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ON A SURFACE OF THE FOURTH ORDER.

LET A, B, C be fixed points; it is required to investigate the nature of the surface, the locus of a point P such that

$$\lambda AP + \mu BP + \nu CP = 0,$$

where λ, μ, ν are given coefficients; the equation depends, it is clear, on the ratios only of these quantities.

The surface is easily seen to be of the fourth order; it is obviously symmetrical in regard to the plane ABC; and the section by this plane, or say the principal section, is a curve of the fourth order, the locus of a point M such that

$$\lambda AM + \mu BM + \nu CM = 0.$$

The curve is considered incidentally by Mr Salmon, p. 125 of his *Higher Plane Curves* [Ed. 3, p. 126 and see also p. 240 *et seq.*]; and he has remarked that the two circular points at infinity are double points on the curve, which is therefore of the eighth class. Moreover, that there are two double foci, since at each of these circular points there are two tangents, each tangent of the one pair intersecting a tangent of the other pair in a double focus; hence, further, that there are four other foci, the points A, B, C, and a fourth point D lying in a circle with A, B, C, and which are such that, selecting any three at pleasure of the points A, B, C, D, the equation of the curve is in respect to such three points of the same form as it is in regard to the points A, B, C.

Consider a given point M, on the principal section, then the equations

$$\frac{BP}{BM} = \frac{CP}{CM}, \qquad \frac{CP}{CM} = \frac{AP}{AM}, \qquad \frac{AP}{AM} = \frac{BP}{BM}$$

belong respectively to three spheres: each of the spheres passes through the point M. The first of the spheres is such that, with respect to it, B and C are the images

each of the other; that is, the centre of the sphere lies on the line BC, and the product of its distances from B and C is equal to the square of the radius; in like manner the second sphere is such that, with regard to it, C and A are the images each of the other; and the third sphere is such that, with regard to it, A and B are the images each of the other. The three spheres intersect in a circle through M at right angles to the principal plane (that is, the three spheres have a common circular section), and the equations of this circle may be taken to be

$$\frac{AP}{AM} = \frac{BP}{BM} = \frac{CP}{CM}.$$

It is clear that the circle of intersection lies wholly on the surface.

The spheres meet the principal plane in three circles, which are the diametral circles of the spheres; these circles are related to each other and to the points A, B, C, in like manner as the spheres are to each other and to the same points. The circles have thus a common chord; that is, they meet in the point M and in another point M': and MM' is the diameter of the circle, the intersection of the three spheres.

It may be shown that M, M' are the images each of the other in respect to the circle through A, B, C. In fact, consider in the first place the two points A, B, and a circle such that, with respect to it, A, B are the images each of the other; take M a point on this circle, and let O be any point on the line at right angles to AB through its middle point, and join OM cutting the circle in M'; then it is easy to see that M, M' are the images each of the other, in regard to the circle, centre O and radius $OA (=OB)$. Hence starting with the points A, B, C and the point M, let O be the centre of the circle through A, B, C, and take M' the image of M in respect to this circle; then considering the circle which passes through M, and in respect to which B, C are images each of the other, this circle passes through M'; and so the circle through M, in respect to which C, A are images each of the other, and the circle through M, in respect to which A, B are images each of the other, pass each of them through M'; that is, the three circles intersect in M'.

It is to be noticed that M', being on the surface, must be on the principal section; that is, the principal section is such that, taking upon it any point M, and taking M' the image of M in regard to the circle through A, B, C, then M' is also on the principal section. It is very easily shown that the curve of the fourth order possesses this property; for M, M' being images each of the other in respect to the circle through A, B, C, then A, B, C are points of this circle, or we have

$$\frac{MA}{M'A} = \frac{MB}{M'B} = \frac{MC}{M'C};$$

that is, the equation

$$\lambda AM + \mu BM + \nu CM = 0$$

being satisfied, the equation

$$\lambda AM' + \mu BM' + \nu CM' = 0$$

is also satisfied.

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The points M, M' of the curve, which are images each of the other in respect to the circle through A, B, C, may be called conjugate points of the curve. The above-mentioned circle, the intersection of the three spheres, is the circle having MM' for its diameter; hence the required surface is the locus of a circle at right angles to the principal plane, and having for its diameter MM', where M and M' are conjugate points of the curve.

In the particular case where the equation of the surface is

$$BC \cdot AP + CA \cdot BP + AB \cdot CP = 0,$$

the principal section is the circle through A, B, C, twice repeated. Any point on the circle is its own conjugate, and the radius of the generating circle of the surface is zero; that is, the surface is the annulus, the envelope of a sphere radius 0, having its centre on the circle through A, B, C. Or attending to real points only, the surface reduces itself to the circle through A, B, C. But this last statement of the solution is an incomplete one. The equation of an annulus, the envelope of a sphere radius c, having its centre on a circle radius unity, is

$$\sqrt{x^2 + y^2} = 1 \pm \sqrt{c^2 - z^2};$$

and hence putting $c = 0$, the equation of the surface is,

$$\sqrt{x^2 + y^2} = 1 \pm zi$$

(if, as usual, $i = \sqrt{-1}$), or, what is the same thing, it is

$$x^2 + y^2 + (z \pm i)^2 = 0;$$

that is, the surface is made up of the two spheres, passing through the points A, B, C, and having each of them the radius zero; or say the two *cone-spheres* through the points A, B, C. In other words, the equation

$$BC \cdot AP + CA \cdot BP + AB \cdot CP = 0$$

is the condition in order that the four points A, B, C, P may lie on a sphere radius zero, or cone-sphere. Using 1, 2, 3, 4 in the place of A, B, C, P to denote the four points, the last-mentioned equation becomes

$$12 \cdot 34 + 13 \cdot 42 + 14 \cdot 23 = 0;$$

and considering 12, &c. as quadratic radicals, the rational form of this equation is

$$\square = \begin{vmatrix} 0 & \overline{12}^2 & \overline{13}^2 & \overline{14}^2 \\ \overline{21}^2 & 0 & \overline{23}^2 & \overline{24}^2 \\ \overline{31}^2 & \overline{32}^2 & 0 & \overline{34}^2 \\ \overline{41}^2 & \overline{42}^2 & \overline{43}^2 & 0 \end{vmatrix} = 0.$$

In my paper "On a Theorem in the Geometry of Position," *Camb. Math. Journ.* vol. II. pp. 267—271 (1841), [1], I obtained this equation, the four points being there considered as lying in a plane, as the relation between the distances of four points in a circle, in addition to the relation

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ 1, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ 1, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ 1, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0,$$

which exists between the distances of any four points in a plane. The present investigation shows the signification of the equation $\square = 0$ between the distances of four points in space; viz. it expresses that the four points lie in a sphere radius zero, or cone-sphere. But the formula in question is in reality included in that given in the paper for the distances of five points in space. For calling the points 0, 1, 2, 3, 4, the relation between the distances of these five points is

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & \overline{01}^2, & \overline{02}^2, & \overline{03}^2, & \overline{04}^2 \\ 1, & \overline{10}^2, & 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2 \\ 1, & \overline{20}^2, & \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2 \\ 1, & \overline{30}^2, & \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2 \\ 1, & \overline{40}^2, & \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0 \end{vmatrix} = 0.$$

Hence if 1, 2, 3, 4 are the centres of spheres radii α, β, γ, δ, and if 0 is the centre of a tangent sphere radius r, we have

$$\overline{01} = r \pm \alpha, \quad \overline{02} = r \pm \beta, \quad \overline{03} = r \pm \gamma, \quad \overline{04} = r \pm \delta;$$

so that, for any given combination of signs, it would at first sight appear that r is determined by a quartic equation; but by means of a simple transformation (indicated to me by Prof. Sylvester) it may be shown that the equation for r is really a quadratic one; moreover, the equation remains unaltered if the signs of α, β, γ, δ and of r, are all reversed; and r^2 has thus in the whole sixteen values. In particular, if α, β, γ, δ are each equal 0, then r^2 is determined by a simple equation (r the radius of the sphere through the four points); and if, moreover, $r = 0$, then we have for the relation between the distances of the four points, the foregoing equation $\square = 0$.

2, *Stone Buildings, W.C., March* 25, 1861.