330.

ON DIFFERENTIAL EQUATIONS AND UMBILICI.

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I.

CONSIDER the integral equation

$$Az^2 + 2Bz + C = 0,$$

where z is the constant of integration: the derived equation is

$$\begin{aligned}\Omega &= (AC' + A'C - 2BB')^2 - 4(AC - B^2)(A'C' - B'^2),\\ &= (CA' - C'A)^2 - 4(AB' - A'B)(BC' - B'C) \quad , = 0;\end{aligned}$$

and if for greater simplicity we write $A = 1$, then the derived equation is

$$\Omega = C'^2 - 4BC'B' + 4CB'^2 = 0,$$

corresponding to the integral equation

$$z^2 + 2Bz + C = 0.$$

Writing the integral equation under the form

$$(z + X)(z + Y) = 0,$$

we have

$$2B = X + Y, \quad C = XY,$$

whence also

$$2B' = X' + Y', \quad C' = XY' + X'Y,$$

and the derived equation becomes

$$\Omega = -(X - Y)^2 X'Y'.$$

Hence if we represent the roots X, Y in the form $P \pm Q\sqrt{\Box}$, so that $P = -B$, $Q\sqrt{\Box} = \sqrt{B^2 - AC}$, Q^2 being the greatest square factor of $B^2 - AC$, then

$$(X - Y)^2 = 4Q^2\Box, \quad X', Y' = P' \pm \left(Q'\sqrt{\Box} + \frac{Q\Box'}{2\sqrt{\Box}}\right),$$

$$X'Y' = P'^2 - \frac{1}{4\Box}(2Q'\Box + Q\Box')^2;$$

and the derived equation is

$$\Omega = -Q^2\{4\Box P'^2 - (2Q'\Box + Q\Box')^2\} = 0.$$

If B, C, &c. are functions of the coordinates (x, y), the equation $z^2 + 2Bz + C = 0$ (z an arbitrary constant) represents a series of curves in the plane of xy; but if we consider z as a coordinate, then the equation represents a surface, and the curves in question are the orthogonal projections on the plane of xy of the sections of the surface by the planes parallel to the plane of xy. To fix the ideas, the plane of xy may be taken to be horizontal, and the ordinates z vertical.

Writing the equation in the form

$$(z + B)^2 - (B^2 - C) = 0,$$

we see that the surface contains upon it the curve $z + B = 0$, $B^2 - C = 0$, which is the line of contact with the circumscribed vertical cylinder: such curve may be termed the envelope, or, when this is necessary, the complete envelope. The equation of the surface has however been taken to be $(z - P)^2 - Q^2\Box = 0$ (viz. it has been assumed that $B = -P$, $B^2 - C = Q^2\Box$); the envelope thus breaks up into the curve, $z - P = 0$, $Q = 0$, taken twice, and the curve $z - P = 0$, $\Box = 0$; the former of these is in general a nodal curve on the surface, and it may be spoken of as the nodal curve; the latter of them is the reduced or proper envelope, or simply the envelope. And the terms nodal curve and envelope may also be applied to the curves $Q = 0$ and $\Box = 0$, which are the projections on the plane of xy of the first-mentioned two curves respectively. There is however a case of higher singularity which it is proper to consider: suppose that Q and $\Box$ have a common factor K, say $Q = KR$, $\Box = K\nabla$; the complete envelope $Q^2\Box = R^2K^3\nabla = 0$ here breaks up into the nodal curve $R = 0$ twice, the cuspidal curve $K = 0$ three times, and the reduced or proper envelope $\nabla = 0$ once.

Reverting for a moment to the form $(z + X)(z + Y) = 0$, the derived equation $\Omega = -(X - Y)^2 X'Y' = 0$ is satisfied by $(X - Y)^2 = 0$; this equation, or say the equation of the envelope, being in fact the singular solution of the differential equation. This assumes however that the differential equation is given in the form in which it is immediately obtained by derivation from the integral equation, without the rejection of factors which are functions of the coordinates (x, y) only; it is proper to consider the reduced equation obtained by rejecting such factors. Thus if X and Y are rational functions, the reduced form is $X'Y' = 0$, which is no longer satisfied by the equation

$(X-Y)^2=0$. In the before-mentioned case where the roots are $P \pm Q\sqrt{\square}$ (or $(X-Y)^2=Q^2\square$), P, Q, and $\square$ being rational functions of (x, y), the derived equation

$$\Omega = -Q^2\{4\square P'^2 - (2Q'\square + Q\square')^2\} = 0$$

divides out by the factor Q^2, but it does not divide out by $\square$; the reduced form is therefore

$$4\square P'^2 - (2Q'\square + Q\square')^2 = 0,$$

which is not satisfied by $Q=0$, while it is still satisfied by $\square = 0$ (since this gives also $\square' = 0$); that is, the nodal curve $Q=0$ is not a solution of the differential equation, but we still have the singular solution $\square = 0$, which corresponds to the reduced or proper envelope. In the case $Q = KR$, $\square = K\nabla$ of a cuspidal curve, the above form of the differential equation becomes

$$4K\nabla P'^2 - \{3KK'R\nabla + K^2(2\nabla R' + \nabla' R)\}^2 = 0,$$

which divides out by K; and, when reduced by the rejection of this factor, it is no longer satisfied by the equation $K=0$, which belongs to the cuspidal curve; that is, neither the nodal curve $R=0$ nor the cuspidal curve $K=0$ is a solution of the differential equation, but we still have the singular solution $\nabla = 0$, which corresponds to the reduced or proper envelope. It would appear that the conclusion may be extended to singularities of a higher nature, viz. the factor corresponding to any singular curve which presents itself as part of the complete envelope divides out from the derived equation; and such singular curve does not constitute a solution of the reduced equation, but we have a singular solution corresponding to the reduced or proper envelope.

II.

Consider the differential equation

$$y(p^2-1) + 2mxp = 0,$$

where, to fix the ideas, $m >$ or $= 1$; the integral equation may be taken to be

$$z = (mx + \sqrt{m^2x^2+y^2})(mx^2 + y^2 + x\sqrt{m^2x^2+y^2})^{m-1};$$

or rather, writing for shortness $\square = m^2x^2 + y^2$, and putting

$$z = (mx + \sqrt{\square})(mx^2 + y^2 + x\sqrt{\square})^{m-1} = P + Q\sqrt{\square},$$

the integral equation is

$$(z-P)^2 - Q^2\square = 0, \text{ or } z^2 - 2Pz + P^2 - Q^2\square = 0,$$

where

$$P^2 - Q^2\square = (m^2x^2 - \square)\{(mx^2+y^2)^2 - x^2\square\}^{m-1} = -y^{2m}\{y^2 + (2m-1)x^2\}^{m-1}.$$

In the particular case $m=1$ the equation is

$$z = x + \sqrt{x^2+y^2}, \text{ or } z^2 - 2zx - y^2 = 0.$$

Before going further, I remark that, m being a positive integer greater than unity, we have

$$z = P + Q\sqrt{\square} = mx(mx^2 + y^2)^{m-1} + \{mx^2 + y^2 + (m-1)mx^2\}(mx^2 + y^2)^{m-2}\sqrt{\square} + \&c.,$$

the subsequent terms being divisible, the rational ones by $\square$, and the irrational ones by $\square\sqrt{\square}$. Hence, observing that

$$mx^2 + y^2 + (m-1)mx^2 = m^2x^2 + y^2 = \square,$$

we see that Q contains the factor $\square$, and the equation $\square = 0$ belongs to a cuspidal curve on the surface. If however $m = 1$, then the equation is $z = x + \sqrt{\square}$, so that $Q = 1$ does not contain the factor $\square$; and $\square = x^2 + y^2 = 0$ is not a singular curve on the surface, but is in fact the reduced or proper envelope.

The curve represented by the integral equation will pass through the origin ($x = 0$, $y = 0$) for the value $z = 0$ of the constant of integration. In fact, for this value, the integral equation becomes

$$-y^{2m}\{y^2 + (2m-1)x^2\}^{m-1} = 0,$$

which belongs to a set of $2m + (m-1) + (m-1)$ lines coinciding with the lines $y = 0$, $y = ix\sqrt{2m-1}$, and $y = -ix\sqrt{2m-1}$ respectively. The directions at the origin are therefore $p = 0$, $p = \pm i\sqrt{2m-1}$, which are the same values of p as are obtained from the differential equation; viz. since this is satisfied identically at the point in question, proceeding to the derived equation, we have

$$p(p^2 - 1) + 2mp = 0,$$

that is

$$p(p^2 + 2m - 1) = 0;$$

but it is to be observed that these values of p are different from the values given by the equation $\square = m^2x^2 + y^2 = 0$, which are $p = \pm im$. The reason is that the curve $\square = 0$ being, as was shown, a cuspidal curve on the surface, the equation $\square = 0$ is not a solution of the differential equation.

If however $m = 1$, then the integral equation gives at the origin no longer three values of p, but only the value $p = 0$. The differential equation however gives, as in the general case, three values; viz. we have $p(p^2 + 1) = 0$; and the values $p = \pm i$ obtained from the factor $p^2 + 1 = 0$ are precisely the values of p obtained from the equation $\square = x^2 + y^2 = 0$, which in the case now under consideration belongs to the reduced or proper envelope of the surface, and *is* therefore the singular solution of the differential equation.

III.

The two curves of curvature which pass through any given point of a surface are distinct curves, not branches of one indecomposable curve. In fact if P, Q are the two curves of curvature for a point A, then for a point A' on P the two curves of curvature will be P, Q'; and if P, Q were branches of an indecomposable curve, then P, Q' would also be branches of an indecomposable curve, and we should have P a branch of two different indecomposable curves, which is of course impossible. In the case of an umbilicus, the two curves P and Q coincide together; or, as we may express it, the curves of curvature through an umbilicus are the duplication of a single, in general indecomposable, curve; and in general this curve has at the umbilicus a *trifid* node. I use this expression to denote a point at which there are three distinct tangents, or, more accurately, three distinct directions of the curve: an ordinary triple point is of necessity a trifid node, but not conversely. The umbilicus of an ellipsoid or other quadric surface is a peculiar exceptional case.

In support of the foregoing conclusions, consider a surface having an umbilicus at the origin, and take $z=0$ as the equation of the tangent plane at that point; the equation of the surface in the neighbourhood of the umbilicus will be

$$z = \tfrac{1}{2}k(x^2+y^2) + \tfrac{1}{6}(ax^3+3bx^2y+3cxy^2+dy^3);$$

so that, writing as usual p and q for the first, and r, s, t for the second, differential coefficients of z, we have

$$\begin{aligned} p &= kx + \tfrac{1}{2}(ax^2+2bxy+cy^2), \\ q &= ky + \tfrac{1}{2}(bx^2+2cxy+dy^2), \\ r &= k + ax + by, \\ s &= bx + cy, \\ t &= k + cx + dy. \end{aligned}$$

The differential equation of the curves of curvature projected on the plane of xy is

$$\left(\frac{dy}{dx}\right)^2 [(1+q^2)s - pqt] + \frac{dy}{dx}[(1+q^2)r - (1+p^2)t] - [(1+p^2)s - pqr] = 0;$$

and substituting therein the foregoing values of p, q, r, s, t, but attending only to the terms of the lowest order in (x, y), and using moreover in the sequel p in the place of $\frac{dy}{dx}$, the equation becomes

$$(bx+cy)(p^2-1) + [(a-c)x + (b-d)y]p = 0;$$

which may be taken as the differential equation of the curves of curvature at and in the neighbourhood of the umbilicus. The equation is satisfied identically by the values $x=0$, $y=0$, which correspond to the umbilicus; and to find p, we have to differentiate the equation, and then substitute these values of x and y; we thus obtain

$$(b+cp)(p^2-1) + [(a-c)+(b-d)p]p = 0,$$

or, what is the same thing,

$$p(a+2bp+cp^2) - (b+2cp+dp^2) = 0,$$

a cubic equation for the determination of p.

I remark that we may without loss of generality write $d=0$: but to simplify the investigation, I suppose in the first instance that we have also $b=0$; this comes to assuming that one of the three planes $ax^3+3bx^2y+3cxy^2+dy^3=0$ bisects the angle formed by the other two planes. The differential equation consequently is

$$cy\,(p^2-1)+(a-c)\,xp=0;$$

or, putting for shortness

$$\frac{a-c}{c}=-2m,$$

it is

$$y\,(p^2-1)+\quad 2mxp=0,$$

which is the differential equation previously considered. Hence, writing now h in the place of z, the equation of the curve of curvature in the neighbourhood of the umbilicus is

$$h=(mx+\sqrt{\square})\,(mx^2+y^2+\sqrt{\square})^{m-1},\ =P+Q\sqrt{\square},$$

where $\square=m^2x^2+y^2$; or, what is the same thing, the equation is

$$h^2-2Ph+P^2-Q^2\square=0;$$

and the equation (in the neighbourhood of the umbilicus) of the curve through the umbilicus is

$$P^2-Q^2\square=-y^{2m}\{y^2+(2m-1)\,x^2\}^{m-1}=0;$$

so that the umbilicus is a trifid node. In the case however of an ellipsoid or other quadric surface, we have $m=1$, so that the equation of the curve of curvature in the neighbourhood of the umbilicus is

$$h=x+\sqrt{x^2+y^2},$$

or, what is the same thing,

$$h^2-2hx-y^2=0:$$

and for the curve through the umbilicus, in the neighbourhood of the umbilicus, the equation is $y^2=0$, so that there is only a single direction of the curve of curvature. The differential equation gives, however, at the umbilicus $p\,(p^2+1)=0$; the value $p=0$ is that which corresponds to the curve of curvature; the other two values $p=\pm i$ correspond to the curve (pair of lines) $x^2+y^2=0$, which is the envelope of the curves of curvature, or, more accurately, the envelope of the projections of the curves of curvature on the tangent plane at the umbilicus.

Blackheath, October 17, 1863.

IV.

The differential equation for the curves of curvature in the neighbourhood of an umbilicus was obtained in a form such as

$$(bx+cy)\,(p^2-1)+2\,(fx+gy)\,p=0;$$

and it was only because this equation did not appear to be readily integrable, that I considered, instead of it, the particular form

$$y(p^2-1)+2mxp=0.$$

But the general equation can be integrated; and the result presents itself in a simple form. For, returning to the differential equation

$$(bx+cy)(p^2-1)+2(fx+gy)p=0,$$

and assuming

$$\frac{bx+cy}{fx+gy}=\frac{-2v}{v^2-1},$$

or

$$(bx+cy)(v^2-1)+2(fx+gy)v=0,$$

we have

$$\frac{p^2-1}{v^2-1}=\frac{p}{v},\text{ or }(p-v)(vp+1)=0,$$

and we may write

$$p-v=0.$$

Assuming also

$$y=ux,\text{ or }u=\frac{y}{x},$$

the relation between u and v is

$$\frac{b+cu}{f+gu}=\frac{-2v}{v^2-1};$$

or, as this may be written,

$$v^2-1+2\frac{f+gu}{b+cu}v=0,$$

giving

$$v=\frac{-(f+gu)-\sqrt{(b+cu)^2+(f+gu)^2}}{b+cu},$$

where for convenience the radical has been taken with a negative sign. We have moreover

$$u=-\frac{b(v^2-1)+2fv}{c(v^2-1)+2gv}.$$

The equation $p-v=0$, substituting for y its value ux, then becomes

$$x\frac{du}{dx}+u-v=0;$$

or, as this may be written,

$$\frac{dx}{x}+\frac{du}{u-v}=0;$$

or, what is the same thing,

$$\frac{dx}{x} + \frac{dv - du}{v - u} - \frac{dv}{v - u} = 0.$$

But

$$v - u = v + \frac{b(v^2 - 1) + 2fv}{c(v^2 - 1) + 2gv} = \frac{V}{c(v^2 - 1) + 2gv},$$

where

$$V = v[c(v^2 - 1) + 2gv] + b^2(v - 1) + 2fv$$
$$= (b + cv)(v^2 - 1) + 2(f + gv)v,$$

and the differential equation takes thus the form

$$\frac{dx}{x} + \frac{dv - du}{v - u} - \frac{[c(v^2 - 1) + 2gv]\,dv}{V} = 0;$$

and hence, writing

$$V = (b + cv)(v^2 - 1) + 2(f + gv)v = c(v - \alpha)(v - \beta)(v - \gamma),$$

and

$$\frac{c(v^2 - 1) + 2gv}{V} = \frac{c(v^2 - 1) + 2gv}{c(v - \alpha)(v - \beta)(v - \gamma)} = \frac{A}{v - \alpha} + \frac{B}{v - \beta} + \frac{C}{v - \gamma},$$

so that

$$A = \frac{c(\alpha^2 - 1) + 2g\alpha}{c(\alpha^2 - 1) + 2g\alpha + 2\{f + (b + g)\alpha + c\alpha^2\}},$$

with the like values for B and C—values which are such that $A + B + C = 1$,—the integral equation is

$$\text{const.} = x(v - u)(v - \alpha)^{-A}(v - \beta)^{-B}(v - \gamma)^{-C},$$

or, substituting for $v - u$ its value, $= \dfrac{c(v - \alpha)(v - \beta)(v - \gamma)}{c(v^2 - 1) + 2gv}$,

$$\text{const.} = x\{c(v^2 - 1) + 2gv\}^{-1}(v - \alpha)^{1-A}(v - \beta)^{1-B}(v - \gamma)^{1-C}.$$

But

$$v = \frac{-(f + gu) - \sqrt{U}}{b + cu};$$

if for shortness $U = (b + cu)^2 + (f + gu)^2$, and thence

$$v^2 = \frac{2(f + gu)^2 + (b + cu)^2 + 2(f + gu)\sqrt{U}}{(b + cu)^2},$$

and

$$c(v^2 - 1) + 2gv = \frac{2(cf - bg)(f + gu + \sqrt{U})}{(b + cu)^2},$$

$$v - \alpha = \frac{-(f + gu) - \sqrt{U} - \alpha(b + cu)}{b + cu}, \text{ \&c.}$$

Substituting these values, and observing that the exponent of $b+cu$ is

$$(-2+1-A+1-B+1-C, =1-A-B-C)=0,$$

the integral equation is

$$\text{const.}=x\,(f+gu+\sqrt{U})^{-1}\times$$

$$\left(f+gu+\alpha\,(b+cu)+\sqrt{U}\right)^{1-A}\left(f+gu+\beta\,(b+cu)+\sqrt{U}\right)^{1-B}\left(f+gu+\gamma\,(b+cu)+\sqrt{U}\right)^{1-C};$$

or, observing that the exponent 1 of x is

$$=-1+(1-A)+(1-B)+(1-C),$$

and putting for shortness $\square=(fx+gy)^2+(bx+cy)^2$, the integral equation finally is

$$\text{const.}=(fx+gy+\sqrt{\square})^{-1}\times$$

$$\left(fx+gy+\alpha\,(bx+cy)+\sqrt{\square}\right)^{1-A}\left(fx+gy+\beta\,(bx+cy)+\sqrt{\square}\right)^{1-B}\left(fx+gy+\gamma(bx+cy)+\sqrt{\square}\right)^{1-C},$$

where the quantities α, β, γ, A, B, C are given by

$$(b+cv)\,(v^2-1)+2\,(f+gv)=c\,(v-\alpha)\,(v-\beta)\,(v-\gamma),$$

$$\frac{c\,(v^2-1)+2gv}{c\,(v-\alpha)\,(v-\beta)\,(v-\gamma)}=\frac{A}{v-\alpha}+\frac{B}{v-\beta}+\frac{C}{v-\gamma}.$$

Consider the curve

$$0=\left(fx+gy+\alpha(bx+cy)+\sqrt{\square}\right)^{1-A}\left(fx+gy+\beta(bx+cy)+\sqrt{\square}\right)^{1-B}\left(fx+gy+\gamma\,(bx+cy)+\sqrt{\square}\right)^{1-C},$$

which corresponds to the value $=0$ of the constant. If, for instance,

$$fx+gy+\alpha\,(bx+cy)+\sqrt{\square}=0,$$

this equation gives

$$(bx+cy)\,\{(bx+cy)\,(\alpha^2-1)+2\,(fx+gy)\,\alpha\}=0\,;$$

or say

$$(bx+cy)\,(\alpha^2-1)+2\,(fx+gy)\,\alpha=0.$$

But we have

$$(b\ \ +c\alpha)\,(\alpha^2-1)+2\,(f\ \ +g\alpha)\,\alpha=0,$$

and the equation therefore is

$$(bx+cy)\,(f+g\alpha)-(fx+gy)\,(b+c\alpha)=0\,;$$

that is

$$(cf-bg)\,(y-\alpha x)=0\,;$$

or simply $y-\alpha x=0$; that is, the directions of the curve at the origin, or point $x=0$, $y=0$, are given by the equations $y-\alpha x=0$, $y-\beta x=0$, $y-\gamma x=0$. This is right, since from the differential equation we obtain at the origin

$$(b+cp)\,(p^2-1)+2\,(f+gp)\,p, =c\,(p-\alpha)\,(p-\beta)\,(p-\gamma), =0.$$

V.

The particular case of the equation

$$y(p^2-1)+2mxp=0$$

is obtained from the general equation by writing therein $b=0$, $c=1$, $g=0$, $f=m$; we have therefore

$$v(v^2+2m-1)=(v-\alpha)(v-\beta)(v-\gamma),$$

or say

$$\alpha=0,\quad \beta=i\sqrt{2m-1},\quad \gamma=-i\sqrt{2m-1};$$

and thence

$$\frac{v^2-1}{v(v^2+2m-1)}=-\frac{1}{2m-1}\frac{1}{v}+\frac{2m}{2m-1}\frac{v}{v^2+2m-1}=\frac{A}{v}+\frac{B}{v+i\sqrt{2m-1}}+\frac{C}{v-i\sqrt{2m-1}},$$

giving

$$A=-\frac{1}{2m-1},\quad B=C=\frac{m}{2m-1}.$$

The integral equation thus is

$$\text{const.}=(mx-\sqrt{\Box})^{-1}(mx+\sqrt{\Box})^{\frac{2m}{2m-1}}\{(mx+i\sqrt{2m-1}\,y+\sqrt{\Box})(mx-i\sqrt{2m-1}\,y+\sqrt{\Box})\}^{\frac{m-1}{2m-1}}$$

where $\Box=m^2x^2+y^2$; or, observing that

$$\begin{aligned}&(mx+i\sqrt{2m-1}\,y+\sqrt{\Box})(mx-i\sqrt{2m-1}\,y+\sqrt{\Box})\\&=(mx+\sqrt{\Box})^2+y^2\\&=2m(mx^2+y^2+x\sqrt{\Box}),\end{aligned}$$

the integral equation is

$$\text{const.}=(mx+\sqrt{\Box})^{\frac{1}{2m-1}}(mx^2+y^2+x\sqrt{\Box})^{\frac{m-1}{2m-1}},$$

or, what is the same thing,

$$\text{const.}=(mx+\sqrt{\Box})(mx^2+y^2+x\sqrt{\Box})^{m-1},$$

the result given in the former part of the present paper.

VI.

I annex the following *à posteriori* verification of the solution

$$\text{const.}=(mx+\sqrt{\Box})(mx^2+y^2+x\sqrt{\Box})^{m-1}$$

of the particular equation

$$y(p^2-1)+2mxp=0.$$

Putting for shortness

$$A = mx + \sqrt{\square},$$
$$B = mx^2 + y^2 + x\sqrt{\square},$$

where it will be remembered that

$$\square = m^2x^2 + y^2,$$

then we have

$$2m B = A^2 + (2m-1)\,y^2.$$

The integral equation may be written

$$h = P + Q\sqrt{\square} = U = AB^{m-1};$$

and we have

$$\frac{U'}{U} = \frac{A'}{A} + (m-1)\frac{B'}{B} = \frac{\Theta}{\sqrt{\square}},$$

if

$$\Theta = \sqrt{\square}\left\{\frac{A'}{A} + (m-1)\frac{B'}{B}\right\}.$$

But we have

$$A'\sqrt{\square} = m\sqrt{\square} + m^2x + yp = mA + yp,$$
$$B'\sqrt{\square} = (2mx + 2yp)\sqrt{\square} + \square + x(m^2x + yp)$$
$$= 2m^2x^2 + y^2 + xyp + (2mx + 2yp)\sqrt{\square}$$
$$= A^2 + pyx + 2\sqrt{\square},$$

and

$$\frac{1}{B} = \frac{2m}{A^2 + (2m-1)\,y^2},$$

and the value of Θ thus is

$$\Theta = \frac{mA + yp}{A} + (2m^2 - 2m)\frac{A^2 + py(x + 2\sqrt{\square})}{A^2 + (2m-1)\,y^2}$$

$$= \frac{1}{A\,[A^2 + (2m-1)\,y^2]}\{(mA + yp)[A^2 + (2m-1)\,y^2] + (2m^2 - 2m)[A^3 + Apy(x + 2\sqrt{\square})]\},$$

where the expression in { } is

$$= (2m^2 - m)\,A\,(A^2 + y^2)$$
$$+ yp\,\{A^2 + (2m-1)\,y^2 + (2m^2 - 2m)\,A\,(x + 2\sqrt{\square})\}.$$

Here the coefficient of yp is $= (2m^2 - m)(A^2 + y^2)$; in fact we have identically

$$A^2 + y^2 - 2A\sqrt{\square} = 0,$$

and thence

$$(2m^2 - 3m + 1)(A^2 + y^2) - 2(2m-1)(m-1)\,A\sqrt{\square} = 0,$$

that is

$$(2m^2 - m - 1)\,A^2 + (2m^2 - 3m + 1)\,y^2 - (2m-2)\,A\,\{A + (2m-1)\sqrt{\square}\} = 0;$$

or

$$(2m^2 - m - 1) A^2 + (2m^2 - 3m + 1) y^2 - (2m^2 - 2m) A (x + 2\sqrt{\square}) = 0,$$

and therefore

$$A^2 + (2m - 1) y^2 + (2m^2 - 2m) A (x + 2\sqrt{\square}) = (2m^2 - m)(A^2 + y^2).$$

Hence the term in { } is

$$= (2m^2 - m)(A^2 + y^2)(A + yp);$$

or, what is the same thing, it is $= (4m^2 - 2m) A\sqrt{\square} (A + yp)$. Hence, restoring for $A^2 + (2m-1) y^2$ its value $2mB$, we find

$$\Theta = \frac{(2m-1)\sqrt{\square}}{B}(A + yp),$$

or

$$\frac{U'}{U} = \frac{2m-1}{B}(A + yp).$$

But writing U_1, U_2 to denote the values corresponding to $+\sqrt{\square}$, $-\sqrt{\square}$ respectively, we have

$$U'_1 = \frac{(2m-1)\,U_1}{B_1}(mx + yp + \sqrt{\square}),$$

$$U'_2 = \frac{(2m-1)\,U_2}{B_2}(mx + yp - \sqrt{\square}),$$

and thence

$$U'_1 U'_2 = \frac{(2m-1)^2\, U_1 U_2}{B_1 B_2}\{(mx + yp)^2 - \square\}$$

$$= \frac{(2m-1)^2\, U_1 U_2}{B_1 B_2}\, y\,\{y\,(p^2 - 1) + 2mxp\}.$$

But we have

$$U' = P' + Q'\sqrt{\square} + \frac{Q\square'}{2\sqrt{\square}} = \frac{1}{2\sqrt{\square}}(2Q'\square + Q\square' + 2P'\sqrt{\square}),$$

and thence

$$U'_1 U'_2 = -\frac{1}{4\square}\{(2Q'\square + Q\square')^2 - 4P'^2\square\},$$

and moreover

$$U_1 U_2 = P^2 - Q^2\square = A_1 A_2 (B_1 B_2)^{m-1},$$

where

$$A A_2 \;= m^2x^2 - \square = -y^2,$$

$$B_1 B_2 \;= (mx^2 + y^2)^2 - \square = y^2\{y^2 + (2m-1)\,x^2\};$$

and we thence find

$$-\frac{1}{4\square}\{(2Q'\square + Q\square')^2 - 4P'^2\square\}$$

$$= \quad (2m-1)^2 A_1 A_2 (B_1 B_2)^{m-2}\, y^2\,\{y\,(p^2 - 1) + 2mxp\}$$

$$= -(2m-1)^2\, y^{2m-1}\,[y^2 + (2m-1)\,x^2]^{m-2}\,\{y\,(p^2 - 1) + 2mxp\}.$$

Hence, the derived equation being

$$Q^2\{(2Q'\square + Q\square')^2 - 4P'^2\square\} = 0,$$

the last preceding equation becomes

$$Q^2\,\square\, y^{2m-1}\{y^2 + (2m-1)\,x^2\}^{m-2}\{y\,(p^2-1) + 2mxp\} = 0.$$

Here, besides the factor Q^2 corresponding to the nodal curve, and the factor $\square$ corresponding to the cuspidal curve, we have the factors y^{2m-1} and $\{y^2+(2m-1)\,x^2\}^{m-2}$; and, rejecting all these, the differential equation in its reduced form is

$$y\,(p^2-1) + 2mxp = 0\,;$$

and the required verification is effected. The occurrence of

$$Q^2\,\square\, y^{2m-1}\{y^2 + (2m-1)\,x^2\}^{m-2}$$

as a factor in the complete derived equation would give rise to some further investigations, but I will not now enter on them.

I remark however that if $m = 1$, viz. if the integral equation be const. $= x + \sqrt{x^2+y^2}$, or say $z = x + \sqrt{x^2+y^2}$, or, what is the same thing,

$$z^2 - 2zx - y^2 = 0,$$

then observing that $y^2 + (2m-1)\,x^2$ is here $= x^2 + y^2$ which is $= \square$, so that

$$\square\,\{y^2 + (2m-1)\,x^2\}^{m-2} = \square\,.\,\square^{-1} = 1,$$

the differential equation in its complete form is

$$y\,(p^2y + 2px - y) = 0\,;$$

so that we have here the factor y which divides out. The last-mentioned result is most readily obtained directly from the equation

$$\Omega = Q^2(2Q'\square + Q\square')^2 - 4P'^2\square = 0,$$

which is the derived equation corresponding to the integral equation $z = P + Q\sqrt{\square}$. We in fact have $P = x$, $Q = 1$, $\square = x^2 + y^2$, and the derived equation thus is

$$(x + yp)^2 - (x^2 + y^2) = 0,$$

that is, $y\,(p^2y + 2px - y) = 0$.

I mention also, in connexion with the foregoing investigation, the integral equation

$$z = x + \sqrt{2x^2 - y^2}, \text{ or } z^2 - 2zx - x^2 + y^2 = 0,$$

for which the derived equation in its complete form is

$$(2x - yp)^2 - (2x^2 - y^2) = 0,$$

or, what is the same thing, $y^2p^2 - 4xyp + 2x^2 + y^2 = 0$, and for which therefore there is no factor to divide out.

VII.

The conics confocal with a given conic form a system similar in its properties to that of the curves of curvature of a quadric surface; and the theory of the last-mentioned system may be studied by means of the system of confocal conics. Consider then the equation

$$\frac{x^2}{a^2+z}+\frac{y^2}{b^2+z}=1,$$

which, if z be an arbitrary parameter, belongs to the conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Treating z as a coordinate, the equation represents a surface of the third order, which is such that its section by any plane parallel to the plane of xy is a conic; and the confocal conics are the projections on the plane of xy, by lines parallel to the axis of z, of the sections of the surface.

The sections by the planes of zx, zy are the parabolas $x^2=z+a^2$ and $y^2=z+b^2$ respectively. When $z>-b^2$, the ordinates in each parabola are real, and these ordinates give the semiaxes of the elliptic section. When $z>-a^2<-b^2$, then only the parabola section in the plane of zx has a real ordinate, and the sections are hyperbolic; and when $z<-a^2$, the section is altogether imaginary. The section in the planes $z=-b^2$ is the pair of coincident lines $y^2=0$, $z=-b^2$, and the section in the plane $z=-a^2$ is the pair of coincident lines $z=-a^2$, $x^2=0$; or, in other words, the plane $z+b^2=0$ touches the surface along the line $y=0$, and the plane $z+a^2=0$ touches the surface along the line $x=0$: this at once appears from the integral form

$$(z+a^2)(z+b^2)-x^2(z+b^2)-y^2(z+a^2)=0.$$

The points $(z=-b^2,\ y=0,\ x=\pm\sqrt{a^2-b^2})$ and $(z=-a^2,\ x=0,\ y=\pm\sqrt{b^2-a^2})$ are conical points; the last two are however imaginary points on the surface. To find the nature of the surface about one of the first-mentioned two points, say the point $(z=-b^2,$ $y=0,\ x=\sqrt{a^2-b^2})$, taking this point for the origin and writing therefore $\sqrt{a^2-b^2}+x$, y and $-b^2+z$ in the place of x, y, z respectively, the equation becomes

$$(a^2-b^2+z)\,z-\left((a^2-b^2)+2x\sqrt{a^2-b^2}+x^2\right)z-(a^2-b^2+z)\,y^2=0,$$

that is

$$z^2-2zx\sqrt{a^2-b^2}-(a^2-b^2)\,y^2-z\,(x^2+y^2)=0\,;$$

so that there is a tangent cone the equation whereof is

$$z^2-2zx\sqrt{a^2-b^2}-(a^2-b^2)\,y^2=0,$$

or, as it may be written,

$$(z-x\sqrt{a^2-b^2})^2-(a^2-b^2)(x^2+y^2)=0.$$

The equation is that of a cone of the second order, meeting the plane of zx in the lines $z=0$, $z=2x\sqrt{a^2-b^2}$ (and therefore such that its sections parallel to the plane of xy are parabolas), and meeting the plane of yz in the lines $z=\pm y\sqrt{a^2-b^2}$ (the origin being at the vertex of the cone or conical point of the surface).

Returning to the original origin, and to the equation of the surface written in the form

$$z^2+z(a^2+b^2-x^2-y^2)+a^2b^2-b^2x^2-a^2y^2=0,$$

calling this for a moment $z^2+2Bz+C=0$, the differential equation is $C'^2-4BB'C'+4CB'^2=0$; or, substituting, this is

$$(b^2x+a^2yp)^2-(a^2+b^2-x^2-y^2)(x+yp)(b^2x+a^2yp)+(a^2b^2-b^2x^2-a^2y^2)(x+yp)^2=0\,;$$

or, reducing, this is

$$(a^2-b^2)\,xy\,\{xy(p^2-1)-(a^2-b^2-x^2+y^2)\,p\}=0,$$

or say

$$xy\,\{xy(p^2-1)-(a^2-b^2-x^2+y^2)\,p\}=0,$$

where the factor xy arises from the level lines $(z+b^2=0,\ y=0)$ and $(z+a^2=0,\ x=0)$. Throwing out this factor, the equation becomes

$$xy(p^2-1)-(a^2-b^2-x^2+y^2)\;p=0,$$

which is satisfied identically by $z+b^2=0$, $y=0$, $x^2=a^2-b^2$. The first derived equation is

$$(xp+y)(p^2-1)+2(x-yp)\,p=0,$$

which for the values in question gives

$$p(p^2+1)=0,$$

where the factor $p=0$ corresponds to the section $y=0$ by the plane $z+b^2=0$: and taking the conical point for origin, and observing that the polar of the line $x=0$, $y=0$ in regard to the tangent cone is $z-x\sqrt{a^2-b^2}=0$, then writing the equation of the tangent cone in the form

$$(z-x\sqrt{a^2-b^2})^2-(a^2-b^2)(x^2+y^2)=0,$$

the two tangent planes through $(x=0,\ y=0)$ are given by the equation $x^2+y^2=0$; and for these planes we have $p^2+1=0$. The factor $p^2+1=0$ determines therefore the directions of the envelope at the conical point.

VIII.

In verification of the equation

$$z=\tfrac{1}{2}k(x^2+y^2)+\tfrac{1}{6}\,ax\,(x^2+y^2)$$

for a quadric surface in the neighbourhood of the umbilicus, I remark that, starting from the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

of an ellipsoid, and taking α, 0, γ as the coordinates of the umbilicus, and θ as the inclination to the axis of x of the tangent to the principal section through the umbilicus, then transforming to the umbilicus as origin and the new axes through that point, viz. the axes of x, z being the tangent and normal in the plane of ac, and the axis of y being at right angles to this (or in the direction of b), the equation becomes

$$\frac{(\alpha + x\cos\theta - z\sin\theta)^2}{a^2} + \frac{y^2}{b^2} + \frac{(\gamma - x\sin\theta - z\cos\theta)^2}{c^2} = 1,$$

or, expanding,

$$\left(\frac{\alpha^2}{a^2} + \frac{\gamma^2}{c^2} - 1\right) + 2x\left(\frac{\alpha\cos\theta}{a^2} - \frac{\gamma\sin\theta}{b^2}\right) - 2z\left(\frac{\alpha\sin\theta}{a^2} + \frac{\gamma\cos\theta}{c^2}\right)$$

$$+ x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{c^2}\right) + \frac{y^2}{b^2} + z^2\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2}\right) - 2zx\sin\theta\cos\theta\left(\frac{1}{a^2} - \frac{1}{c^2}\right) = 0.$$

But we have

$$\alpha = a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad \gamma = \frac{c\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}},$$

$$\tan\theta = \frac{c}{a}\frac{\sqrt{a^2 - b^2}}{\sqrt{a^2 - c^2}},$$

and thence

$$\sin\theta = \frac{c\sqrt{a^2 - b^2}}{b\sqrt{a^2 - c^2}} = \frac{c}{ba}\alpha, \quad \cos\theta = \frac{a\sqrt{b^2 - c^2}}{b\sqrt{a^2 - c^2}} = \frac{a}{bc}\gamma;$$

and substituting these values, the equation becomes

$$-2z\frac{b}{ca} + \frac{x^2}{b^2} + \frac{y^2}{b^2} + z^2\frac{(a^2 + c^2)b^2 - a^2c^2}{a^2b^2c^2} + 2\frac{\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}}{b^2ca}zx = 0,$$

or, what is the same thing,

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2) + \frac{1}{b^3}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}\,zx + \frac{a^2b^2 + c^2b^2 - a^2b^2}{2b^3ac}z^2;$$

whence approximately

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2),$$

and thence to the third order in x, y,

$$z = \frac{ca}{b^3}\cdot\tfrac{1}{2}(x^2 + y^2) + \frac{ca}{2b^6}\sqrt{a^2 - b^2}\sqrt{a^2 - c^2}\,x(x^2 + y^2),$$

which is of the form in question.

5, *Downing Terrace, Cambridge, November* 2, 1863.