115.

NOTE ON THE PORISM OF THE IN-AND-CIRCUMSCRIBED POLYGON.

THE equation of a conic passing through the points of intersection of the conics

$$U = 0, \quad V = 0$$

is of the form

$$wU + V = 0,$$

where w is an arbitrary parameter. Suppose that the conic touches a given line, we have for the determination of w a quadratic equation, the roots of which may be considered as parameters for determining the line in question. Let one of the values of w be considered as equal to a given constant k, the line is always a tangent to the conic

$$kU + V = 0;$$

and taking $w = p$ for the other value of w, p is a parameter determining the particular tangent, or, what is the same thing, the point of contact of this tangent.

Suppose the tangent meets the conic $U = 0$ (which is of course the conic corresponding to $w = \infty$) in the points P, P', and let θ, ∞ be the parameters of the point P, and θ', ∞ the parameters of the point P'. It follows from my "Note on the Geometrical representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$," [113][1] and from the theory of invariants, that if $\square \xi$ represent the "Discriminant" of $\xi U + V$

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[1] I take the opportunity of correcting an obvious error in the note in question, viz. $a^2 + b^2 + c^2 - 2bc - 2ca - 2ab$ is throughout written instead of (what the expression should be) $b^2c^2 + c^2a^2 + a^2b^2 - 2a^2bc - 2b^2ca - 2c^2ab$. [This correction is made, *ante* p. 55.]

(I now use the term discriminant in the same sense in which determinant is sometimes used, viz. the discriminant of a quadratic function $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ or $(a, b, c, f, g, h)(x, y, z)^2$, is the determinant $k = abc - af^2 - bg^2 - ch^2 + 2fgh$), and if

$$\Pi\xi = \int_\infty \frac{d\xi}{\sqrt{\Box\xi}},$$

then the following theorem is true, viz.

"If (θ, ∞), (θ', ∞) are the parameters of the points P, P' in which the conic $U = 0$ is intersected by the tangent, the parameter of which is p, of the conic $kU + V = 0$, then the equations

$$\Pi\theta = \Pi p - \Pi k,$$

$$\Pi\theta' = \Pi p + \Pi k,$$

determine the parameters θ, θ' of the points in question." And again,—

"If the variable parameters θ, θ' are connected by the equation

$$\Pi\theta' - \Pi\theta = 2\Pi k,$$

then the line PP' will be a tangent to the conic $kU + V = 0$." Whence, also,—

"If the sides of a triangle inscribed in the conic $U = 0$ touch the conics

$$k\,U + V = 0,$$

$$k'\,U + V = 0,$$

$$k''U + V = 0,$$

then the equation

$$\Pi k + \Pi k' + \Pi k'' = 0$$

must hold good between the parameters k, k', k''."

And, conversely, when this equation holds good, there are an infinite number of triangles inscribed in the conic $U = 0$, and the sides of which touch the three conics; and similarly for a polygon of any number of sides.

The algebraical equivalent of the transcendental equation last written down is

$$\begin{vmatrix} 1, & k, & \sqrt{\Box k} \\ 1, & k', & \sqrt{\Box k'} \\ 1, & k'', & \sqrt{\Box k''} \end{vmatrix} = 0;$$

let it be required to find what this becomes when $k = k' = k'' = 0$, we have

$$\sqrt{\Box k} = A + Bk + Ck^2 + \dots,$$

and substituting these values, the determinant divides by

$$\begin{vmatrix} 1, & k, & k^2 \\ 1, & k', & k'^2 \\ 1, & k'', & k''^2 \end{vmatrix},$$

the quotient being composed of the constant term C, and terms multiplied by k, k', k''; writing, therefore, $k = k' = k'' = 0$, we have $C = 0$ for the condition that there may be inscribed in the conic $U = 0$ an infinity of triangles circumscribed about the conic $V = 0$; C is of course the coefficient of ξ^2 in $\sqrt{\Box\xi}$, i.e. in the square root of the discriminant of $\xi U + V$; and since precisely the same reasoning applies to a polygon of any number of sides,—

THEOREM. The condition that there may be inscribed in the conic $U = 0$ an infinity of n-gons circumscribed about the conic $V = 0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of the square root of the discriminant of $\xi U + V$ vanishes. [This and the theorem p. 90 are erroneous, see *post*, 116].

It is perhaps worth noticing that $n = 2$, i.e. the case where the polygon degenerates into two coincident chords, is a case of exception. This is easily explained.

In particular, the condition that there may be in the conic [1]

$$ax^2 + by^2 + cz^2 = 0$$

an infinity of n-gons circumscribed about the conic

$$x^2 + y^2 + z^2 = 0,$$

is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1 + a\xi)(1 + b\xi)(1 + c\xi)}$$

vanishes; or, developing each factor, the coefficient of ξ^{n-1} in

$$(1 + \tfrac{1}{2} a\xi - \tfrac{1}{8} a^2\xi^2 + \tfrac{1}{16} a^3\xi^3 - \tfrac{5}{64} a^4\xi^4 + \&c.)(1 + \tfrac{1}{2} b\xi - \&c.)(1 + \tfrac{1}{2} c\xi - \&c.)$$

vanishes.

Thus, for a triangle this condition is

$$a^2 + b^2 + c^2 - 2bc - 2ca - 2ab = 0;$$

for a quadrangle it is

$$a^3 + b^3 + c^3 - bc^2 - b^2c - ca^2 - c^2a - ab^2 - a^2b + 2abc = 0,$$

which may also be written

$$(b + c - a)(c + a - b)(a + b - c) = 0;$$

and similarly for a pentagon, &c.

[1] I have, in order to present this result in the simplest form, purposely used a notation different from that of the note above referred to, the quantities $ax^2+by^2+cz^2$ and $x^2+y^2+z^2$ being, in fact, interchanged.

Suppose the conics reduce themselves to circles, or write

$$U = \quad x^2 + y^2 - R^2 = 0,$$
$$V = (x - a)^2 + y^2 - r^2 = 0;$$

R is of course the radius of the circumscribed circle, r the radius of the inscribed circle, and a the distance between the centres. Then

$$\xi U + V = (\xi + 1, \ \xi + 1, \ -\xi R^2 - r^2 + a^2, \ 0, \ -a, \ 0)(x, \ y, \ 1)^2,$$

and the discriminant is therefore

$$-(\xi + 1)^2 (\xi R^2 + r^2 - a^2) - (\xi + 1) a^2 = -(1 + \xi) \{r^2 + \xi (r^2 + R^2 - a^2) + \xi^2 R^2\}.$$

Hence,

THEOREM. The condition that there may be inscribed in the circle $x^2 + y^2 - R^2 = 0$ an infinity of n-gons circumscribed about the circle $(x - a)^2 + y^2 - r^2 = 0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1 + \xi) \{r^2 + \xi (r^2 + R^2 - a^2) + \xi^2 R^2\}}$$

may vanish.

Now

$$(A + B\xi + C\xi^2)^{\frac{1}{2}} = \sqrt{A} \left\{ 1 + \tfrac{1}{2}B \frac{\xi}{A} + (\tfrac{1}{2}AC - \tfrac{1}{8}B^2) \frac{\xi^2}{A^2} + \ldots \right\},$$

or the quantity to be considered is the coefficient of ξ^{n-1} in

$$(1 + \tfrac{1}{2}\xi - \tfrac{1}{8}\xi^2 \ldots) \left\{ 1 + \tfrac{1}{2}B \frac{\xi}{A} + (\tfrac{1}{2}AC - \tfrac{1}{8}B^2) \frac{\xi^2}{A^2} + \ldots \right\},$$

where, of course,

$$A = r^2, \quad B = r^2 + R^2 - a^2, \quad C = R^2.$$

In particular, in the case of a triangle we have, equating to zero the coefficient of ξ^2,

$$(A - B)^2 - 4AC = 0;$$

or substituting the values of A, B, C,

$$(a^2 - R^2)^2 - 4r^2 R^2 = 0,$$

that is

$$(a^2 - R^2 + 2Rr)(a^2 - R^2 - 2Rr) = 0;$$

the factor which corresponds to the proper geometrical solution of the question is

$$a^2 - R^2 + 2Rr = 0,$$

Euler's well-known relation between the radii of the circles inscribed and circumscribed in and about a triangle, and the distance between the centres. I shall not now discuss the meaning of the other factor, or attempt to verify the formulæ which have been given by Fuss, Steiner and Richelot, for the case of a polygon of 4, 5, 6, 7, 8, 9, 12, and 16 sides. See Steiner, *Crelle*, t. II. [1827] p. 289, Jacobi, t. III. [1828] p. 376; Richelot, t. V. [1830] p. 250; and t. XXXVIII. [1849] p. 353.

2 *Stone Buildings, July* 9, 1853.