From the *Philosophical Magazine*, vol. XXVII. (1864), pp. 437—440.]

CONSIDERING the coefficients (a, b, c, d, e) as linear functions of the coordinates x, y, z, w, then the equation

$$\text{Disct. } (a, b, c, d, e \between t, 1)^4 = 0,$$

or, as it may be written,

$$(ae - 4bd + 3c^2)^3 - 27\,(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0$$

represents, as is known, a developable surface or "torse," having for its edge of regression (or cuspidal curve) the sextic curve the equations whereof are

$$\begin{aligned} ae \ - 4bd + 3c^2 &= 0, \\ ace + 2bcd - ad^2 - b^2e - c^3 &= 0\,; \end{aligned}$$

and for its nodal curve, a curve the equations whereof (equivalent to two independent relations between the coordinates) are

$$\frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e}\,;$$

or, as these may also be written,

$$\begin{aligned} a^2d - 3abc + 2b^3 &= 0, \\ a^2e + 2abd - 9ac^2 + 6b^2c &= 0, \\ abe - 3acd + 2b^2d &= 0, \\ ad^2 - b^2e &= 0, \\ ade - 3bce + 2bd^2 &= 0, \\ ae^2 + 2bde - 9c^2e + 6cd^2 &= 0, \\ be^2 - 3cde + 2d^3 &= 0\,; \end{aligned}$$

which curve is in fact an excubo-quartic,—viz. a quartic curve the partial intersection of a quadric surface and a cubic surface, having in common two non-intersecting right lines. To show that this is so, I remark that the coefficients a, b, c, d, e, quà linear functions of the four coordinates, satisfy a linear equation which may be taken to be

$$a+b+c+d+e=0\,;$$

this being so, the first form shows that the curve in question lies on the quadric surface

$$ac-b^2+\tfrac{1}{2}(ad-bc)+\tfrac{1}{6}(ae+2bd-3c^2)+\tfrac{1}{2}(be-cd)+ce-d^2=0,$$

or, as this equation may also be written,

$$c\left(a-\tfrac{1}{2}b-\tfrac{1}{2}c-\tfrac{1}{2}d+e\right)-b^2+\tfrac{1}{2}ad+\tfrac{1}{6}(ae+2bd)+\tfrac{1}{2}be-d^2=0.$$

Substituting for c its value, this equation is

$$-(a+e+b+d)\left(\tfrac{3}{2}a+\tfrac{3}{2}e\right)-b^2+\tfrac{1}{2}ad+\tfrac{1}{6}(ae+2bd)+\tfrac{1}{2}be-d^2=0,$$

or, what is the same thing,

$$9(a+e+b+d)(a+e)+6(b^2+d^2)-3(ad+be)-(ae+2bd)=0.$$

Hence, finally, the equation of the quadric surface is

$$9a^2+17ae+9e^2+6b^2-2bd+6d^2+9ab+9de+6ad+6be=0\,;$$

and the curve lies also on the cubic surface

$$ad^2-b^2e=0.$$

It only remains to show that these surfaces have in common two right lines, and to find the equations of these lines.

The cubic surface is a skew surface or "scroll" such that the equations of any generating line are $d-\theta b=0$, $e-\theta^2 a=0$, where θ is an arbitrary parameter. But considering the two lines

$$(d-\theta_1 b=0,\quad e-\theta_1^2 a=0),\quad (d-\theta_2 b=0,\quad e-\theta_2^2 a=0),$$

the general equation of the quadric surface through these two lines may be written

$$\begin{aligned}
&A\;.\;(d-\theta_1 b)(d-\theta_2 b)\\
&+B\;.\;(e-\theta_1^2 a)(e-\theta_2^2 a)\\
&+C\;.\;(d-\theta_1 b)(e-\theta_2^2 a)+(d-\theta_2 b)(e-\theta_1^2 a)\\
&+\frac{D}{\theta_1-\theta_2}\{(d-\theta_1 b)(e-\theta_2^2 a)-(d-\theta_2 b)(e-\theta_1^2 a)\}=0
\end{aligned}$$

or, expanding and reducing,

$$\begin{aligned}
&A\{\, d^2 - (\theta_1 + \theta_2)\, bd + \theta_1\theta_2 b^2\} \\
&+ B\{\, e^2 - (\theta_1^2 + \theta_2^2)\, ea + \theta_1^2\theta_2^2 a^2\} \\
&+ C\{2de - (\theta_1^2 + \theta_2^2)\, ad - (\theta_1 + \theta_2)\, be + \theta_1\theta_2(\theta_1 + \theta_2)\, ab\} \\
&+ D\{\qquad (\theta_1 + \theta_2)\, ad - \qquad be - \qquad \theta_1\theta_2\, ab\} = 0,
\end{aligned}$$

which, if θ_1, θ_2 are the roots of the equation $\theta^2 - \frac{1}{3}\theta + 1 = 0$, and therefore $\theta_1 + \theta_2 = \frac{1}{3}$, $\theta_1\theta_2 = 1$, and $\theta_1^2 + \theta_2^2 = -\frac{17}{9}$, is

$$\begin{aligned}
&A\,(\, d^2 - \tfrac{1}{3}\, db + b^2) \\
&+ B\,(\, e^2 + \tfrac{17}{9}\, ae + a^2) \\
&+ C\,(2de + \tfrac{17}{9}\, ad - \tfrac{1}{3}\, be + \tfrac{1}{3}\, ab) \\
&+ D\,(\qquad \tfrac{1}{3}\, ad - \quad be - \quad ab) = 0.
\end{aligned}$$

Putting $A = 6$, $B = 9$, $C = \frac{9}{2}$, $D = -\frac{15}{2}$, this is

$$\begin{aligned}
&9\,(\, a^2 + \tfrac{17}{9}\, ae + e^2) \\
&+ 6\,(\, b^2 - \tfrac{1}{3}\, bd + d^2) \\
&+ \tfrac{9}{2}\,(\tfrac{1}{3}ab + 2\, de + \tfrac{17}{9}\, ad - \tfrac{1}{3}\, be) \\
&+ \tfrac{15}{2}\,(\, ab \qquad - \tfrac{1}{3}\, ad + \quad be) = 0,
\end{aligned}$$

which is the before-mentioned quadric surface; hence the quadric surface and the cubic surface intersect in the two lines

$$(d - \theta_1 b = 0,\ \ e - \theta_1^2 a = 0),\quad (d - \theta_2 b = 0,\ \ e - \theta_2^2 a = 0)$$

(where θ_1, θ_2 are the roots of the quadric equation $\theta^2 - \frac{1}{3}\theta_1 + 1 = 0$); and they consequently intersect also in an excubo-quartic curve, which is the theorem required to be proved.

Blackheath, March 26, 1864.