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ON THE TRANSCENDENT $\operatorname{gd} u = \frac{1}{i} \log \tan (\frac{1}{4}\pi + \frac{1}{2}ui)$.

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THE elliptic functions which correspond to the modulus $k=1$ reduce themselves, as is well known, to circular functions. The case in question plays implicitly an important part in the general theory, and it has been particularly studied by Gudermann, and by Dr Booth in connexion with his theory of parabolic logarithms. But in the absence of a notation corresponding to that used for elliptic functions in general, the theory has not, it appears to me, been exhibited in its proper form. The defect is very easily supplied: using for $\operatorname{am} u$, to the modulus 1, the notation $\operatorname{gd} u$ (Gudermannian of u), then if

$$u = \int_0 \frac{d\phi}{\cos\phi} = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

we have

$$\phi = \operatorname{gd} u;$$

and hence, observing that the equation between u and ϕ is

$$u = \log \frac{1 + \frac{1}{2}\tan\phi}{1 - \frac{1}{2}\tan\phi},$$

or, what is the same thing,

$$\tan \tfrac{1}{2}\phi = \frac{e^u - 1}{e^u + 1},$$

and that we have

$$\log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}ui) = \log \frac{1 + \tan \frac{1}{2}ui}{1 - \tan \frac{1}{2}ui}$$

$$= \log \frac{\cos \frac{1}{2}ui + \sin \frac{1}{2}ui}{\cos \frac{1}{2}ui - \sin \frac{1}{2}ui}$$

$$= \log \frac{e^{\frac{1}{2}u} + e^{-\frac{1}{2}u} + i(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})}{e^{\frac{1}{2}u} + e^{-\frac{1}{2}u} - i(e^{\frac{1}{2}u} - e^{-\frac{1}{2}u})}$$

$$= \log \frac{e^u + 1 + i(e^u - 1)}{e^u + 1 - i(e^u - 1)}$$

$$= \log \frac{1 + i \tan \frac{1}{2}\phi}{1 - i \tan \frac{1}{2}\phi} = \log e^{i\phi} = i\phi,$$

or if

$$u = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

then

$$\phi = \frac{1}{i} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}ui);$$

and substituting for ϕ its value, we obtain

$$\text{gd}\, u = \frac{1}{i} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}ui),$$

which is the definition of the transcendent $\text{gd}\, u$. It is to be noticed that $\text{gd}\, u$, although exhibited in an imaginary form, is a real function of u; and, moreover, that it is an odd function, viz. we have

$$\text{gd}(-u) = -\text{gd}(u),$$

and therefore also

$$\text{gd}(0) = 0.$$

The original equation,

$$u = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

written under the form

$$u = i \frac{1}{i} \log \tan \left(\tfrac{1}{4}\pi + \tfrac{1}{2} i \frac{\phi}{i}\right),$$

shows that we have

$$u = i\, \text{gd}\left(\frac{\phi}{i}\right) = i\, \text{gd}(-i\phi);$$

or substituting for ϕ its value $\text{gd}\, u$, we have

$$u = i\, \text{gd}(-i\, \text{gd}\, u),$$

which may also be written

$$iu = \text{gd}(i\, \text{gd}\, u);$$

so that $\text{gd}\, u$ is a quasi-periodic function of the second order—a property which has not, at least obviously, any analogue in the general theory. We have

$$\cos \text{gd}\, u = \tfrac{1}{2}(e^{i\,\text{gd}\, u} + e^{-i\,\text{gd}\, u})$$

$$= \tfrac{1}{2}\left(\frac{1 + \tan \frac{1}{2}ui}{1 - \tan \frac{1}{2}ui} + \frac{1 - \tan \frac{1}{2}ui}{1 + \tan \frac{1}{2}ui}\right)$$

$$= \frac{1 + \tan^2 \frac{1}{2}ui}{1 - \tan^2 \frac{1}{2}ui} = \frac{1}{\cos ui};$$

and in like manner

$$\sin \operatorname{gd} u = \frac{1}{2i}\left(e^{i \operatorname{gd} u} - e^{-i \operatorname{gd} u}\right)$$

$$= \frac{1}{2i}\left(\frac{1+\tan\frac{1}{2}ui}{1-\tan\frac{1}{2}ui} - \frac{1-\tan\frac{1}{2}ui}{1+\tan\frac{1}{2}ui}\right)$$

$$= \frac{2\tan\frac{1}{2}ui}{i\left(1-\tan^2\frac{1}{2}ui\right)} = \frac{\sin ui}{i \cos ui},$$

or, as these equations may also be written,

$$\sec \operatorname{gd} u = \cos ui = \tfrac{1}{2}\left(e^{u} + e^{-u}\right),$$

$$\tan \operatorname{gd} u = \frac{1}{i}\sin ui = \tfrac{1}{2}\left(e^{u} - e^{-u}\right);$$

and from these equations we have

$$\sec \operatorname{gd}(u+v) = \sec \operatorname{gd} u \,.\, \sec \operatorname{gd} v + \tan \operatorname{gd} u \,.\, \tan \operatorname{gd} v,$$

$$\tan \operatorname{gd}(u+v) = \tan \operatorname{gd} u \,.\, \sec \operatorname{gd} v + \tan \operatorname{gd} v \,.\, \sec \operatorname{gd} v;$$

or, what is the same thing,

$$\sin \operatorname{gd}(u+v) = \frac{\sin \operatorname{gd} u + \sin \operatorname{gd} v}{1 + \sin \operatorname{gd} u \,.\, \sin \operatorname{gd} v},$$

$$\cos \operatorname{gd}(u+v) = \frac{\cos \operatorname{gd} u \,.\, \cos \operatorname{gd} v}{1 + \sin \operatorname{gd} u \,.\, \sin \operatorname{gd} v};$$

which forms are at once obtainable from the formulæ

$$\sin \operatorname{am}(u+v) = \frac{\sin \operatorname{am} u \cos \operatorname{am} v \,\Delta \operatorname{am} v + \sin \operatorname{am} v \cos \operatorname{am} u \,\Delta \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

$$\cos \operatorname{am}(u+v) = \frac{\cos \operatorname{am} u \cos \operatorname{am} v - \sin \operatorname{am} u \sin \operatorname{am} v \,\Delta \operatorname{am} u \,\Delta \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

$$\Delta \operatorname{am}(u+v) = \frac{\Delta \operatorname{am} u \,\Delta \operatorname{am} v - k^2 \sin \operatorname{am} u \sin \operatorname{am} v \cos \operatorname{am} u \cos \operatorname{am} v}{1 - k^2 \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v},$$

observing that for $k = 1$ we have $\Delta \operatorname{am} = \cos \operatorname{am}$, and that the numerators and denominator each of them divide by

$$1 - \quad \sin \operatorname{am} u \sin \operatorname{am} v.$$

2, *Stone Buildings, W.C., May* 7, 1862.