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ON CERTAIN MULTIPLE INTEGRALS CONNECTED WITH THE THEORY OF ATTRACTIONS.

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It is easy to deduce from Mr Boole's formula, given in my paper "On a Multiple Integral connected with the theory of Attractions," *Journal*, t. II. [1847], pp. 219—223, [44], the equation

$$\int \frac{d\xi\, d\eta \ldots}{[(\xi-\alpha)^2+(\eta-\beta)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{fg \ldots \pi^{\frac{1}{2}n}}{\theta_1{}^n \Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)} \int_\epsilon^\infty \frac{s^{q-1}(\theta_1{}^2-\sigma)^q\, ds}{\sqrt{\left\{\left(s+\frac{f^2}{\theta_1{}^2}\right)\left(s+\frac{g^2}{\theta_1{}^2}\right)\ldots\right\}}}$$

where n is the number of variables of the multiple integral, and the condition of the integration is

$$\frac{(\xi-\alpha_1)^2}{f^2}+\frac{(\eta-\beta_1)^2}{g^2}+\ldots \overset{=}{<} 1\,;$$

also where

$$\sigma = \frac{(\alpha-\alpha_1)^2}{s+\frac{f^2}{\theta_1{}^2}}+\frac{(\beta-\beta_1)^2}{s+\frac{g^2}{\theta_1{}^2}}\ldots+\frac{v^2}{s},$$

and ϵ is the positive root of

$$\theta_1{}^2 = \frac{(\alpha-\alpha_1)^2}{\epsilon+\frac{f^2}{\theta_1{}^2}}+\frac{(\beta-\beta_1)^2}{\epsilon+\frac{g^2}{\theta_1{}^2}}\ldots+\frac{v^2}{\epsilon}.$$

Suppose $f=g\ldots=\theta_1$, and write $(\alpha-\alpha_1)^2+\ldots=k^2$, we obtain

$$\int \frac{d\xi \ldots}{[(\xi-\alpha)^2+\ldots v^2]^{\frac{1}{2}n-q}} = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)} \int_\epsilon^\infty \frac{s^{q-1}(\theta_1{}^2-\sigma)^q\, ds}{(1+s)^{\frac{1}{2}n}},$$

the limiting condition for the multiple integral being

$$(\xi-\alpha_1)^2+\ldots \gtreqless \theta_1^2,$$

and the function σ, and limit ϵ, being given by

$$\sigma=\frac{k^2}{1+s}+\frac{v^2}{s},\quad \theta_1^2=\frac{k^2}{1+\epsilon}+\frac{v^2}{\epsilon},$$

ϵ denoting, as before, the positive root. Observing that the quantity under the integral sign on the second side vanishes for $s=\epsilon$, there is no difficulty in deducing, by a differentiation with respect to θ_1, the formula

$$\int\frac{d\Sigma}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{2\theta_1\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q)}\int_\epsilon^\infty\frac{s^{q-1}(\theta_1^2-\sigma)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n}},$$

where $d\Sigma$ is the element of the surface $(\xi-\alpha_1)^2+\ldots=\theta_1^2$, and the integration is extended over the entire surface.

A slight change of form is convenient. We have

$$\theta_1^2-\sigma=\theta_1^2-\frac{k^2}{1+s}-\frac{v^2}{s}=\frac{1}{s(1+s)}(\theta_1^2s^2+\chi s-v^2),$$

if we suppose

$$\chi=\theta_1^2-k^2-v^2.$$

The formulæ then become

$$\int\frac{d\xi\ldots}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)}\int_\epsilon^\infty\frac{(\theta_1^2s^2+\chi s-v^2)^q\,ds}{s(1+s)^{\frac{1}{2}n+q}},$$

$$\int\frac{d\Sigma}{[(\xi-\alpha)^2\ldots+v^2]^{\frac{1}{2}n-q}}=\frac{2\pi^{\frac{1}{2}n}\theta_1}{\Gamma(\frac{1}{2}n-q)\,\Gamma q}\int_\epsilon^\infty\frac{(\theta_1^2s^2+\chi s-v^2)^{q-1}\,ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

in which ϵ is the positive root of the equation

$$\theta_1^2\epsilon^2+\chi\epsilon-v^2=0.$$

I propose to transform these formulæ by means of the theory of images; it will be convenient to investigate some preliminary formulæ. Suppose $\lambda^2=\alpha^2+\beta^2\ldots$, $\lambda_1^2=\alpha_1^2+\beta_1^2\ldots$; also consider the new constants $a, b, \ldots, a_1, b_1, \ldots, u, f_1$, determined by the equations

$$\frac{\delta^2\alpha}{\lambda^2+v^2}=a,\quad \frac{\delta^2\alpha_1}{\lambda_1^2-\theta_1^2}=a_1,$$
$$\vdots\qquad\qquad\vdots$$
$$\frac{\delta^2 v}{\lambda^2+v^2}=u,\quad \frac{\delta^2\theta_1}{\lambda_1^2\sim\theta_1^2}=f_1,$$

where δ is arbitrary. Then, putting

$$l^2=a^2+b^2\ldots,\quad l_1^2=a_1^2+b_1^2\ldots,$$

it is easy to see that

$$(\lambda^2+\upsilon^2)(l^2+u^2)=\delta^4, \quad (\lambda_1^2-\theta_1^2)(l_1^2-f_1^2)=\delta^4,$$

and

$$\frac{\delta^2 a}{l^2+u^2}=\alpha, \quad \frac{\delta^2 a_1}{l_1^2-f_1^2}=\alpha_1,$$
$$\vdots \qquad \vdots$$
$$\frac{\delta^2 u}{l^2+u^2}=\upsilon, \quad \frac{\delta^2 f_1}{l_1^2\sim f_1^2}=\theta_1.$$

Proceeding to express the single integrals in terms of the new constants, we have in the first place $k^2=\delta^4 k^2$, where

$$k^2=\left(\frac{a}{l^2+u^2}-\frac{a_1}{l_1^2-f_1^2}\right)^2+\dots;$$

or if we write

$$aa_1+bb_1\dots=ll_1\cos\omega,$$

we have

$$k^2=\frac{l^2}{(l^2+u^2)^2}+\frac{l_1^2}{(l_1^2-f_1^2)^2}-\frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)}.$$

Hence also $\chi=\delta^4 j$, where

$$j=\frac{f_1^2}{(l_1^2-f_1^2)^2}-k^2-\frac{u^2}{(l^2+u^2)^2},$$

whence

$$-j=\frac{1}{l^2+u^2}+\frac{1}{l_1^2-f_1^2}+\frac{2ll_1\cos\omega}{(l^2+u^2)(l_1^2-f_1^2)},$$

$$=\frac{1}{(l^2+u^2)(l_1^2-f_1^2)}\{p^2+u^2-f_1^2\},$$

where $p^2=l^2+l_1^2-2ll_1\cos\omega$, that is

$$p^2=(a-a_1)^2+(b-b_1)^2+\dots;$$

consequently $\theta_1^2 s^2+\chi s-\upsilon^2=\delta^4\Pi$, where Π is given by

$$\Pi=\frac{f_1^2}{(l_1^2-f_1^2)^2}s^2-\frac{(p^2+u^2-f_1^2)}{(l^2+u^2)(l_1^2-f_1^2)}s-\frac{u^2}{(l^2+u^2)^2};$$

and it is clear that ϵ will be the positive root of

$$0=\frac{f_1^2}{(l_1^2-f_1^2)^2}\epsilon^2-\frac{(p^2+u^2-f_1^2)}{(l^2+u^2)(l_1^2-f_1^2)}\epsilon-\frac{u^2}{(l^2+u^2)^2}.$$

It may be noticed that, in the particular case of $u=0$, the roots of this equation are 0, and $\dfrac{(p^2-f_1^2)(l_1^2-f_1^2)}{l^2f_1^2}$. Consequently if $p^2-f_1^2$ and $l_1^2-f_1^2$ are of opposite signs, we have $\epsilon=0$; but if $p^2-f_1^2$ and $l_1^2-f_1^2$ are of the same sign, $\epsilon=\dfrac{(p^2-f_1^2)(l_1^2-f_1^2)}{l^2f_1^2}$.

In order to transform the double integrals, considering the new variables $x, y, \ldots$, I write $x^2+y^2 \ldots = r^2$ and

$$\xi = \frac{\delta^2 x}{r^2}, \ldots$$

whence also, if $\xi^2+\eta^2+\ldots=\rho^2$ (which gives $r\rho=\delta^2$), we have

$$x = \frac{d^2\xi}{\rho^2}, \ldots;$$

also it is immediately seen that

$$(\xi-\alpha)^2+\ldots\, v^2 = \frac{\delta^4}{(l^2+u^2)\,r^2}\{(x^2-a)^2+\ldots+u^2\},$$

$$(\xi-\alpha_1)^2\ldots-\theta_1^{\,2} = \frac{\delta^4}{(l_1^{\,2}-f_1^{\,2})\,r^2}\{(x-a_1)^2+\ldots-f_1^2\};$$

and from the latter equation it follows that the limiting condition for the first integral is $(x-a_1)^2+\ldots \gtreqless f_1^2$ (there is no difficulty in seeing that the sign < in the former limiting condition gives rise here to the sign >), and that the second integral has to be extended over the surface $(x-a_1)^2+\ldots=f_1^2$. Also if dS represent the element of this surface, we may obtain

$$d\xi\, d\eta \ldots = \frac{\delta^{2n}}{r^{2n}}\,dx\,dy\ldots,\quad d\Sigma = \frac{\delta^{2n-2}}{r^{2n-2}}\,dS;$$

and, combining the above formulæ, we obtain

$$\int \frac{dx\,dy\ldots}{(x^2+y^2\ldots)^{\frac{1}{2}n+q}\,\{(x-a)^2+(y-b)^2\ldots+u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-q)\,\Gamma(q+1)\,(l^2+u^2)^{\frac{1}{2}n-q}}\int_\epsilon^\infty \frac{\Pi^q ds}{s\,(1+s)^{\frac{1}{2}n+q}},$$

the limiting condition of the multiple integral being

$$(x-a_1)^2+(y-b_1)^2\ldots \gtreqless f_1^{\,2};$$

and

$$\int \frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n+q-1}\,\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-q}}$$

$$= \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n-q)\,\Gamma q\,(l^2+u^2)^{\frac{1}{2}n-q}\,(l_1^{\,2}-f_1^{\,2})}\int_\epsilon^\infty \frac{\Pi^{q-1}ds}{(1+s)^{\frac{1}{2}n+q-1}},$$

where dS is the element of the surface $(x-a_1)^2+(y-b_1)^2\ldots=f_1^2$, and the integration extends over the entire surface. In these formulæ, l, l_1, p, Π denote as follows:

$$l^2=a^2+b^2+\ldots,\quad l_1^{\,2}=a_1^{\,2}+b_1^{\,2}+\ldots,\quad p^2=(a-a_1)^2+(b-b_1)^2+\ldots,$$

$$\Pi = \frac{f_1^{\,2}}{(l_1^{\,2}-f_1^{\,2})^2}s^2 - \frac{(p^2+u^2-f_1^{\,2})}{(l_1^{\,2}-f_1^{\,2})\,(l^2+u^2)}s - \frac{u^2}{(l^2+u^2)^2};$$

and ϵ is the positive root of the equation $\Pi=0$.

The only obviously integrable case is that for which in the second formula $q=1$; this gives

$$\int \frac{dS}{(x^2+y^2\ldots)^{\frac{1}{2}n}\{(x-a)^2+(y-b)^2+u^2\}^{\frac{1}{2}n-1}} = \frac{2\pi^{\frac{1}{2}n} f_1}{\Gamma(\frac{1}{2}n)(l^2+u^2)^{\frac{1}{2}n-1}(l_1^2 \sim f_1^2)(1+\epsilon)^{\frac{1}{2}n-1}}.$$

In the case of $u=0$, we have, as before, when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of opposite signs, $\epsilon=0$, and therefore $1+\epsilon=1$; but when $p^2-f_1^2$ and $l_1^2-f_1^2$ are of the same sign, the value before found for ϵ gives

$$1+\epsilon = \frac{1}{l^2 f_1^2}\{l^2 f_1^2 + (p^2-f_1^2)(l_1^2-f_1^2)\}.$$

Consider the image of the origin with respect to the sphere $(x-a_1)^2+(y-b_1)^2\ldots=f_1^2$, the coordinates of this image are

$$\frac{a_1}{l_1^2}(l_1^2-f_1^2),\quad \frac{b_1}{l_1^2}(l_1^2-f_1^2), \ldots,$$

and consequently, if μ be the distance of this image from the point $(a, b \ldots)$, we have

$$\mu^2 = \{a - \frac{a}{l_1^2}(l_1^2-f_1^2)\}^2 + \ldots$$

$$= \frac{1}{l_1^2}\{l^2 f_1^2 + (p^2-f_1^2)(l_1^2-f_1^2)\};$$

whence, by a simple reduction,

$$1+\epsilon = \frac{l_1^2 \mu^2}{l^2 f_1^2},$$

or the values of the integral are

$$p^2-f_1^2 \text{ and } l_1^2-f_1^2 \text{ opposite signs, } I = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\,\frac{f_1}{l^{n-2}(l_1^2 \sim f_1^2)},$$

$$p^2-f_1^2 \text{ and } l_1^2-f_1^2 \text{ the same sign, } I = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\,\frac{f_1^{\,n-1}}{l_1^{\,n-2}\mu^{n-2}(l_1^2 \sim f_1^2)},$$

where μ is the distance from the point $(a, b \ldots)$ of the image of the origin with respect to the sphere $(x-a_1)^2+\ldots-f_1^2=0$.

Stone Buildings, August 6, 1850.